



Quaternionic Reformulation of Dyonic Field Equations

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ABSTRACT: Keeping in view the importance of quaternions and their roles in various problems of physics, in this paper, we have made an attempt to reformulate the quantum equations of dyons in terms of simple, compact and consistent notations of quaternions. Establishing the connection between the quaternion basis elements and the Pauli spin matrices, we have reformulated the generalized wave function, generalized four-potential, generalized current, Lorentz force equation of motion associated with dyons in simple and compact quaternion notations. It has been shown that quaternionic form of generalized potential generalized current and Lorentz force equation of motion not only compact and simple but also remains invariant under quaternion transformations showing that the presented formulation is manifestly covariant. Generalized Maxwell-Dirac equation, generalized field tensor and continuity equation have also been reformulated by means of consistent quaternion notation in simple and compact forms. It has been shown that the reformulation of quantum equations by means of quaternions of dyons reproduces the theory of electric charge (magnetic monopole) in the absence of magnetic monopole (electric charge) on dyons.

Keywords: Quaternionic reformulation, Generalized Maxwell-Dirac equation, quaternions of dyons, quaternion quantum mechanics

I. INTRODUCTION

Quaternions were invented by Hamilton [1] to extend the theory of complex variables to three dimensions. Recently, there has been a revival in the formulation of natural laws within the framework of general quaternion algebra and many of the basic physical equations [2-6] have been reformulated by means of quaternions. The quaternion quantum mechanics (QQM) have been investigated by Finkelstein *et al.* [7]. Adler [8] proposed the idea of algebraic quaternion generalization of classical Yang-Mill's theory. Soucek [9] used the quaternion quantum mechanics to describe the theory of tachyons (particles travel faster than light). Keeping in the view the possibility of extension of quaternion formalism to curved spaces [10] and that of quaternion number to 16-dimensional algebra, some authors [3-6] developed quaternionic formulation for extended Maxwell's equations. Quaternions also play an important role in classical field theories [8] and extend the manifold structure. Quaternion representation of the Lorentz group for classical physical application has recently been developed by Abony *et al.* [11] and maintained the relativistic covariance even for the existence of the magnetic monopole. In order to understand the theoretical existence of monopoles (dyons) and keeping in view of their recent potential importance, with the fact that formalism necessary to describe has been clumsy and not manifestly covariant the quaternionic form of generalized field of dyons have been developed in unique, simple, compact and consistent manner [12]. Rajput *et al.* [13] has developed the unique consistent quaternionic formulation for dyons, which reproduces to usual electrodynamics in absence of magnetic charge.

Keeping in view the importance of quaternions and their roles in various problems of physics, in this paper, we have made an attempt to reformulate the quantum equations of dyons in terms of simple, compact and consistent notations of quaternions. Establishing the connection between the quaternion basis elements and the Pauli spin matrices, we have reformulated the generalized wave function, generalized four-potential, generalized current, Lorentz force equation of motion associated with dyons in simple and compact quaternion notations. It has been shown that quaternionic form of generalized potential generalized current and Lorentz force equation of motion not only compact and simple but also remains invariant under quaternion transformations showing that the presented formulation is manifestly covariant. Generalized Maxwell-Dirac equation, generalized field tensor and continuity equation have also been reformulated by means of consistent quaternion notation in simple and compact forms. It has been shown that the reformulation of quantum equations by means of quaternions of dyons reproduces the theory of electric charge (magnetic monopole) in the absence of magnetic monopole (electric charge) on dyons.

II. DYONS AND THEIR FIELD EQUATIONS

Following Rajput *et al.* [13], let us define the generalized charge on dyons as

$$q = e - ig \quad \dots(1)$$

where e and g are electric and magnetic charges respectively. Starting with the idea of two four-potentials [14] $\{A^\mu\} = \{\phi^e, \vec{A}\}$ and $\{B^\mu\} = \{\phi^g, \vec{B}\}$ respectively associated with electric and magnetic charges we define the generalized four-potential $\{V^\mu\} = \{\phi, \vec{V}\}$ of dyons as $V_\mu = A_\mu - i B_\mu$ $\dots(2)$

The electric and magnetic fields of dyons are defined in terms of components of electric and magnetic potentials as

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi_e - \vec{\nabla} \times \vec{B} \quad \& \quad \vec{H} = -\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \phi_g + \vec{\nabla} \times \vec{A} \quad \dots(3)$$

These electric and magnetic fields of dyons are dual invariant under duality transformations given by

$$\vec{E} \rightarrow \vec{E} \cos\theta + \vec{H} \sin\theta \quad \& \quad \vec{H} \rightarrow -\vec{E} \sin\theta + \vec{H} \cos\theta \quad \dots(4)$$

along with then following dual transformations between potentials

$$\{A^\mu\} \rightarrow \{A^\mu\} \cos\theta + \{B^\mu\} \sin\theta \quad \& \quad \{B^\mu\} \rightarrow -\{A^\mu\} \sin\theta + \{B^\mu\} \cos\theta \quad \dots(5)$$

Apart from equation (4), the expression of generalized electric and magnetic fields given by equation (3) are symmetrical. Here, both the electric and magnetic fields of dyons may be written in terms of longitudinal and transverse components while the magnetic field is purely transverse in case of pure electric charge. Let us introduce the generalized electromagnetic vector field of dyons as complex quantity i.e.

$$\vec{\psi} = \vec{E} - i \vec{H} \quad \dots(6)$$

which satisfies the following generalized Dirac Maxwell's (GDM) [13]

$$\vec{\nabla} \cdot \vec{\psi} = J_0 \quad \& \quad \frac{\partial \vec{\Psi}}{\partial t} - i \vec{J} \quad \dots(7)$$

where J_0 and \vec{J} are the generalized charge and current source densities of dyons which are given by

$$\begin{aligned} J_\mu &= j_\mu - i k_\mu \cong (\rho, -\vec{J}), \\ \rho &= \rho_e - i \rho_g; \vec{J} = \vec{j} - i \vec{k} \end{aligned} \quad \dots(8)$$

The relation between generalized vector field and potential of dyons is given as

$$\vec{\psi} = -\frac{\partial \vec{V}}{\partial t} - \text{grad} V_0 - i \text{curl } \vec{V} \quad \dots(9)$$

The tensorial form of generalized Dirac Maxwell's (GDM) is defined as

$$F_{\mu\nu, \nu} = j_\mu \quad \& \quad \tilde{F}_{\mu\nu, \nu} = k_\mu \quad \dots(10)$$

where

$$\begin{aligned} F_{\mu\nu} &= E_{\mu\nu} - \tilde{H}_{\mu\nu} \quad \& \quad \tilde{F}_{\mu\nu} = H_{\mu\nu} - \tilde{E}_{\mu\nu} \\ E_{\mu\nu} &= A_{\mu, \nu} - A_{\nu, \mu} \quad \& \quad H_{\mu\nu} = B_{\mu, \nu} - B_{\nu, \mu} \\ E_{\mu\nu} &= \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} E^{\rho\sigma} \quad \& \quad \tilde{H}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} H^{\rho\sigma} \end{aligned} \quad \dots(11)$$

From equation (7), we obtain a vector parameter S (say) i.e.

$$S = \vec{\Psi} = -\frac{\partial \vec{J}}{\partial t} - \text{grad } J_0 - i \text{curl } \vec{J} \quad \dots(12)$$

where \square represents the D'Alembertian operator and is given as

$$\frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \quad \dots(13)$$

Defining generalized field tensor as

$$G_{\mu\nu} = F_{\mu\nu} - i \tilde{F}_{\mu\nu}; \quad \dots(14)$$

one gets the following generalized field equations of dyons i.e.

$$G_{\mu\nu, \nu} = j_\mu \quad \& \quad \tilde{G}_{\mu\nu, \nu} = 0 \quad \dots(15)$$

where

$$G_{\mu\nu} = V_{\mu, \nu} - V_{\nu, \mu} \quad \dots(16)$$

is called the generalized electromagnetic field tensor of dyons. In terms of four-potential $\{V_\mu\}$ the field equation(15) may be written as

$$V_{\mu, \nu} = J_{\mu, \nu} \quad \dots(17)$$

Equations (10) and (15) are invariant under duality transformation [13]

$$\begin{aligned} (F, \tilde{F}) &= (F \cos\theta + \tilde{F} \sin\theta, -F \sin\theta + \tilde{F} \cos\theta) \\ (j_\mu, k_\mu) &= (j_\mu \cos\theta + k_\mu \sin\theta, -j_\mu \sin\theta + k_\mu \cos\theta) \end{aligned} \quad \dots(18)$$

where $\frac{g}{e} = \frac{A_\mu}{B_\mu} = \frac{k_\mu}{j_\mu} = \frac{F_{\mu\nu}}{\tilde{F}_{\mu\nu}} = \tan \theta$.

Thus, the generalized charge of dyons may also be written as

$$q = |q| \exp(-i\theta) \quad \dots(19)$$

The Lorentz force equation of motion for dyons is

$$f_{\mu} = m_o \frac{d^2 x_{\mu}}{d\tau^2} = \text{Re} (q^* G_{\mu\nu} v^{\nu}) \quad \dots(20)$$

where Re denotes real part, $\frac{d^2 x_{\mu}}{d\tau^2}$ is the four-acceleration and v^{ν} is the four-velocity of the particle.

III. QUATERNION ANALYSIS OF DYONIC FIELDS

Taking the curl of second part of equation (7), we get

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \vec{\Psi}) &= \vec{\nabla} \times \left(-i \frac{\partial \vec{\Psi}}{\partial t} - i \vec{J} \right) \Rightarrow \vec{\nabla} (\vec{\nabla} \cdot \vec{\Psi}) - \nabla^2 \vec{\Psi} = -i \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{\Psi}) - i \vec{\nabla} \times \vec{J} \\ \Rightarrow \vec{\nabla} J_0 - \nabla^2 \vec{\Psi} &= -\frac{\partial^2 \vec{\Psi}}{\partial t^2} - \frac{\partial \vec{J}}{\partial t} - i \vec{\nabla} \times \vec{J} \Rightarrow \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \vec{\Psi} = -\frac{\partial \vec{J}}{\partial t} - \vec{\nabla} J_0 - i \vec{\nabla} \times \vec{J} \\ \Rightarrow \quad \vec{\Psi} &= -\frac{\partial \vec{J}}{\partial t} - \text{grad } J_0 - i \text{curl } \vec{J} = S \text{ (say)} \quad \dots(21) \end{aligned}$$

where \square is called the D'Alembertian operator and we have used the vector properties and 1st part of equation (7). From (9) and (21), we see that the roles of \vec{V} and V_0 in $\vec{\Psi}$ and \vec{J} and J_0 in $\square \vec{\Psi}$ are same. Therefore, J must be related with V by the same operator \square .

In order to write the field equations (9) and (21) in quaternionic form we define the quaternion in terms of Pauli spin matrices as

$$q = q_0 + \sigma_1 q_1 + \sigma_2 q_2 + \sigma_3 q_3 \quad \dots(22)$$

where $\sigma_i = ie_i$, $\sigma_i^2 = 1$ & $\sigma_i \sigma_j = \delta_{ij} + i \varepsilon_{ijk} \sigma_k$.

The four-potential $\{V_{\mu}\}$ and four-current $\{J_{\mu}\}$ of electromagnetic field may also be written as quaternion valued in the following way

$$V = V_0 + \sigma_1 V_1 + \sigma_2 V_2 + \sigma_3 V_3 \quad \dots(23)$$

and

$$J = J_0 + \sigma_1 J_1 + \sigma_2 J_2 + \sigma_3 J_3 \quad \dots(24)$$

The quaternion differential operator D can be written as

$$D = \partial_0 + \sigma_1 \partial_1 + \sigma_2 \partial_2 + \sigma_3 \partial_3 \quad \dots(25)$$

Operating the equations (23) and (24) by (25) we have

$$\begin{aligned} D V &= (\partial_0 + \sigma_1 \partial_1 + \sigma_2 \partial_2 + \sigma_3 \partial_3) (V_0 + \sigma_1 V_1 + \sigma_2 V_2 + \sigma_3 V_3) \\ &= (\partial_0 V_0 + \text{div} V) + \sigma_1 \{ \partial_0 V_1 + \partial_1 V_0 + i(\nabla \times V)_1 \} \\ &\quad + \sigma_2 \{ \partial_0 V_2 + \partial_2 V_0 + i(\nabla \times V)_2 \} + \sigma_3 \{ \partial_0 V_3 + \partial_3 V_0 + i(\nabla \times V)_3 \} \\ &= \Psi_0 - \sigma_1 \Psi_1 - \sigma_2 \Psi_2 - \sigma_3 \Psi_3 \quad \dots(26) \end{aligned}$$

Using, $\Psi_0 = \partial_0 V_0 + \text{div } \vec{V} = 0$ (due to Lorentz condition) and $\Psi_j = -\partial_0 V_j - \partial_j V_0 - i(\vec{\nabla} \times \vec{V})_j$ from equation (9), we get

$$D V = -\Psi \quad \dots(27)$$

Similarly

$$\begin{aligned} D J &= (\partial_0 + \sigma_1 \partial_1 + \sigma_2 \partial_2 + \sigma_3 \partial_3) (\partial_0 J_0 + \sigma_1 J_1 + \sigma_2 J_2 + \sigma_3 J_3) \\ &= (\partial_0 J_0 + \text{div} J) + \sigma_1 \{ \partial_0 J_1 + \partial_1 J_0 + i(\nabla \times J)_1 \} \\ &\quad + \sigma_2 \{ \partial_0 J_2 + \partial_2 J_0 + i(\nabla \times J)_2 \} + \sigma_3 \{ \partial_0 J_3 + \partial_3 J_0 + i(\nabla \times J)_3 \} \\ &= S_0 - \sigma_1 S_1 - \sigma_2 S_2 - \sigma_3 S_3 \end{aligned}$$

Here $S_0 = (\partial_0 J_0 + \text{div } \vec{J}) = 0$ (due to equation of continuity) and

$$S = -\partial_0 J_j - \partial_j J_0 - i(\vec{\nabla} \times \vec{J})_j \text{ (eqn.(21))}$$

Therefore

$$D J = -S \text{ (since } S = \sigma_1 S_1 + \sigma_2 S_2 + \sigma_3 S_3 \text{)} \quad \dots(28)$$

Now the quaternion conjugate of D, V, J, Ψ and S are define as

$$\overline{D} = \partial_0 - \sigma_1 \partial_1 - \sigma_2 \partial_2 - \sigma_3 \partial_3 \quad \dots(29a)$$

$$\overline{V} = V_0 - \sigma_1 V_1 - \sigma_2 V_2 - \sigma_3 V_3 \quad \dots(29b)$$

$$\overline{J} = J_0 - \sigma_1 J_1 - \sigma_2 J_2 - \sigma_3 J_3 \quad \dots(29c)$$

$$\overline{\Psi} = \Psi_0 - \sigma_1 \Psi_1 - \sigma_2 \Psi_2 - \sigma_3 \Psi_3 \quad \dots(29d)$$

$$\overline{S} = S_0 - \sigma_1 S_1 - \sigma_2 S_2 - \sigma_3 S_3 \quad \dots(29e)$$

Operating Ψ by \overline{D} we get

$$\begin{aligned} \overline{D} \Psi &= (\partial_0 - \sigma_1 \partial_1 - \sigma_2 \partial_2 - \sigma_3 \partial_3) (\sigma_1 \Psi_1 + \sigma_2 \Psi_2 + \sigma_3 \Psi_3) \\ &= -\partial_1 \Psi_1 - \partial_2 \Psi_2 - \partial_3 \Psi_3 + \sigma_1 \{ \partial_0 \Psi_1 - i(\partial_2 \Psi_3 - \partial_3 \Psi_2) \} \\ &\quad + \sigma_2 \{ \partial_0 \Psi_2 - i(\partial_3 \Psi_1 - \partial_1 \Psi_3) \} + \sigma_3 \{ \partial_0 \Psi_3 - i(\partial_1 \Psi_2 - \partial_2 \Psi_1) \} \\ &= -\vec{\nabla} \cdot \vec{\Psi} + \sigma_1 \{ \partial_0 \Psi_1 - i(\nabla \times \Psi)_1 \} + \sigma_2 \{ \partial_0 \Psi_2 - i(\nabla \times \Psi)_2 \} \\ &\quad + \sigma_3 \{ \partial_0 \Psi_3 - i(\nabla \times \Psi)_3 \} \\ &= -J_0 - \sigma_1 J_1 - \sigma_2 J_2 - \sigma_3 J_3 \\ \Rightarrow \overline{D} \Psi &= -J \quad \dots(30) \end{aligned}$$

Similarly

$$\begin{aligned} \overline{(DV)} &= (V_0 - \sigma_1 V_1 - \sigma_2 V_2 - \sigma_3 V_3) (\partial_0 - \sigma_1 \partial_1 - \sigma_2 \partial_2 - \sigma_3 \partial_3) \\ &= (\partial_0 V_0 + \text{div} V) - \sigma_1 \{ \partial_0 V_1 + \partial_1 V_0 + i(\nabla \times V)_1 \} \\ &\quad - \sigma_2 \{ \partial_0 V_2 + \partial_2 V_0 + i(\nabla \times V)_2 \} - \sigma_3 \{ \partial_0 V_3 + \partial_3 V_0 + i(\nabla \times V)_3 \} \\ &= \Psi_0 + \sigma_1 \Psi_1 + \sigma_2 \Psi_2 + \sigma_3 \Psi_3 \end{aligned}$$

$$\text{Therefore, } \overline{(DV)} = \Psi \quad \dots(31)$$

$$\begin{aligned} \text{Likewise } \overline{(DJ)} &= (\partial_0 J_0 - \sigma_1 J_1 - \sigma_2 J_2 - \sigma_3 J_3) (\partial_0 - \sigma_1 \partial_1 - \sigma_2 \partial_2 - \sigma_3 \partial_3) \\ &= \partial_0 J_0 + \text{div} J - \sigma_1 \{ \partial_0 J_1 + \partial_1 J_0 + i(\nabla \times J)_1 \} \\ &\quad - \sigma_2 \{ \partial_0 J_2 + \partial_2 J_0 + i(\nabla \times J)_2 \} - \sigma_3 \{ \partial_0 J_3 + \partial_3 J_0 + i(\nabla \times J)_3 \} \\ &= S_0 + \sigma_1 S_1 + \sigma_2 S_2 + \sigma_3 S_3 \end{aligned}$$

$$\text{Therefore, } \overline{(DJ)} = S \quad \dots(32)$$

Now we may express J and S in terms of V and Ψ . Here, the temporal component of J may be written as

$$\begin{aligned} J_0 &= \vec{\nabla} \cdot \vec{\Psi} = \partial_1 \Psi_1 + \partial_2 \Psi_2 + \partial_3 \Psi_3 \\ &= \partial_1 \{ -\partial_0 V_1 - \partial_1 V_0 - i(\partial_2 V_3 - \partial_3 V_2) \} + \partial_2 \{ -\partial_0 V_2 - \partial_2 V_0 - i(\partial_3 V_1 - \partial_1 V_3) \} \\ &\quad + \partial_3 \{ -\partial_0 V_3 - \partial_3 V_0 - i(\partial_1 V_2 - \partial_2 V_1) \} \\ &= -\partial_0 (\partial_1 V_1 + \partial_2 V_2 + \partial_3 V_3) - \partial_1^2 V_0 - \partial_2^2 V_0 - \partial_3^2 V_0 \\ &= -\partial_0 (\text{div} V) - (\partial_1^2 + \partial_2^2 + \partial_3^2) V_0 = -\partial_0 (-\partial_0 V_0) - (\nabla^2) V_0 \quad [\because \partial_0 V_0 + \text{div} V = 0] \\ &= \partial_0^2 V_0 - \nabla^2 V_0 = (\partial_0^2 - \nabla^2) V_0 = V_0 \end{aligned}$$

Thus the temporal components of J is related with the temporal component of V .

The spatial component of J can be expressed of as

$$\begin{aligned} J_1 &= -\partial_0 \Psi_1 + i(\nabla \times \Psi)_1 = -\partial_0 \{ -\partial_0 V_1 - \partial_1 V_0 - i(\nabla \times V)_1 \} + i \{ \partial_2 \Psi_3 - \partial_3 \Psi_2 \} \\ &= \partial_0^2 V_1 - \partial_2^2 V_1 - \partial_3^2 V_1 + \partial_1 (\partial_0 V_0 + \partial_2 V_2 + \partial_3 V_3) \\ &= \partial_0^2 V_1 - \partial_1^2 V_1 - \partial_2^2 V_1 - \partial_3^2 V_1 + \partial_1 (\partial_0 V_0 + \partial_1 V_1 + \partial_2 V_2 + \partial_3 V_3) \\ &= (\partial_0^2 - \nabla^2) V_1 + \partial_1 (\partial_0 V_0 + \text{div} V) = V_1 \quad [\because \partial_0 V_0 + \text{div} V = 0] \end{aligned}$$

Similarly $J_2 = V_2$ & $J_3 = V_3$

Thus the spatial components of J are related with the spatial component of V . Therefore we can write

$$V = J \quad \dots(33)$$

The temporal component of S i.e. S_0 is zero due to continuity equation. The spatial components of S are expressed as

$$\begin{aligned} S_1 &= -\partial_0 J_1 - \partial_1 J_0 - i(\nabla \times J)_1 \\ &= -\partial_0 \{-\partial_0 \Psi_1 + i(\nabla \times \Psi)_1\} - \partial_1 (\nabla \cdot \Psi) \\ &\quad - i[\partial_2 \{-\partial_0 \Psi_3 + i(\nabla \times \Psi)_3\} - \partial_3 \{-\partial_0 \Psi_2 + i(\nabla \times \Psi)_2\}] \\ &= \partial_0^2 \Psi_1 - i\partial_0 \partial_2 \Psi_3 + i\partial_0 \partial_3 \Psi_2 - \partial_1^2 \Psi_1 - \partial_1 \partial_3 \Psi_3 - i\partial_2 \partial_0 \Psi_3 \\ &\quad + \partial_2 \partial_1 \Psi_2 - \partial_2^2 \Psi_1 - i\partial_3 \partial_0 \Psi_2 - \partial_3^2 \Psi_1 + \partial_3 \partial_1 \Psi_3 - \partial_1 \partial_2 \Psi_2 \\ &= \partial_0^2 \Psi_1 - \partial_1^2 \Psi_1 - \partial_2^2 \Psi_1 - \partial_3^2 \Psi_1 = (\partial_0^2 - \nabla^2) \Psi_1 = \Psi_1 \end{aligned}$$

Similarly $S_2 = \Psi_2$ & $S_3 = \Psi_3$

Thus, the spatial components of S are related with the spatial components of Ψ and we can write

$$\Psi = S \quad \dots(34)$$

Now operating \bar{D} by D and D by \bar{D} we get,

$$\begin{aligned} D \bar{D} &= (\partial_0 + \sigma_1 \partial_1 + \sigma_2 \partial_2 + \sigma_3 \partial_3) (\partial_0 - \sigma_1 \partial_1 - \sigma_2 \partial_2 - \sigma_3 \partial_3) \\ &= \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 = (\partial_0^2 - \nabla^2) \Psi_1 = \end{aligned} \quad \dots(35a)$$

$$\begin{aligned} \bar{D} D &= (\partial_0 - \sigma_1 \partial_1 - \sigma_2 \partial_2 - \sigma_3 \partial_3) (\partial_0 + \sigma_1 \partial_1 + \sigma_2 \partial_2 + \sigma_3 \partial_3) \\ &= \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 = (\partial_0^2 - \nabla^2) \Psi_1 = \end{aligned} \quad \dots(35b)$$

Therefore

$$D \bar{D} = \bar{D} D = \quad \dots(36)$$

Similarly

$$V \bar{V} = \bar{V} V = V_0^2 - |\vec{V}|^2 \quad \dots(37)$$

$$J \bar{J} = \bar{J} J = J_0^2 - |\vec{J}|^2 \quad \dots(38)$$

In order to write the field equation (7) and Lorentz force equation (20) as quaternion valued, we may write the generalized field tensor $G_{\mu\nu}$, four-velocity $\{v^\nu\}$ and four-force $\{f_\mu\}$ in the following quaternion notations as;

$$G = \sigma_0 G_0 + \sigma_1 G_1 + \sigma_2 G_2 + \sigma_3 G_3 \quad \dots(39a)$$

$$v = \sigma_0 v_0 + \sigma_1 v_1 + \sigma_2 v_2 + \sigma_3 v_3 \quad \dots(39b)$$

$$f = \sigma_0 f_0 + \sigma_1 f_1 + \sigma_2 f_2 + \sigma_3 f_3 \quad \dots(39c)$$

where

$$G_n = \sigma_0 G_{n0} + \sigma_1 G_{n1} + \sigma_2 G_{n2} + \sigma_3 G_{n3} \quad (n=0,1,2,3) \quad \dots(39d)$$

The elements (terms) of equation (39a) i.e. the components of G are not real but they are also quaternion showing that not only the vectors but tensors can be represented as quaternions. The components of G i.e. G_0 , G_1 , G_2 , and G_3 may be written in the following quaternionic forms

$$G_0 = \sigma_0 G_{00} + \sigma_1 G_{01} + \sigma_2 G_{02} + \sigma_3 G_{03} \quad \dots(40a)$$

$$G_1 = \sigma_0 G_{10} + \sigma_1 G_{11} + \sigma_2 G_{12} + \sigma_3 G_{13} \quad \dots(40b)$$

$$G_2 = \sigma_0 G_{20} + \sigma_1 G_{21} + \sigma_2 G_{22} + \sigma_3 G_{23} \quad \dots(40c)$$

$$G_3 = \sigma_0 G_{30} + \sigma_1 G_{31} + \sigma_2 G_{32} + \sigma_3 G_{33} \quad \dots(40d)$$

where

$$G_{00} = G_{11} = G_{22} = G_{33} = 0 \quad (\text{due to anti-symmetric nature of field tensor}) \quad \dots(40e)$$

Operating quaternion differential operator D given by equation (26) to G given by (40), we get

$$\begin{aligned} D G_0 &= (\partial_0 + \sigma_1 \partial_1 + \sigma_2 \partial_2 + \sigma_3 \partial_3) (\sigma_1 G_{01} + \sigma_2 G_{02} + \sigma_3 G_{03}) \\ &= \sigma_1 \partial_0 G_{01} + \sigma_2 \partial_0 G_{02} + \sigma_3 \partial_0 G_{03} + \partial_1 G_{01} + i\sigma_3 \partial_1 G_{02} - i\sigma_2 \partial_1 G_{03} \\ &\quad - i\sigma_3 \partial_2 G_{01} + \partial_2 G_{02} + i\sigma_1 \partial_2 G_{03} + i\sigma_2 \partial_3 G_{01} - i\sigma_1 \partial_3 G_{03} + \partial_3 G_{03} \quad \dots(41a) \end{aligned}$$

and

$$\begin{aligned}\overline{G_0} \overline{D} &= (-\sigma_1 G_{01} - \sigma_2 G_{02} - \sigma_3 G_{03})(\partial_0 - \sigma_1 \partial_1 - \sigma_2 \partial_2 - \sigma_3 \partial_3) \\ &= -\sigma_1 \partial_0 G_{01} + \partial_1 G_{01} + i\sigma_3 \partial_2 G_{01} - i\sigma_2 \partial_3 G_{01} - \sigma_2 \partial_0 G_{02} - i\sigma_3 \partial_1 G_{02} \\ &\quad + \partial_2 G_{02} + i\sigma_1 \partial_3 G_{03} - \sigma_3 \partial_0 G_{03} + i\sigma_2 \partial_1 G_{03} - i\sigma_1 \partial_2 G_{03} + \partial_3 G_{03} \quad \dots(41b)\end{aligned}$$

Adding equation (41a) and (41b) we have

$$D G_0 + \overline{G_0} \overline{D} = 2[\partial_1 G_{01} + \partial_2 G_{02} + \partial_3 G_{03}] = 2 J_0$$

Using $[A, B] = \frac{1}{2}[\overline{A} B + \overline{B} A]$, thus we get

$$[\overline{D}, G_0] = \frac{1}{2}[D G_0 + \overline{G_0} \overline{D}] = J_0 \quad \dots(42a)$$

Also

$$\begin{aligned}D G_1 &= (\partial_0 + \sigma_1 \partial_1 + \sigma_2 \partial_2 + \sigma_3 \partial_3)(G_{10} + \sigma_2 G_{12} + \sigma_3 G_{13}) \\ &= \partial_0 G_{10} + \sigma_2 \partial_0 G_{12} + \sigma_3 \partial_0 G_{13} + \sigma_1 \partial_1 G_{10} + i\sigma_3 \partial_1 G_{12} - i\sigma_2 \partial_1 G_{13} + \sigma_2 \partial_2 G_{10} + \partial_2 G_{12} \\ &\quad + i\sigma_1 \partial_2 G_{13} + \sigma_2 \partial_3 G_{10} - i\sigma_1 \partial_3 G_{12} + \partial_3 G_{13}\end{aligned}$$

and

$$\begin{aligned}\overline{G_1} \overline{D} &= (G_{10} - \sigma_2 G_{12} - \sigma_3 G_{13})(\partial_0 - \sigma_1 \partial_1 - \sigma_2 \partial_2 - \sigma_3 \partial_3) \\ &= \partial_0 G_{10} - \sigma_1 \partial_1 G_{10} - \sigma_2 \partial_2 G_{10} - \sigma_3 \partial_3 G_{10} - \sigma_2 \partial_0 G_{12} - i\sigma_3 \partial_1 G_{12} + \partial_2 G_{12} + i\sigma_1 \partial_3 G_{12} \\ &\quad - \sigma_3 \partial_0 G_{13} + i\sigma_2 \partial_1 G_{13} - i\sigma_1 \partial_2 G_{13} + \partial_3 G_{13}\end{aligned}$$

$$\text{Thus } D G_1 + \overline{G_1} \overline{D} = 2[\partial_0 G_{10} + \partial_2 G_{12} + \partial_3 G_{13}] = 2 J_1$$

Therefore

$$[\overline{D}, G_1] = \frac{1}{2}[D G_1 + \overline{G_1} \overline{D}] = J_1 \quad \dots(42b)$$

Similarly

$$[\overline{D}, G_2] = \frac{1}{2}[D G_2 + \overline{G_2} \overline{D}] = J_2 \quad \dots(42c)$$

$$[\overline{D}, G_3] = \frac{1}{2}[D G_3 + \overline{G_3} \overline{D}] = J_3 \quad \dots(42d)$$

Also

$$\begin{aligned}D G &= (\partial_0 + \sigma_1 \partial_1 + \sigma_2 \partial_2 + \sigma_3 \partial_3)(G_0 + \sigma_1 G_1 + \sigma_2 G_2 + \sigma_3 G_3) \\ &= (\partial_0 G_0 + \partial_1 G_1 + \partial_2 G_2 + \partial_3 G_3) + \sigma_1 \{\partial_0 G_1 + \partial_1 G_0 + i(\partial_2 G_3 - \partial_3 G_2)\} \\ &\quad + \sigma_2 \{\partial_0 G_2 + \partial_2 G_0 + i(\partial_3 G_1 - \partial_1 G_3)\} + \sigma_3 \{\partial_0 G_3 + \partial_3 G_0 + i(\partial_1 G_2 - \partial_2 G_1)\}\end{aligned}$$

and

$$\begin{aligned}\overline{G} \overline{D} &= (G_0 - \sigma_1 G_1 - \sigma_2 G_2 - \sigma_3 G_3)(\partial_0 - \sigma_1 \partial_1 - \sigma_2 \partial_2 - \sigma_3 \partial_3) \\ &= (\partial_0 G_0 + \partial_1 G_1 + \partial_2 G_2 + \partial_3 G_3) - \sigma_1 \{\partial_0 G_1 + \partial_1 G_0 + i(\partial_2 G_3 - \partial_3 G_2)\} \\ &\quad - \sigma_2 \{\partial_0 G_2 + \partial_2 G_0 + i(\partial_3 G_1 - \partial_1 G_3)\} - \sigma_3 \{\partial_0 G_3 + \partial_3 G_0 + i(\partial_1 G_2 - \partial_2 G_1)\}\end{aligned}$$

Thus

$$D G + \overline{G} \overline{D} = 2[\partial_0 G_0 + \partial_1 G_1 + \partial_2 G_2 + \partial_3 G_3]$$

Therefore

$$\begin{aligned}[\overline{D}, G] &= \frac{1}{2}[D G + \overline{G} \overline{D}] = \partial_0 G_0 + \partial_1 G_1 + \partial_2 G_2 + \partial_3 G_3 \\ &= \partial_0[\sigma_1 G_{01} + \sigma_2 G_{02} + \sigma_3 G_{03}] + \partial_1[G_{10} + \sigma_2 G_{12} + \sigma_3 G_{13}] \\ &\quad + \partial_2[G_{20} + \sigma_1 G_{21} + \sigma_3 G_{23}] + \partial_3[G_{30} + \sigma_1 G_{31} + \sigma_2 G_{32}] \\ &= \sigma_0[\partial_1 G_{01} + \partial_2 G_{02} + \partial_3 G_{03}] + \sigma_1[\partial_0 G_{10} + \partial_2 G_{12} + \partial_3 G_{13}] \\ &\quad + \sigma_2[\partial_0 G_{20} + \partial_1 G_{21} + \partial_3 G_{23}] + \sigma_3[\partial_0 G_{30} + \partial_1 G_{31} + \partial_2 G_{32}] \\ &= J_0 + \sigma_1 J_1 + \sigma_2 J_2 + \sigma_3 J_3 \\ \Rightarrow [\overline{D}, G] &= J \quad \dots(43)\end{aligned}$$

which is quaternionic form of equation (15).

Also

$$\begin{aligned} D J + \bar{J} \bar{D} &= 2[\partial_0 J_0 + \partial_1 J_1 + \partial_2 J_2 + \partial_3 J_3] \\ &= 2[\partial_0 J_0 + \text{div} J] = 0 \quad (\text{due to continuity equation}) \end{aligned}$$

Thus

$$[\bar{D}, J] = 0 \quad \dots(44)$$

which is nothing but the continuity equation in quaternionic form. From equations (39) and (40), the force equation (20) may be written in terms of following simple and compact notations of quaternions as

$$f = q[v, G] = \frac{1}{2} q[\bar{v} G + \bar{G} v] \quad \dots(45)$$

IV. DISCUSSION

The generalized wave function have been expressed in complex form in terms of electric and magnetic field components of generalized four- potentials associated with dyons. Taking the curl of $\bar{\nabla} \times \bar{\Psi}$ we have obtained the relation between the components of generalized four-current with the generalized field associated with dyons namely a new generalized vector field \bar{S} . As such the S behaves with the components of generalized four-current in the same manner as $\bar{\Psi}$ is identified with the components of generalized four-potential of dyons. The electromagnetic four potentials and four-currents have also been expressed in quaternion form . We have also written the differential operator in quaternion form and operated the four potentials and four-currents by quaternion differential operator . After using Lorentz gauge condition and continuity equation we have obtained the equation $D V = -\Psi$ and $D J = -S$ which are the quaternionic form of generalized potential and generalized four-current of dyons respectively. As such these two equations are quantum equations respectively for generalized four-potential and generalized four-current of dyons. These equations are the short hand notation and written in compact and consistent way. These two equations remain invariant under quaternion transformations and as such the quaternionic formulation becomes manifestly covariant. After taking the quaternion conjugate of D, V, J, Ψ and S and operating those with one another we have obtained the relations between generalized vector field and the components of generalized four-current of dyons and hence establishes the quaternionic form of generalized Maxwell-Dirac equation of dyons in simple, compact and consistent manner. On the other hand, we have describe the quaternion conjugate forms of generalized potential and current of dyons. The temporal and spatial components of four current have been expressed as the D'Alembertian operator of the temporal and spatial components of tour potentials. The generalized field tensor, four-velocity, components of field tensor and Lorentz force equation have also been written is quaternionic form.

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