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Some Type of Fuzzy Game Problem and its Solutions

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ABSTRACT: In various fields such as Operations Research, Management Sciences, and Control theory the concept of Fuzzy Game Theory has gained significant attention. This paper focuses on the manipulation of triangular fuzzy numbers and offers a method for solving Fuzzy Game Problems that incorporate these types of fuzzy numbers. It outlines the operations involved with triangular fuzzy numbers and demonstrates how they can be effectively applied to game theory scenarios. By applying the minimax-maximin principle in such settings, we aim to provide a structured method for solving Fuzzy Game Problems with a higher degree of certainty in the presence of fuzziness.

Keywords: Fuzzy sets, Triangular Fuzzy numbers, Fuzzy ranking, Fuzzy Game problem.

INTRODUCTION

Game theory is a mathematical framework that analyzes competitive situations, where individuals or organizations with opposing goals make strategic decisions. Bellman and Zadeh (1970), it focuses on understanding the decision-making processes of participants, or "adversaries," in these scenarios. Game theory is especially useful when two or more entities with conflicting objectives interact and must decide how to act based on their opponents' possible actions. Campos (1989) study fuzzy linear Programming models to solve fuzzy matrix games. Nayak *et al.* (2020) study brain tumor detection and extraction using type-2 fuzzy with morphology. Selvakumari and Lavanya (2015). Given an approach for solving fuzzy game problem. Tanu and Sah (2022), study fuzzy linear programming problems and solutions using ranking function and simplex method. Yadav and Pareek (2014) given Game theory framework for corrugated packaging in polyester sector. Mathur and Kumar (2023), given some application to real life problems, stability analysis of within host dengue model incorporating the Impact of cell mediated and innate immune reactions. For basic theory in fuzzy game theory, an interested readers may refer to Dubois and Prade (1980). We explore algorithms and solutions for fuzzy game problems involving triangular fuzzy numbers, along with a ranking method based on the r-cut procedure for triangular fuzzy numbers. The main emphasis is on the minimax principle, which suggests that each player will make decisions aimed at minimizing their maximum possible loss (or maximizing their minimum possible gain). This principle plays a crucial role in strategic decision-making, where players seek the most favorable outcomes under uncertainty.

Additionally, the paper discusses how an opponent's understanding of the first player's tendencies can influence decision-making. If the second player deduces probabilities different from those assumed in an optimal mixed strategy, they can still use this knowledge to their advantage, reducing the first player's expected payoff. We specifically analyze solutions to such fuzzy games using pure strategies, employing the minimax-maximin principle. This approach helps in determining optimal strategies in uncertain environments, where fuzzy numbers represent the vagueness or imprecision of available information.

FUZZY NUMBER

A fuzzy set F defined on the set of real numbers R is said to be a fuzzy number if its membership function $\mu_F : R \to [0 , 1]$ has the following characteristics.

(i) F is normal, that is there exists an $x \in R$ such that $\mu_F(x) = 1$.

(ii) F is convex, that is for every $x_1, x_2 \in R$,

$\mu_F(\lambda x_1 + (1 - \lambda) x_2) \geq \min\{\mu_F(x_1), \mu_F(x_2)\}, \lambda \in [0 , 1]$

(iii) μ_F is upper semi-continuous.

(iv) Sup (F) is bounded in R.

TRIANGULAR FUZZY NUMBER

A fuzzy number T = (l, m, u) in R is said to be a triangular fuzzy number if its membership function $\mu_T : R \to [0,1]$ has the following characteristics.

$$\mu_T(u)=\begin{cases}\frac{x-l}{m-l} \text{ if } l \le x \le m \\ \frac{u-x}{u-m} \text{ if } m < x \le u \\ 0, \text{otherwise}\end{cases}$$

This means:

- For values between l and m, the membership function increases linearly.
- For values between m and u, the membership function decreases linearly.
- Outside of the interval [l, u], the membership function is zero, meaning the values are not considered part of the fuzzy set.

Visualization:

A triangular fuzzy number can be visualized as a triangle where:

- The base of the triangle spans the range from l to u.
- The peak of the triangle is located at m, representing the most likely value.

Example: 1

Consider the triangular fuzzy number F= (2, 5, 8).

This means:

- The lower bound is 2.
- The peak or most likely value is 5.
- The upper bound is 8.

RANKING OF TRIANGULAR FUZZY NUMBER

Ranking of triangular fuzzy numbers involves comparing fuzzy numbers to determine which one is "larger" or "smaller" based on their values. Lious and Wang (1992) introduce and study Ranking fuzzy numbers with integral value. Salahshour and Abbasbandy (2011), study Ranking fuzzy numbers using fuzzy maximizing-minimizing points. Several methods exist for ranking triangular fuzzy numbers, and one of the common approaches is the Graded Mean Method. Ranking using the Graded Mean method is a technique used for comparing triangular fuzzy numbers by converting them into a crisp value. The basic idea is to compute the graded mean (also known as the graded mean value) of the fuzzy number. This method combines the parameters of the triangular fuzzy number (lower bound l, peak m, and upper bound u) in a way that reflects the overall "central tendency" of the fuzzy number, considering the fuzziness or uncertainty.

Formula for the Graded Mean Method:

Given a triangular fuzzy number T = (l, m, u), where

- l = Lower bound
- m = Peak (modal value)
- u = Upper bound

For every T = (l, m, u) in F(R), the ranking function T: F(R) F(R) by graded mean is defined as

$$\Re(T) = \left(\frac{l+4m+u}{6}\right)$$

For any two triangular fuzzy numbers $T_1 = (l_1, m_1, u_1)$ and $T_2 = (l_2, m_2, u_2)$ in F(R), we have the following comparison.

(i) $T_1 < T_2$ if and only if $\Re(T_1) < \Re(T_2)$.

(ii) $T_1 > T_2$ if and only if $\Re(T_1) > \Re(T_2)$.

(iii) $T_1 = T_2$ if and only if $\Re(T_1) = \Re(T_2)$.

(iv) $T_1 - T_2 = 0$ if and only if $\Re(T_1) - \Re(T_2) = 0$.

(v) If $\Re(T) > 0$ them $T > 0$.

(vi) If $\Re(T) = 0$ them $T = 0$.

r - CUT OF A TRIANGULAR FUZZY NUMBER

The r-cut of a fuzzy number provides a crisp interval for each level of membership. For a triangular fuzzy number T = (a, b, c), the r-cut is the interval corresponding to a membership level r, where $0 \le r \le 1$. For a triangular fuzzy number T= (a, b, c), the r-cut is defined as the interval

$[x_1(r), x_2(r)]$, where:

$x_1(r) = a + (b - a)\, r$ for $0 \le r \le 1$ and $x_2(r) = c - (c - b)\, r$.

This means that for each membership level r, the fuzzy number is represented by a crisp interval $[x_1(r), x_2(r)]$.

ARITHMETIC OPERATIONS ON TRIANGULAR FUZZY NUMBER USING r- CUT

Let us now consider the arithmetic operations like addition, subtraction, multiplication, and division between two triangular fuzzy numbers using the r-cut method.

Let $\tilde{A} = (a_1, b_1, c_1)$ and $\tilde{B} = (a_2, b_2, c_2)$ be two triangular fuzzy numbers. Their r-cuts are: $\tilde{A}_r = [a_{1r}, c_{1r}]$, $\tilde{B}r = [a_{2r}, c_{2r}]$.

(i) Addition of Two Triangular Fuzzy Numbers:

$\tilde{A} + \tilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$.

The r-cut of the sum is: $(\tilde{A} + \tilde{B})\, r = [a_{1r} + a_{2r}, c_{1r} + c_{2r}]$.

(ii) Subtraction of Two Triangular Fuzzy Numbers:

$\tilde{A} - \tilde{B} = (a_1 - a_2, b_1 - b_2, c_1 - c_2)$.

The r-cut of the difference is: $(\tilde{A} - \tilde{B})\, r = [a_{1r} - a_{2r}, c_{1r} - c_{2r}]$.

(iii) Multiplication of Two Triangular Fuzzy Numbers:

Multiplication of fuzzy number is more complex. The r-cut of the product is:

$(\tilde{A} \bullet \tilde{B})r = [\min(a_{1r} \bullet a_{2r}, a_{1r} \bullet c_{2r}, c_{1r} \bullet a_{2r}, c_{1r} \bullet c_{2r}), \max(a_{1r} \bullet a_{2r}, a_{1r} \bullet c_{2r}, c_{1r} \bullet a_{2r}, c_{1r} \bullet c_{2r})]$.

(iv) Division of Two Triangular Fuzzy Numbers:

Division of fuzzy number is also complex. The r-cut of the quotient is:

$(\tilde{A} / \tilde{B})r = [\min(a_{1r} / a_{2r}, a_{1r} / c_{2r}, c_{1r} / a_{2r}, c_{1r} / c_{2r}), \max(a_{1r} / a_{2r}, a_{1r} / c_{2r}, c_{1r} / a_{2r}, c_{1r} / c_{2r})]$.

Example 2:

Let $\tilde{A} = (1, 2, 3)$ and $\tilde{B} = (2, 3, 4)$. Compute $\tilde{A} + \tilde{B}$ using r-cut for r = 0.5.

For $\tilde{A}$:

$a_{1r} = 1 + 0.5\,(2 - 1) = 1.5$ and $c_{1r} = 3 - 0.5\,(3 - 2) = 2.5$.

So, $\tilde{A}\, 0.5 = [1.5, 2.5]$.

For $\tilde{B}$:

a_{2r} = 2 + 0.5 (3 − 2) = 2.5 and c_{2r} = 4 − 0.5 (4 − 3) = 3.5.
So, $\tilde{B}$ 0.5 = [2.5, 3.5]
Therefore,
Addition: $(\tilde{A} + \tilde{B})0.5$ = [1.5 + 2.5, 2.5 + 3.5] = [4, 6].
Subtraction: $(\tilde{A} - \tilde{B})0.5$ = [1.5 - 2.5, 2.5 - 3.5] = [-1, -1].
Multiplication:
$(\tilde{A} \bullet \tilde{B})0.5$ = [min (1.5 • 2.5, 1.5 • 3.5, 2.5 • 2.5, 2.5 • 3.5), max (1.5 • 2.5, 1.5 • 3.5, 2.5 • 2.5, 2.5 • 3.5)]
= [min (3.75, 5.25, 6.25, 8.75), max (3.75, 5.25, 6.25, 8.75)] = [3.75, 8.75].
Division:
$(\tilde{A} / \tilde{B})0.5$ = [min (1.5 / 2.5, 1.5 / 3.5, 2.5 / 2.5, 2.5 / 3.5), max (1.5 / 2.5, 1.5 / 3.5, 2.5 / 2.5, 2.5 / 3.5)]
= [min (0.6, 0.42857, 1, 0.71429), max (0.6, 0.42857, 1, 0.71429)] = [0.42857, 1].

MATHEMATICAL FORMULATION OF A FUZZY GAME PROBLEM

In a two-player game, Player A has m pure strategies denoted as $A_1, A_2, \ldots, A_m$ and Player B has n pure strategies denoted as $B_1, B_2, \ldots, B_n$. Each player selects their strategy from their respective sets of pure strategies. It is assumed that Player A is always the gainer, and Player B is always the loser. This means that all payoffs are expressed in terms of Player A's gains.
If Player A chooses strategy A_i and Player B chooses strategy B_j, the payoff for Player A is represented by a_{ij}. This value a_{ij} indicates the amount Player A gains from Player B under the given strategy combination. The payoff matrix $A = [a_{ij}]$ is an $m \times n$ matrix that encapsulates all possible outcomes of the game, with rows corresponding to Player A's strategies and columns corresponding to Player B's strategies.
Here's a breakdown of the components:
1. Players: Player A: Has m pure strategies, denoted as $A_1, A_2, \ldots., A_m$.
Player B: Has n pure strategies, denoted as $B_1, B_2, \ldots., B_n$.
2. Payoff Matrix: The payoff matrix for Player A is an m × n matrix where each element a_{ij} represents the payoff to Player A when:
Player A chooses strategy A_i.
Player B chooses strategy B_j.
Since this is a zero-sum game, the payoff to Player B is $-a_{ij}$ (*i.e.*, what Player A gains, Player B loses).
3. Assumptions: The game is zero-sum: The gain of Player A is exactly the loss of Player B, and vice versa. All payoffs are from the perspective of Player A.
4. Payoff Matrix Representation: The payoff matrix P for Player A can be written as:

P =	a_{11}	a_{12}	…	a_{1n}
	a_{21}	a_{22}	…	a_{2n}
	⋮	⋮	⋱	⋮
	a_{m1}	a_{m2}	…	a_{mn}

Here:
Rows correspond to Player A's strategies $A_1, A_2, \ldots., A_m$.
Columns correspond to Player B's strategies $B_1, B_2, \ldots.,$ Bn.
The entry a_{ij} is the payoff to Player A when Player A chooses A_i and Player B chooses B_j.
Key Concepts in Zero-Sum Games
1. Pure Strategy Nash Equilibrium: A pair of strategies (A_i, B_j) is a Nash equilibrium if:
Player A cannot improve their payoff by unilaterally changing their strategy.
Player B cannot improve their payoff (*i.e.*, reduce Player A's payoff) by unilaterally changing their strategy.
2. Mixed Strategies: If no pure strategy Nash equilibrium exists, players may randomize their strategies using mixed strategies.
A mixed strategy for Player A is a probability distribution over $A_1, A_2, \ldots,$ Am.
A mixed strategy for Player B is a probability distribution over $B_1, B_2, \ldots, B_n$.
3. Minimax Theorem: In zero-sum games, the minimax theorem states that the maximum payoff Player A can guarantee (maximin value) is equal to the minimum payoff Player B can force Player A to accept (minimax value).
This common value is called the value of the game.
Example 3
Suppose:
Player A has 2 strategies: A_1, A_2.
Player B has 3 strategies: B_1, B_2, B_3.
The payoff matrix P for Player A might look like:

P =	3	1	4
	2	5	0

In the given game scenario, Player A has two strategies, A_1 and A_2, while Player B has three strategies, B_1, B_2, and B_3. The payoffs are defined from Player A's perspective, meaning they represent the gains for Player A and the corresponding losses for Player B.
1. If Player A selects strategy A_1 and Player B selects strategy B_2, the payoff to Player A is 1. This implies that Player A gains 1 unit, and Player B loses 1 unit in this outcome.
2. If Player A selects strategy A_2 and Player B selects strategy B_3, the payoff to Player A is 0. This indicates that there is no gain for Player A and no loss for Player B in this particular outcome.
These payoffs can be represented in a payoff matrix, where the rows correspond to Player A's strategies and the columns correspond to Player B's strategies. The specific payoffs mentioned above would occupy the corresponding cells in the matrix. This framework is commonly used in game theory to analyze strategic interactions between players.

COMPUTATIONAL PROCEDURE FOR SOLVING FUZZY GAME PROBLEM

In this solution, we address a fuzzy game problem where the strategies of the players are represented using triangular fuzzy numbers.

Step 1: Check for the Existence of a Saddle Point

Begin by examining the payoff matrix to determine if a saddle point exists. A saddle point is present when the maximum value in the row minima equals the minimum value in the column maxima. If a saddle point is found, the game's value and optimal strategies are directly identified. If no saddle point exists, proceed to the next step.

Step 2: Compare Column Strategies for Dominance

1. Direct Comparison of Columns: Compare each pair of columns in the payoff matrix. If every element in Column A is less than or equal to the corresponding elements in Column B, then Column A dominates Column B. In this case, Column B can be eliminated from the matrix.

2. Iterative Column Comparison: Repeat the comparison process for all remaining columns, eliminating any dominated columns. Continue this until no further columns can be removed based on direct dominance.

Step 3: Compare Row Strategies for Dominance

1. Direct Comparison of Rows: Analyze each pair of rows in the matrix. If every element in Row A is greater than or equal to the corresponding elements in Row B, then Row A dominates Row B. Row B can then be eliminated from the matrix.

2. Iterative Row Comparison: Conduct similar comparisons across all remaining rows, removing any that are dominated by others. Continue this process until no further rows can be eliminated.

3. Reduction to a Single Cell: If, after the elimination process, the game reduces to a single cell, this cell provides the value of the game along with the optimal strategies for the players. If the game matrix cannot be reduced to a single cell, proceed to the next step.

Step 4: Applying Mixed Strategy Dominance: Dominance is not limited to pure strategies. A strategy can also be dominated if it is inferior to a combination (or average) of two or more other pure strategies. In such cases, compare the given strategy with linear combinations of other strategies. If it proves to be consistently worse, it can be eliminated from the matrix.

By following these steps, the complexity of the payoff matrix can be reduced, simplifying the determination of optimal strategies and the game's value.

Remark: A game is called fair when its value is zero. This means whatever one player wins, the other player loses by the same amount, so neither player ends up with an advantage. Both players have equal chances, making the game balanced and fair for both sides.

EXAMPLE OF A FUZZY GAME PROBLEM

Let us consider the following fuzzy game problem:

	B_1	B_2
A_1	4.21	4
A_2	5	5.72

By using the measure convert the above problem into a crisp value problem.

Here:

- Rows represent Player A's strategies: A_1, A_2.
- Columns represent Player B's strategies: B_1, B_2.
- The entry a_{ij} is the payoff to Player A when Player A chooses A_i and Player B chooses B_j.

Step 1: Find the Maximin Value for Player A:

The maximin value is the maximum of the minimum payoffs for Player A across all their strategies.

1. For each row (Player A's strategy), find the minimum payoff:

- Row 1 (A_1): Minimum payoff = min (4.21, 4) = 4.
- Row 2 (A_2): Minimum payoff = min (5, 5.72) = 5.

2. The maximin value is the maximum of these minimum payoffs:

- Maximin value = max (4, 5) = 5.

Step 2: Find the Minimax Value for Player B:

The minimax value is the minimum of the maximum payoffs for Player B across all their strategies.

1. For each column (Player B's strategy), find the maximum payoff:

- Column 1 (B_1): Maximum payoff = max (4.21, 5) = 5.
- Column 2 (B_2): Maximum payoff = max (4, 5.72) = 5.72.

2. The minimax value is the minimum of these maximum payoffs:

- Minimax value = min (5, 5.72) = 5.

Step 3: Check for Saddle Point:

A saddle point exists if the maximin value equals the minimax value.

- Maximin value = 5
- Minimax value = 5

Since the maximin value equals the minimax value, the game has a saddle point. The value of the game is 5.

Step 4: Optimal Strategies:

- Player A's optimal strategy: Choose the strategy corresponding to the maximin value. Here, Player A should always choose A_2 (since the maximin value of 5 corresponds to A_2).
- Player B's optimal strategy: Choose the strategy corresponding to the minimax value. Here, Player B should always choose B_1 (since the minimax value of 5 corresponds to B_1).

Final Solution:

- Value of the game: 5.
- Player A's optimal strategy: Always choose A_2.
- Player B's optimal strategy: Always choose B_1.

CONCLUSIONS

This paper presents a straightforward approach to solving fuzzy game problems where the players' strategies are represented by triangular fuzzy numbers. By applying a ranking method to these fuzzy numbers, we convert the fuzzy payoffs into a crisp payoff matrix. Analyzing this matrix reveals the presence of a saddle point, indicating that both players can adopt pure strategies as their optimal choices. In cases where the payoffs remain in fuzzy form, defuzzification techniques, such as the centroid method, are employed to transform the fuzzy values into crisp numbers. Once defuzzified, the game is solved using traditional game theory methods, following the same steps as with crisp payoffs.

FUTURE SCOPE

- Extension to Other Fuzzy Numbers: Develop methods for games with trapezoidal, Gaussian, or other types of fuzzy numbers.
- Multi-Objective Fuzzy Games: Incorporate multiple objectives for each player.
- Dynamic Fuzzy Games: Study games where strategies and payoffs evolve over time.
- Machine Learning Integration: Use machine learning algorithms to approximate solutions for large-scale fuzzy games.
- Applications: Apply fuzzy game theory to real-world problems like supply chain management, economics, and cyber security.

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REFERENCES

Bellman, R. E. & Zadeh, L. A. (1970). Decision-making in a fuzzy environment. *Management Science, 17*(4), B-141-164.

Campos, L. (1989). Fuzzy linear programming models to solve fuzzy matrix games. *Fuzzy sets and systems*, *32*(3), 275-289.

Dubois, D. and Prade, H. (1980). Fuzzy Sets and Systems, Theory and applications, Academic Press, New York.

Lious, T. S. and Wang, M. J. (1992). Ranking fuzzy numbers with integral value. *Fuzzy sets and systems*, *50*(3), 247-255.

Mathur, K. S. and Kumar, B. (2023). Stability Analysis of Within Host Dengue Model Incorporating the Impact of Cell Mediated and Innate Immune Reactions. *Biological Forum – An International Journal, 15*(6), 674-678.

Nayak, D. R., Padhy, N. and Swain, B. K. (2020). Brain Tumour Detection and Extraction using Type-2 Fuzzy with Morphology. *International Journal on Emerging Technologies, 11*(2), 840–844.

Salahshour, S. and Abbasbandy, T. A. (2011). Ranking fuzzy numbers using fuzzy maximizing-minimizing points. *Aix-les-Bains, France,* 763-769.

Selvakumari, K. and Lavanya, S. (2015). An approach for solving fuzzy game problem. *Indian Journal of Science and Technology, 8*(15), 1-6.

Tanu, K. and Sah, A. K. (2022). Fuzzy linear programming problems and solutions using ranking function and simplex method. *International J. Scientific Res. Sci. Tech., 9*(6), 738-743.

Yadav, R. and Pareek, S. (2014). Game theory Framework for Corrugated Packaging in Polyester Sector. *International Journal of Theoretical & Applied Sciences*, *6*(2), 94-100.

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