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Prime Labeling of Cyclotic Graph and Union of Cyclotic Graphs

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ABSTRACT: A graph G of order n is said to be prime if there is a bijective function from the set of vertices to the first n natural numbers such that labels of adjacent vertices are relatively prime. In the present work, we investigate some results for primality of recently introduced cyclotic graph G(n,k) for different values of n and k. We also investigate some condition under which arbitrary union of some specific cyclotic graph will be prime.

Keywords: Cyclotic graph, Disjoint unoin of graphs, Prime labeling.

Abbreviations: gcd, greatest common divisor.

I. INTRODUCTION

We consider only undirected and non-trivial graph $G=(V(G),E(G))$ with the vertex set $V(G)$ and edge set $E(G)$. For various graph theoretic notations and terminology we follow Gross and Yellen [4], whereas for number theoretic results we follow Burton [1].

Definition 1.1 Let $G=(V(G),E(G))$ be a graph with n vertices. A bijection $f:V(G)\rightarrow\{1,2,3,\dots,n\}$ is called a prime labeling if for each edge $e=uv$ in $E(G)$, gcd$\big(f(u),f(v)\big)=1$. A graph which admits a prime labeling is called a prime graph.

The notion of a prime labeling was introduced by Roger Entringer. Many researchers have studied prime labeling for a good number of graphs listed in Gallian [3]. Entringer conjectured that all trees have prime labeling which is not settled till today. In this paper we investigate some results about primality of cyclotic graph.

Prajapati and Singh [6] introduced cyclotic graph and studied some of its basic properties like planarity, hamiltonianity etc. and also proved some results about number of edges of cyclotic graph. Thenafter, Shrimali and Singh [5] proved some results about neighborhood-prime labeling of cyclotic graph.

Definition 1.2 The cyclotic graph $G(n,k)$ with a positive integer $n>1$ and integer k such that $0\le k\le n-1$ are defined to be a graph with $V\big(G(n,k)\big)=\{v_i:1\le i\le n\}$ and $E\big(G(n,k)\big)=E_O\cup E_I$ where $E_O=\left\{v_{2i}v_{2i+1}:1\le i\le\left\lfloor\frac{n}{2}\right\rfloor\right\}$ and $E_I=\left\{v_1v_{1+(k+1)},v_{1+(k+1)}v_{1+2(k+1)},\dots,v_{n-2k-1}v_{n-k},v_{n-k}v_1\right\}$, here subscripts are taken modulo n. The elements of E_O are called outer edges and the elements of E_I are called inner edges. Cycle formed by all the inner edges is called inner cycle and i_e denotes the number of inner edges in graph $G(n,k)$.

Definition 1.3 An independent set of vertices in a graph G is a set of mutually non-adjacent vertices.

Definition 1.4 The independence number of a graph G is the maximum cardinality of an independent set of vertices. It is denoted by $\alpha(G)$.

Following lemma is useful for proving certain graphs are not prime. The proof of this lemma is available in [2].

Lemma 1.1 Let G be a prime graph of order n then the independence number $\alpha(G)\ge\left\lfloor\frac{n}{2}\right\rfloor$.

II. RESULTS AND DISCUSSION

Theorem 2.1 The cyclotic graph $G(n,0)$ is prime.

Proof: Let $G=G(n,0)$ and $v_1,v_2,\dots v_n$ be the vertices of G. Let $E(G)=E_O\cup E_I$, where $E_O=\left\{v_{2i}v_{2i+1}:1\le i\le\left\lfloor\frac{n}{2}\right\rfloor\right\}$ and $E_I=\{v_1v_2,v_2v_3,\dots,v_{n-1}v_n,v_nv_1\}$. Define a bijective map $f:V(G)\rightarrow\{1,2,3,\dots,n\}$ as $f(v_i)=i$. From the definition of f the labels of two consecutive vertices are relatively prime. So f is prime labeling. Hence, $G(n,0)$ is a prime graph.

Theorem 2.2 The cyclotic graph $G(n,n-1)$ is prime.

Proof: Let $G(n,n-1)$ and $v_1,v_2,\dots v_n$ be the vertices of G. Let $E(G)=E_O\cup E_I$, where $E_O=\left\{v_{2i}v_{2i+1}:1\le i\le\left\lfloor\frac{n}{2}\right\rfloor\right\}$ and $E_I=\{v_1v_1\}$. Clearly, a bijective function $f:V(G)\rightarrow\{1,2,3,\dots,n\}$ defined by $f(v_i)=i$ becomes a prime labeling. Hence, $G(n,n-1)$ is a prime graph.

Theorem 2.3 If $(k+1)|n$, $k>1$, then the cyclotic graph $G(n,k)$ is prime.

Proof: Let $G=G(n,k)$ and $V(G)$ be the vertex set of G. We denote the set of vertices of inner edges of G by $V(E_I)$ and $d(v_i,v_j)$ denotes the distance between vertices v_i and v_j.

case 1: n is even

Let $H=V_1\cup V_2$, where $V_1=\{v_i:v_i\in V(E_I)\}$ and $V_2=\left\{v_i:d(v_i,v_j)=1\text{ for some }v_j\in V_1\text{ and }v_i\notin V_1\right\}$.

Clearly, $H\subset V(G)$. Also, $|V_1|=\frac{n}{k+1}$ and $|V_2|=\frac{n}{k+1}$

Therefore, $|H|=\frac{n}{k+1}+\frac{n}{k+1}=\frac{2n}{k+1}$. (1)

For $1\le i,j\le n$, Define a bijective map $f:V(G)\rightarrow\left\{1,2,3,\dots,\frac{2n}{k+1},\frac{2n}{k+1}+1,\dots,n\right\}$ as follows.

If $v_j \in H$ then,

$$f(v_j) = \begin{cases} 1 & \text{if } j = 1 \\ 2i+1 & \text{if } j = 1 + i(k+1) \\ 2i & \text{if } j = 2 + i(k+1) \text{ where } i(k+1) \text{ is odd} \\ 2i & \text{if } j = i(k+1) \text{ where } i(k+1) \text{ is even} \end{cases}$$

Now, we have to use numbers from the set $\left\{\frac{2n}{k+1} + 1, \dots, n\right\}$ to label vertices of $G(n,k)$ which are not in H. From (1), we have $n - \frac{2n}{k+1} = n\left(\frac{k-1}{k+1}\right)$ vertices which are not in H. By the definition of $G(n,k), \deg(v_i) = 1$ if $v_i \in V(G(n,k)) - H$. Therefore, they form $\frac{n}{2}\left(\frac{k-1}{k+1}\right)$ edges. Now, label end vertices of each edge from these edges by any consecutive integers from $\left\{\frac{2n}{k+1} + 1, \dots, n\right\}$. Hence, $G(n,k)$ is prime. For example, prime labeling of $G(12,2)$ is shown in the Fig. 1.

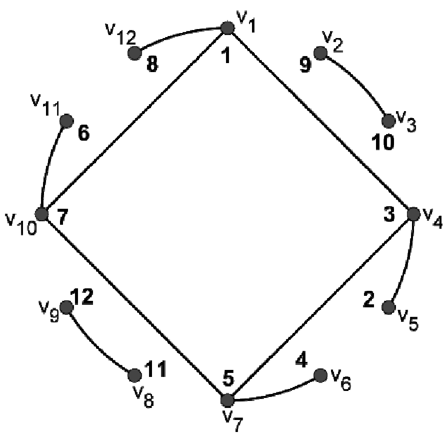

Fig. 1. Prime labeling of $G(12,2)$.

case 2: n is odd

Let $H = V_1 \cup V_2$, where $V_1 = \{v_i : v_i \in V(E_I)\}$ and $V_2 = \{v_i : d(v_i, v_j) = 1 \text{ for some } v_j \in V_1 \text{ and } v_i \notin V_1\}$.

Clearly, $H \subset V(G)$. Also, $|V_1| = \frac{n}{k+1}$ and $|V_2| = \frac{n}{k+1} - 1$

Therefore, $|H| = \frac{n}{k+1} + \frac{n}{k+1} - 1 = \frac{2n}{k+1} - 1$. (2)

For $1 \le i,j \le n$, Define a bijective map $f: V(G) \to \left\{1,2,3,\dots,\frac{2n}{k+1} - 1, \frac{2n}{k+1}, \dots, n\right\}$ as follows.

If $v_j \in H$ then,

$$f(v_j) = \begin{cases} 1 & \text{if } j = 1 \\ 2i+1 & \text{if } j = 1 + i(k+1) \\ 2i & \text{if } j = 2 + i(k+1) \text{ where } i(k+1) \text{ is odd} \\ 2i & \text{if } j = i(k+1) \text{ where } i(k+1) \text{ is even} \end{cases}$$

Now, we have to use numbers from the set $\left\{\frac{2n}{k+1}, \dots, n\right\}$ to label vertices of $G(n,k)$ which are not in H. From (2), we have $n - \left(\frac{2n}{k+1} - 1\right) = n\left(\frac{k-1}{k+1}\right) + 1$ vertices which are not in H. By definition of $G(n,k), \deg(v_i) = 1$ if $v_i \in V(G(n,k)) - H$. Therefore, they form $\frac{1}{2}\left(n\left(\frac{k-1}{k+1}\right) + 1\right)$ edges. Now, label end vertices of each edge from these edges by any consecutive integers from $\left\{\frac{2n}{k+1}, \dots, n\right\}$. Hence, $G(n,k)$ is prime. For example, prime labeling of $G(15,2)$ is shown in the Fig.2.

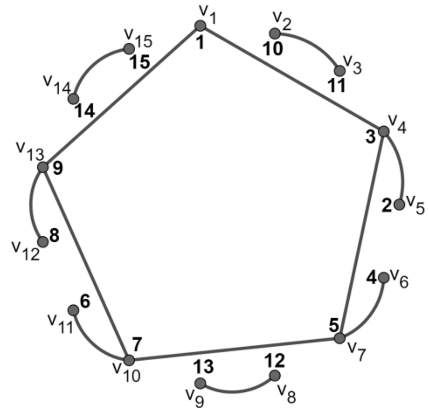

Fig. 2. Prime labeling of $G(15,2)$.

Theorem 2.4 The cyclotic graph $G(n,1)$ is prime if n is even.

Proof: Let $v_1, v_2, \dots v_n$ be the vertices of $G(n,1)$. Define a bijective function $f: V(G(n,1)) \to \{1,2,3,\dots,n\}$ as $f(v_i) = i$. Now, labels of adjacent pair of vertices are either consecutive integer or consecutively odd integers or one of the label is 1. Hence, $G(n,1)$ is prime graph when n is even.

Illustration 2.1 Prime labeling of $G(8,1)$ is shown in the Fig. 3.

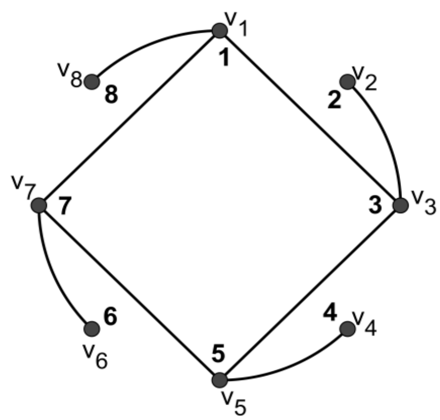

Fig. 3. Prime labeling of G(8,1).

One can observe that, if sum of two positive integers a and b is prime, then $\gcd(a,b) = 1$. Now, we use this result to prove following two theorems.

Theorem 2.5 The cyclotic graph $G(n,1)$ is prime if $n + 2$ is odd prime.

Proof: Let $v_1, v_2, \dots v_n$ be the vertices of $G(n,1)$. Define a bijective function $f: V(G(n,1)) \to \{1,2,3,\dots,n\}$ as

$$f(v_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd} \\ n + 1 - \frac{i}{2} & \text{if } i \text{ is even} \end{cases}$$

Since v_3 and v_{n-1} are adjacent to v_1 and $f(v_1) = 1$, $\gcd(f(v_1), f(v_3)) = \gcd(f(v_1), f(v_{n-1})) = 1$. For adjacent vertices v_2 and v_n, $\gcd(f(v_2), f(v_n)) = \gcd\left(n, \frac{n+1}{2}\right) = 1$.

Now, for remaining pairs of adjacent vertices one can observe that either their labels are consecutive integers or sum of their labels is $n+2$ which is prime as in hypothesis. Hence, we can say that $G(n,1)$ is prime graph.

Illustration 2.2 Prime labeling of $G(11,1)$ is shown in the Fig. 4.

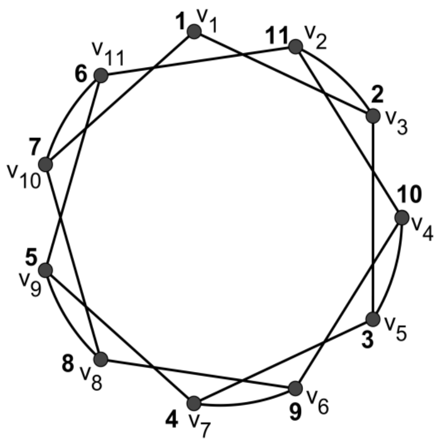

Fig. 4. Prime labeling of G(11,1)

Theorem 2.6 The cyclotic graph $G(n,1)$ is prime if n is odd prime.

Proof: Let $v_1, v_2, \dots v_n$ be the vertices of $G(n,1)$. Define a bijecive function $f: V(G(n,1)) \to \{1,2,3,\dots,n\}$ as

$$f(v_i) = \begin{cases} \frac{2n-i+1}{2} \text{ if } i \text{ is odd} \\ \frac{i}{2} \text{ if } i \text{ is even} \end{cases}$$

Since v_3 and v_{n-1} are adjacent to v_1 and $f(v_1) = n$ which is prime, therefore $\gcd(f(v_1), f(v_3)) = \gcd(f(v_1), f(v_{n-1})) = 1$. Now, v_2 and v_n are adjacent vertices and $f(v_2) = 1$, hence $\gcd(f(v_2), f(v_n)) = 1$. For remaining pair of adjacent vertices, one can observe that labels of adjacent vertices of $G(n,1)$ are either consecutive integers or their sum is n which is prime. Hence, $G(n,1)$ is prime graph.

Theorem 2.7 The Graph $G = \cup_{i=1}^{m} G(n,1)$ is never prime if n is odd integer and $m > 1$.

Proof: One can easily observe that total number of vertices of G are nm and if n is odd integer then the independence number $\alpha(G) = m\left(\frac{n-1}{2}\right)$.

Now,

$$\alpha(G) = m\left(\frac{n-1}{2}\right)$$
$$= \frac{nm}{2} - \frac{m}{2} < \frac{nm}{2} - \frac{1}{2} \le \left\lfloor \frac{nm}{2} \right\rfloor = \left\lfloor \frac{|V(G)|}{2} \right\rfloor$$

Hence, by Lemma 1.1, G is not a prime graph.

III. CONCLUSION

In this paper we have studied prime labeling for cyclotic graph $G(n,0), G(n,n-1)$ and $G(n,1)$ for different values of n and $G(n,k)$ if $(k+1)|n, k > 1$.

IV. FUTURE SCOPE

Prime labeling of cyclotic graph $G(n,k)$ can be studied for different values of n and k which are still open to study. Prime labeling of disjoint union or one point union of cyclotic graph $G(n,k)$ can also be studied for different values of n and k.

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Conflict of Interest: It is confirmed that no one of the two authors has any conflict of interest associated with the publication of this paper.

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