Fixed Point Theorems on Partial b-Metric Space using Fuzzy Mapping

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ABSTRACT: In the present work, we prove some fixed point theorems on partial b-metric space using fuzzy mapping for single and pair of mappings. These theorems extends the work of Shoaib et al. published in Fixed Point Theory and Application in 2018. One example is also given in the support of our result.

Keywords: Fixed Point Theorem, Fuzzy mapping, b-metric space, partial metric space

I. INTRODUCTION

Theory of the fixed point plays a key role in the various disciplines. This theory has many applications in pure as well as in applied mathematics. In 1922 Stefan Banach [1] proposed the idea of fixed point in contraction principle known as Banach Contraction Principle and it is most important and useful result in the fixed point theory. It got importance due to its application in the iterations work with the help of computer.

The fuzzy set notion was studied by Zadeh [2] in 1965. Then Butnariu [3] and Weiss [4] proposed the concept of fuzzy mapping and proved various fixed point results using this mapping. Later, the fuzzy contraction mapping introduced by Heilpern [5] and by using the fuzzy contraction mapping a fixed point theorem has been proved which is a fuzzy analogue of Nadler’s[6] fixed point theorem for the multivalued mapping.

The generalization of usual metric space named as b-metric space was presented by the Bakhtin [7] and Czerwik [8] in 1998. After that many results are given in β-generalized weak contractive multi-functions and b-metric spaces [9-11].

Steve G Matthew [12] proposed the Partial metric space in 1994 and Banach contraction theorem was proved under the conditions of partial metric space.

In 2014, Satish Shukla [13] proposed the partial b-metric and using the Kannan type mapping, a fixed point theorem for Banach contraction was proved on this space.

In this present work, we introduce a fuzzy mapping in partial b-metric space and also some fixed point theorems are proved on the partial b-metric space using the fuzzy mapping for single as well as for the two mappings.

Definition 1.1[14] Let \( Y = \emptyset \) and \( t \geq 1 \) and function \( d_p: Y \times Y \rightarrow [0, \infty) \) is b-metric when it satisfies the below mentioned properties:

(i) \( d_p(e_1', f_1') \geq 0 \quad \forall e_1', f_1' \in Y \)
(ii) \( d_p(e_1', f_1') = 0 \quad \text{iff} \quad e_1' = f_1' \)
(iii) \( d_p(e_1', f_1') \leq t[d_p(e_1', i_1') + d_p(i_1', f_1')] \quad \forall e_1', i_1', f_1' \in Y \)

\( t \geq 1 \) is called the coefficient of \( d_p \).

Definition 1.2[15] Let \( Y \neq \emptyset \) and function \( d_p: Y \times Y \rightarrow [0, \infty) \) is known as partial b-metric space and also some fixed point theorems are proved on the partial b-metric space using the fuzzy mapping for single as well as for the two mappings.
where \( B_p(e_1, r') = \{ f_1 \in Y; d'_p (e_1, f_1) \leq r' \} \).

Next we are giving one Example of Partial b-Metric Space.

**Example 1.1** Let \( Y = Q^+ \cup \{ 0 \} \) and a function 
\[ d'_p : Y \times Y \to Y \]
by 
\[ d'_p (r_1'', r_2'') = \| r_1'' - r_2'' \| \] then we show that 
\( (Y, d'_p) \) be partial b-metric space.

**Proof:** 
(i) 
\[ d'_p (r_1'', r_2'') = \| r_1'' - r_2'' \| = d'_p (r_1', r_2') \]
(ii) 
\[ d'_p (r_1'', r_1') = \| r_1'' - r_1' \| \leq \| r_1'' - r_1' \| \]
\[ \leq d'_p (r_1'', r_2'') \]
\[ d'_p (r_1'', r_1') \leq d'_p (r_1'', r_2'') \]
(iii) 
\[ d'_p (r_1'', r_2'') = \| r_1'' - r_2'' \| = \| r_1'' - r_1' \| = d'_p (r_1''', r_1''') \]
(iv) 
\[ d'_p (r_1'', r_3''') = \| r_1'' - r_3''' \| = \| r_1'' - r_3''' \| \]
\[ \leq d'_p (r_2'', r_3''') \]
\[ \leq d'_p (r_2'', r_2'') \]
\[ d'_p (r_1'', r_3''') \leq d'_p (r_2'', r_3''') \]
then, it satisfies all the conditions of partial b-metric space. So, \( (Y, d'_p) \) is a partial b-metric space.

**Definition 1.4[17]** Let \( (Y, d'_p) \) is partial b-metric space and let a sequence \( (y_n) \in Y \) \( y \in Y \).

(i)  
\[ \text{Sequence} \ (y_n) \ is \ convergent \ to \ y \ if} \lim_{n \to \infty} d'_p (y_n, y) \ exists \ and \ it \ is \ finite. \]

(ii)  
\[ \text{In} \ (Y, d'_p) \ sequence \ (y_n) \ is \ a \ cauchy \ sequence \ if} \lim_{n,m \to \infty} d'_p (y_n, y_m) \ exists \ and \ it \ is \ finite. \]

(iii)  
\[ \text{Let} \ (Y, d'_p) \ be \ the \ complete \ partial b-metric \ space \ if} \text{every \ cauchy \ sequence} \ (y_n) \ \text{in } Y \ \text{there} \exists \ y \ \text{in } Y \ \text{s.t.} \]
\[ \lim_{n,m \to \infty} d'_p (y_n, y_m) = \lim_{n \to \infty} d'_p (y_n, y) = \| y - y \| \]

The limit of convergent sequence may not be unique in partial b-metric space.

**Definition 1.5[15]** Let \( A \) be subset of \( N \) is said to be closed in \( (N, p) \) if it is closed w.r.t. \( r_p \). Then \( A \) subset is known as bounded in \( (N, d'_p) \) if there is \( n_0 \in N \) and \( M > 0 \) such that 
\[ a \in B_p (n_0, M) \ \forall \ a \in A \text{ i.e.,} \]
\[ d'_p (n_0, a) < d'_p (n_0, n_0) + M \]
\[ \forall \ a \in A. \]

Let \( CB_p (N) \) is collection of all closed, non-empty and bounded subsets of \( N \) w.r.t. the partial b-metric \( d'_p \).

For \( A \in CB_p (N) \), \( \exists \) \( a \) such that 
\[ d'_p (x, A) = \inf_{y \in A} d'_p (x, y) = \|
\[ d'_p (x, A) = \inf_{y \in A} d'_p (x, y). \]

In \( A \), there is least one best approximation if each \( y_1 \in N \), then \( A \) is said to be proximal set. The set of all proximal subsets of \( N \) is denoted by \( P(N) \).

**Definition 1.6[18]** The function 
\[ H'_p : P(N) \times P(N) \to [0, \infty), \]
\[ H'_p (A, B) = \max \{ \sup_{x \in B} d'_p (a, B), \sup_{y \in A} d'_p (A, b) \} \]
is called partial Hausdorff b-metric space on \( P(N) \).

A function with domain \( N \) be fuzzy set in \( N \) and the values belongs to \( [0, 1] \), then the group of all the fuzzy sets in \( N \) is \( F(N) \). If a fuzzy set \( K \) and \( y_1 \in \gamma \), then \( K(y_1) \) is function value which is known as the grade of membership of \( y_1 \) in \( K \). Then \( \gamma \)-level set of the fuzzy set is represented by \( [K]_\gamma \) and defined as follows:
\[ [K]_\gamma = \{ y_1 : K(y_1) \geq \gamma \} \text{ where } \gamma \in (0, 1) \]

Let \( N \neq \emptyset \) and \( J \) is metric space. Then a mapping \( V \) is fuzzy mapping if mapping \( V \) from \( N \) into \( F(J) \). Fuzzy mapping \( V \) be the fuzzy subset on \( N \times J \) with membership function \( V(y_1)(x) \) be grade of membership of \( x \) in \( V(y_1) \). Here, we represent the \( \alpha \)-level set of \( V(y_1) \) as \( V(y_1) \)_\alpha[16].

**Definition 1.7[16]** A point \( y_1 \in N \) is known as the fuzzy fixed point of the fuzzy mapping \( \gamma : N \to F(N) \) if \( \exists \alpha \in (0, 1) \), such that \( y_1 \in [\gamma(y_1)]_\alpha \).

**Lemma 1.1[18]** Let two non-empty proximal subsets \( V \) and \( W \) of partial b-metric space \( (\gamma, d'_p) \) and if \( v^* \in V \) then 
\[ d'_p (v^*, W) \leq H(V, W) \]

**Lemma 1.2[18]** Let \( (x, d'_p) \) be partial b-metric space and let \( (P(N), H'_p) \) is partial Hausdorff b-metric space. Then, \( \forall V, W \in P(N) \) and for each \( v^* \in V \), there exists \( w^* \in W \) satisfying
\[ d'_p (v^*, W) = d'_p (v^*, w^*) \]
then,
\[ H'_p (V, W) \geq d'_p (a, w^*) \]

**II. MAIN RESULT**

In this section, our purpose is to prove some fixed point theorems under fuzzy mapping for single as well as for two self-mappings.

**Definition 2.1:** Let \( (X, d'_p) \) be partial b-metric space with the constant \( t \geq 1 \) and a mapping \( V : X \to F(X) \) is known as the multi valued generalized contraction if
\[ H'_p (V(x), V(y)) \leq a d'_p (x, y) + b d'_p (x, y) \cdot \alpha \]
\[ + \frac{a_2 d'_p (x, y) \cdot \alpha}{d'_p (y)} + \frac{a_3 d'_p (x, y) \cdot \alpha}{d'_p (y, x) \cdot \alpha} \]
\[ + \frac{a_4}{d'_p (x, y, x) \cdot \alpha} + \frac{a_5}{d'_p (y, x, y) \cdot \alpha} \]
for all \( x, y \in X \) and \( \alpha \geq 0 \), \( i = 1, 2, \ldots, 5 \) with \( t a_1 + a_2 + \frac{a_4}{3} + \frac{a_5}{2} < 1 \).
Theorem 2.1 Let \((Y,d'_{pb})\) be the complete partial b-metric space with constant \(t \geq 1\). Let \(V: Y \to F(Y)\) is fuzzy mapping, and let \(m_0\) is any arbitrary point in \(Y\). Let there \(\exists \alpha(m) \in (0,1)\) \(\forall\) \(m \in Y\) and \(V\) satisfies the following conditions:

\[
H_{d'_{pb}}(\{Vm\}_{a(m)},\{Vn\}_{a(m)}) \leq a_1 d'_{pb}(m,\{Vn\}_{a(m)}) + a_2 d'_{pb}(n,\{Vm\}_{a(m)}) + a_3 d'_{pb}(m,\{Vn\}_{a(m)}) + a_4 \left(\frac{d'_{pb}(m,\{Vn\}_{a(m)}) + d'_{pb}(n,\{Vm\}_{a(m)})}{3t}\right)
\]

And \(d'_{pb}(m_0,\{Vn\}_{a(m)}) \leq \nu(1-tv)r \forall m,n \in B_{d'_{pb}}(m_0,r), r > 0\) and \(t = 1\), where \(\nu = \frac{1}{\sum_{i=1}^{n}a_i} \leq 1\). Then, there exist \(m^*\) in \(B_{d'_{pb}}(m_0,r)\) such that \(m^* \in \{Vn\}_{a(m)}\).

Proof: Let \(m_0\) be any arbitrary point in \(Y\) such that \(m_1 \in \{Vm_0\}_{a(m_0)}\). Let a sequence \(\{m_i\}\) of points in \(Y\) such that \(m_i \in \{Vm_{i-1}\}_{a(m_{i-1})}\).

First, we show that \(m_i \in B_{d'_{pb}}(m_0,r) \forall i \in N\)

\[d'_{pb}(m_0, m_i) = d'_{pb}(m_0, \{Vn\}_{a(m)}) \leq \nu(1-tv)r < r \Rightarrow m_i \in B_{d'_{pb}}(m_0, r).\]

Now, by using Lemma 1.1 we get

\[
d'_{pb}(m_j, m_{j+1}) = H_{d'_{pb}}(\{Vm_{j-1}\}_{a(m_{j-1})},\{Vm_j\}_{a(m_j)}) \leq a_1 d'_{pb}(m_{j-1},\{Vm_{j-1}\}_{a(m_{j-1})}) + a_2 d'_{pb}(m_{j},\{Vm_{j}\}_{a(m_j)}) + a_3 d'_{pb}(m_{j-1},\{Vm_{j-1}\}_{a(m_{j-1})}) + a_4 \left(\frac{d'_{pb}(m_{j-1},\{Vm_{j-1}\}_{a(m_{j-1})}) + d'_{pb}(m_{j},\{Vm_{j}\}_{a(m_j)})}{3t}\right)
\]

Now, we show that \(d'_{pb}(m_h, m_h) = 0\)

Then, \(d'_{pb}(m_h, m_h) = H_{d'_{pb}}(\{Vm_{h-1}\}_{a(m_{h-1})},\{Vm_{h-1}\}_{a(m_{h-1})}) \leq a_1 d'_{pb}(m_{h-1},\{Vm_{h-1}\}_{a(m_{h-1})}) \leq 0.\)

\[
\sum_{m=1}^{j} d'_{pb}(m_h, m_h) = 0
\]
By completeness, there exist a point $m_{i+1}$ in $Y$ such that

$$
lim_{i \to \infty} d'(m_{i}, m_{i+1}) = 0
$$

Now, we show that sequence $(m_{i})$ is a Cauchy sequence. Since $(m_{i})$ converges to $m^*$ as $i \to \infty$, we have

$$
lim_{i \to \infty} d'(m_{i}, m^*) = 0
$$

Hence, sequence $(m_{i})$ is Cauchy sequence in $Y$. By completeness, there exist a point $m^*$ which belongs to $Y$. Since, $(m_{i})$ converges to $m^*$ as $i \to \infty$

$$
\lim_{i \to \infty} d'(m_{i}, m^*) = d'(m^*, m^*) = 0
$$

(3)

Now, we show that $m^*$ be the fixed point of $T$.

$$
d'(m^*, [Vm^*]_{a(m^*)}) \leq \frac{t}{1 - tv} [d'(m_{i+1}, m_{i+1}) + d'(m_{i}, m_{i})]
$$

(4)

5. Conclusion

In this paper, we have introduced and studied the concept of $b$-distance on a $b$-metric space. We have also established some fixed point results for the $b$-distance $d'$ on $b$-metric space. The obtained results are new and generalize and improve many known results in the literature. These results can be applied in various fields of mathematics and its applications.
Consider \( r'' = 1 \) and \( r = 4 \), then \( B_{d''_{pb}}(r'',r) = [0.5] \).

Let \( a_1 = \frac{1}{50}, a_2 = \frac{1}{40}, a_3 = \frac{1}{30}, a_4 = \frac{1}{20}, a_5 = \frac{1}{10} \) Then

\[
H_{d''_{pb}}(r''_1)_{a(r''_1)} = \frac{1}{50} + \frac{1}{30} + \frac{1}{60} \leq \frac{1}{10} = 0.1
\]

and

\[
\| r'' - s'' \| = \frac{1}{10}
\]

Then by theorem 2.1

\[
d''_{pb}(r''_1), V(r''_1)_{a(r''_1)} \leq v(1 - tv)r
\]

where \( v = \frac{a_1 + a_3 + a_5}{3} = \frac{1}{10} \).

Then all the conditions of theorem 2.1 is satisfied, \( \exists 0 \in B_{d''_{pb}}(r''_1,0,v) \). Then 0 is the fixed point of V.

**Theorem 2.2** Let \((Y,d''_{pb})\) is complete partial b-metric space with constant \( \geq 1 \). Let \( V,Y \rightarrow F(Y)\) is fuzzy mapping and let \( x_0 \) is any arbitrary point in Y. Suppose there exists \( \alpha (x), \alpha (y) \in (0,1) \forall x \in Y \) satisfying the following conditions:

\[
H_{d''_{pb}}(VX_{a(x)}) \leq \alpha (x) + d''_{pb}(x,X_{a(x)}) + a_3 d''_{pb}(y,Y_{a(y)}) + a_4 d''_{pb}(y,Y_{a(y)}) + \frac{a_5}{2T}d''_{pb}(y,x)
\]

and

\[
d''_{pb}(x_0,X_{a(x_0)}) \leq v(1 - tv)r \forall x \in Y, v \in B_{d''_{pb}}(x_0,r), r > 0\text{ and }t < 1,
\]

where \( v = \frac{a_1 + a_3 + a_5 + \frac{a_2}{3}}{2} \). Also, \( a_i \geq 0 \), where \( i = 1,2,3,4,5 \). Also, \( a_i \geq 0 \) and \( 2a_1 + 2a_2 + 2a_3 + a_4 < 2 \text{ and } 2a_1 + 2a_2 + 2a_3 + a_4 < 2 \text{ where } \sum_{i=1}^{4} a_i < 1 \). Then, there exist \( x^* \) in \( B_{d''_{pb}}(x_0,r) \) such that the common fixed point of V and W is \( x^* \).

**Proof:** Let \( x_1 \) is any arbitrary point in Y s.t. \( x_1 \in X \). Let a sequence \( \{x_n\} \) of points in X such that \( x_{2i+1} \in \{x_{2i-1}\}_{a(x_{2i-1})}, x_{2i+2} \in \{X_{2i+1}\}_{a(x_{2i+1})} \).

First we show that \( x_n \in B_{d''_{pb}}(x_0,0T) \forall n \in N \)

\[
d''_{pb}(x_0,x_1) = d''_{pb}(x_0,X_{a(x_0)}) \leq v(1 - tv)r < r
\]

\( x_1 \in B_{d''_{pb}}(x_0,0T) \). Let \( x_{2j}, x_{2j+1}, \ldots, x_{2j+l-1} \in B_{d''_{pb}}(x_0,0T), j \in N \).

If \( l = 2 + \frac{1}{2} \) then \( l = \frac{3}{2} \).

Now, by using lemma 1.1 we get
\[d_{pb}(x_{2i+1}, x_{2i+2}) = Hd'_{pb}\left([Vx_{2i}]a_v(x_{2i+1}), [Wx_{2i+1}]a_p(x_{2i+1})\right)\]
\[\leq a_1d'_{pb}(x_{2i+1}, x_{2i+2}) + \frac{a_4}{2t}d'_{pb}(x_{2i+1}, x_{2i+1})\]
\[+a_3\left[d'_{pb}(x_{2i+1}, x_{2i+1}) + d'_{pb}(x_{2i+1}, x_{2i+1})\right]\]
\[\leq a_1d'_{pb}(x_{2i+1}, x_{2i+2}) + \frac{a_4}{2t}d'_{pb}(x_{2i+1}, x_{2i+1})\]

Then by using 4th condition of partial b-metric space, we get
\[d'_{pb}(x_{2i+1}, x_{2i+2}) \leq a_2d'_{pb}(x_{2i+1}, x_{2i+1}) + \frac{a_4}{2t}d'_{pb}(x_{2i+1}, x_{2i+1})\]
\[\leq \left(a_1 + a_2t + \frac{a_4}{2t}\right)d'_{pb}(x_{2i+1}, x_{2i+1})\]
\[+ a_3\left[d'_{pb}(x_{2i+1}, x_{2i+1}) + d'_{pb}(x_{2i+1}, x_{2i+1})\right]\]
\[\leq \left(a_1 + a_2t + \frac{a_4}{2t}\right)d'_{pb}(x_{2i+1}, x_{2i+1})\]

Then by adding equation (4) and (5), we get
\[2d'_{pb}(x_{2i+1}, x_{2i+1}) \leq (a_1 + a_2t + \frac{a_4}{2t})d'_{pb}(x_{2i+1}, x_{2i+1})\]
\[+ a_3\left[d'_{pb}(x_{2i+1}, x_{2i+1}) + d'_{pb}(x_{2i+1}, x_{2i+1})\right]\]
\[\leq \left(a_1 + a_2t + \frac{a_4}{2t}\right)d'_{pb}(x_{2i+1}, x_{2i+1})\]

As, \(v = \frac{(a_1 + a_2t + \frac{a_4}{2t})}{2} < \frac{1}{2}\)

Then by the equation (6), we get
\[d'_{pb}(x_{2i+1}, x_{2i+1}) \leq \frac{1}{1 - v}\]

On the same way, if \(j = 2l + 2\) where \(l = 0, 1, 2, \ldots, \frac{l-1}{2}\)
\[d'_{pb}(x_{2i+1}, x_{2i+3}) \leq \frac{1}{1 - v}\]

and by eq. (9) \(d'_{pb}(x_{2i+1}, x_{2i+3}) \leq \frac{1}{1 - v}\)

then by adding equation (9) and (10), we have
\[d'_{pb}(x_{j}, x_{j+1}) \leq \frac{1}{1 - v}\]

Now, \(d'_{pb}(x_{0}, x_{j+1}) \leq \frac{1}{1 - v}\)

Then, \(d'_{pb}(x_{0}, x_{j}) = 0\)
Now, we show that sequence \( \{x_n\} \) is Cauchy sequence.

So, equation (11) can be written as

\[
d'_p(x_w, x_w) \leq v^i d'_p(x_0, x_1)
\]

Let \( i \) and \( w \) be two positive integers, \( i < w \)

\[
d'_p(x_{i+w}, x_w) \leq d'_p(x_{i+w}, x_i) + \sum_{k=1}^{i+w} d'_p(x_{i+k}, x_i)
\]

\[
d'_p(x_{i+w}, x_{i+w}) \leq \left[ tv' d'_p(x_0, x_1) + \sum_{k=1}^{i+w-1} d'_p(x_{i+k}, x_i) \right]
\]

\[
d'_p(x_0, x_1) \leq tv' + t^2 v' + \cdots + t^{i+w} v'\sum_{k=1}^{i+w-1} d'_p(x_0, x_1)
\]

\[
d'_p(x_w, x_{w+1}) \leq tv' + t^2 v' + \cdots + t^{w-1} v'\sum_{k=1}^{w-1} d'_p(x_0, x_1)
\]

\[
d'_p(x_w, x_{w+1}) \leq t v' + t v'^2 + \cdots + t^{w-1} v'\sum_{k=1}^{w-1} d'_p(x_0, x_1)
\]

\[
d'_p(x_{w+1}, x_w) \leq tv' + t v'^2 + \cdots + t^{w-1} v'\sum_{k=1}^{w-1} d'_p(x_0, x_1)
\]

\[
d'_p(x_{w+1}, x_w) \leq tv' + t v'^2 + \cdots + t^{w-1} v'\sum_{k=1}^{w-1} d'_p(x_0, x_1)
\]

\[
\lim_{w \to \infty} d'_p(x_w, x^*) = d'_p(x^*, x^*) = \lim_{w \to \infty} d'_p(x_w, x_w) = 0
\]

Now, we need to prove \( x^* \) is the common fixed point of \( V \) and \( W \).

\[
d'_p(x^*, [Wx^*]_{aw}(x^*)) \leq t \left[ d'_p(x_{w+1}, [Wx^*]_{aw}(x^*)) \right]
\]

\[
d'_p(x^*, [Wx^*]_{aw}(x^*)) \leq \left( 1 - t \right) d'_p(x_{w+1}, [Wx^*]_{aw}(x^*))
\]

Then by taking \( w \to \infty \), we get

\[
\lim_{w \to \infty} d'_p(x^*, [Wx^*]_{aw}(x^*)) = 0
\]

So, \( [Wx^*]_{aw}(x^*) E x^* \)

This implies that \( x^* \) is a fixed point of \( W \).

On same way we can prove that \( x^* \) is the unique fixed point of \( V \) and \( W \).

III. CONCLUSIONS

We have introduced the fuzzy mapping concept in partial b-metric space. Some fixed point theorems are proved to get the unique common fixed point. This result extends the result of Shoaib et al. [18].

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CONFLICT OF INTEREST

Author has no conflict of interest.
REFERENCES