



## On the Supermagic Labelings of Forests

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**ABSTRACT:** An edge-magic labeling of a  $(p,q)$ -graph  $G$  is a bijection function  $f:V(G)\cup E(G) \rightarrow \{1,2,\dots,p+q\}$  such that  $f(u) + f(v) + f(uv) = C_f$  is a constant for every edge  $uv$  of  $G$ . If such an edge-magic labeling exists,  $G$  is said to be edge-magic and  $C_f$  is called the valence of  $f$ . Further, if  $f(V(G)) = \{1,2,\dots,p\}$  then  $f$  is a super edge-magic labeling of  $G$  and  $G$  is said to be supermagic and if  $f(E(G)) = \{1,2,\dots,q\}$  then  $f$  is a supermagic labeling of  $G$  and  $G$  is said to be supermagic.

This chapter is devoted to the study of the supermagic properties of certain classes of forests such as  $K_{1,m} \cup K_{1,n}$ ,  $K_{1,2} \cup K_{1,n}$ ,  $P_m \cup K_{1,n}$ ,  $2P_n$ ,  $K_{1,m} \cup 2nP_2$ . We are also interested in it since most of the forests referred to in this chapter, have each two components and thus show that bipartite graphs within even number of components may be supermagic.

### I. INTRODUCTION

The subject of edge-magic labeling of graphs had its origins three decades ago in the work of Kotzig and Rosa [4,5] on what they called magic valuations of graphs, which are also commonly known as edge-magic total labeling[7]. Interest in these labeling has lately been rekindled by a paper on the subject due to Ringel and Llado[6]. Shortly after this, Enomoto, Llado, Nakamigawa and Ringel[3] defined a more restrictive form of edge-magic labelings namely super edge-magic labeling which Wallis[7] refers to as strong edge-magic total labeling. Analogously Akka and Warad[1] defined another restrictive form of edge-magic labeling, namely supermagic labelings. For a  $(p,q)$ -graph  $G = (V,E)$  a bijective function  $f:V \cup E \rightarrow \{1,2,\dots,p+q\}$  is an edge-magic labeling of  $G$  if  $f(u) + f(v) + f(uv)$  is a constant  $c_f$  (called valence of  $f$ ) for any edge  $uv \in E$ . A graph that admits such a labeling is an edge-magic graph. In [3] Enomoto, Llado, Nakamigawa and Ringel defined an edge-magic labeling  $f$  of a graph  $G$  to be super edge-magic if it has the extra property that  $f(V(G)) \rightarrow \{1,2,\dots,p\}$  and said to be supermagic if  $f(E(G)) \rightarrow \{1,2,\dots,q\}$ . Thus, a super edge-magic (supermagic) is a graph that admits a super edge-magic (supermagic) labeling. Lately, super edge-magic labeling and super edge-magic graphs have been called by Wallis [7] as strongly edge-magic total labeling and strongly super edge-magic total graphs respectively. In the similar way, as we did in the case of supermagic labeling and supermagic graphs. This chapter is mainly

devoted to the study of the supermagic properties of certain classes of forests. We are also interested in it since most of the forests referred to in this chapter have each two components and thus show that bipartite graphs with an even number of components may be supermagic. The next characterization found in [2] has proved to be very useful and therefore we state it as Lemma 1.1.

**Lemma1.1:** A  $(p,q)$ -graph  $G$  is supermagic if and only if there exists a bijective function  $f:V(G) \rightarrow \{q+1, q+2, \dots, p+q\}$  such that the set  $S = \{f(u) + f(v) : uv \in E(G)\}$  which consists of  $q$  consecutive integers.

In such a case,  $f$  extends to a supermagic labeling of  $G$  with valence  $\lambda = q + s$  where  $s = \min(S)$  and  $S = \{\lambda - (i)\}_{i=1}^q$ . Moreover,  $\sum_{v \in V(G)} f(v) \deg v = qs + \binom{q}{2}$ . Therefore, it is clear that due to Lemma 1.1, it suffices to exhibit the vertex labeling in order to identify a supermagic graph.

**Lemma1.2:** [6] If  $G$  is a  $(p,q)$ -graph, where  $q$  is even,  $p + q = 2(\text{mod}4)$  and every vertex of  $G$  has odd degree, then  $G$  is not edge-magic.

### II. MAIN RESULTS

This section is devoted to the study of the supermagic properties of certain classes of forests  $K_{1,m} \cup K_{1,n}$ ,  $K_{1,2} \cup K_{1,n}$ ,  $P_m \cup K_{1,n}$ ,  $2P_n$ ,  $K_{1,m} \cup 2nP_2$ .

**Theorem2.1:** If  $m$  is a multiple of  $n+1$ , then the forest  $K_{1,m} \cup K_{1,n}$  is super magic.

**Proof:** Let  $V(F) = \{x, y\} \cup \{u_i / 1 \leq i \leq m\} \cup \{v_i / 1 \leq i \leq n\}$

and  $E(F) = \{xu_i / 1 \leq i \leq m\} \cup \{yv_i / 1 \leq i \leq n\}$ . Then consider the vertex labeling  $f: V(F) \rightarrow \{m+n+1, m+n+2, \dots, 2(m+n+1)\}$  such that  $f(x) = 2m+2n+1-\alpha$ ,  $f(y) = 2m+2n+2$ ,  $f(u_i) = 2m+2n+4 - (\alpha+2i)$  for  $i = 1, 2, \dots, m$  and  $f(v_i) = 2(m+n+1) - (\alpha+1)(i+1)$  for  $i = 1, 2, \dots, n$  where  $\alpha = m/(n+1)$ . Therefore by Lemma 1.1,  $f$  extends to a supermagic labeling of  $F$  with valence  $4m + 4n + 3 - \alpha$ .

**Theorem 2.2:** The forest  $F \cong K_{1,2} \cup K_{1,n}$  is supermagic if and only if  $n$  is a multiple of 3. Furthermore, there are essentially only two supermagic labelings of  $F$ .

**Proof:** Let the vertex and edge sets be  $V(F) = \{u\} \cup \{v_i / 1 \leq i \leq n\} \cup \{w_1, w_2, w_3\}$

and  $E(F) = \{uv_i / 1 \leq i \leq n\} \cup \{w_1w_2, w_1w_3\}$ . Define  $f: V(F) \rightarrow \{n+3, n+4, \dots, 2n+6\}$  be an arbitrary supermagic labeling of  $F$  such that  $f(u) = \alpha$  and  $\{f(w_1), f(w_2), f(w_3)\} = \{i, j, k\}$  where  $i < j < k$ .

Let  $S = \{f(x) + (y) / xy \in E(F)\}$  and  $L = \{\alpha+n+3, \alpha+n+4, \dots, \alpha+2n+6\}$  where  $|S| = n + 2$  and  $|L| = n + 4$ . Clearly  $S - \{f(w_1) + f(w_2), f(w_1) + f(w_3)\} = L - \{\alpha+\alpha, \alpha+i, \alpha+j, \alpha+k\}$ . Thus,  $\{\alpha+n+3, \alpha+2n+6\} \subset \{2\alpha, \alpha+i, \alpha+k\}$ , since by removing  $2\alpha, \alpha+i, \alpha+j$  and  $\alpha+k$  from  $L$ . We obtain  $S = \{f(w_1)+f(w_2), f(w_1)+f(w_3)\}$  which is a set of consecutive integers minus two elements. This implies that  $\{2n+6, n+3\} \subset \{\alpha, i, j\}$ .

Now we prove that  $i = 2n+6$  and  $k = n+3$ . To do this, it suffices to verify that  $\alpha \notin \{2n+6, n+3\}$ . Let  $\beta = f(w_1)$  then since  $\deg w_1 = 2$ ,  $\deg u = n$  and  $f(u) = \alpha$ . It follows by Lemma 1.1 that

$$\sum_{t=n+3}^{2n+6} t + \alpha(n-1) + \beta = (n+2)s + \binom{n+2}{2}$$

where  $s = \min(S)$ . Hence  $\frac{3n^2+21n+36}{2} + \alpha(n-1) + \beta - \frac{(n+2)(n+1)}{2} = (n+2)s$

$$\therefore s = \frac{(n^2 + 9n + 17) + \alpha(n-1) + \beta}{n+2}$$

Now suppose to the contrary that  $\alpha = n+3$ . Then  $s = \frac{\beta}{n+2} + 2n + 7$ , so  $n+2$  divides  $\beta$  which gives that  $\beta = n+2$ . This in turn leads us to conclude that  $s = 2n+8$ . Furthermore, the vertex  $u$  which is labeled  $n+3$  cannot be adjacent to the vertices labeled  $n+4$  or  $n+5$ ; for otherwise  $s = 2n+7$  or  $2n+8$ .

Thus,  $\{n+4, n+5, 2n+6\} = \{i, j, k\}$  which is impossible.

Next, assume to the contrary that  $\alpha = 2n+6$ .

Then  $s = \frac{\beta-3}{n+2} + 3n + 7$  and consequently  $n+2$  divides  $\beta-3$  which gives that  $\beta-3 = 0$  since  $n+2 \geq 3$  and  $n+3 \leq \beta \leq 2n+6$ . Thus,  $\beta = 3$  and  $s = n+3$ . Then either  $f(w_2) = n+3$  or  $f(w_3) = n+3$ . This gives that  $f(w_1) + f(w_2) = n+6$  or  $f(w_1) + f(w_3) = n+6$  and  $n+6 < s = 2n+6$  which is a contradiction.

Finally, since the vertices  $w_2$  and  $w_3$  are indistinguishable, we discuss the following three cases.

**Case I:** If  $f(w_1) = 2n+6$ ,  $f(w_2) = n+3$  and  $f(w_3) = j$ , then  $\{3n+9, j+2n+6\} = \{\alpha+j, 2\alpha\}$ . Thus,  $3n+9 = \alpha+j$  and  $j + 2n + 6 = 2\alpha$  gives that  $\alpha = \frac{5n}{3} + 5$ . Hence  $n$  is a multiple of 3. Therefore, by taking  $\alpha = \frac{5n}{3} + 5$ ,  $f(w_1) = 2n+6$ ,  $f(w_2) = n+3$  and  $f(w_3) = (4n/3) + 4$ .

We get the same supermagic labeling of  $F$  as the proof of Theorem 2.1.

**Case II:** If  $f(w_1) = n+3$ ,  $f(w_2) = 2n+6$  and  $f(w_3) = j$ , then  $\{3n+4, j+n+3\} = \{\alpha+j, 2\alpha\}$ . Thus,  $3n+9 = \alpha+j$  and  $j + n + 3 = 2\alpha$  give that  $\alpha = \frac{4n}{3} + 4$ . Hence  $n$  is a multiple of 3. Now it is easy to verify that if we take  $f(u) = \frac{4n}{3} + 4$ ,  $f(w_1) = n+3$ ,  $f(w_2) = 2n+6$  and  $f(w_3) = \frac{5n}{3} + 5$ , we get the same supermagic labeling of  $F$  by assigning the remaining labels to all other vertices of  $F$ .

**Case III:** If  $f(w_1) = j$ ,  $f(w_2) = 2n+6$  and  $f(w_3) = n+3$  then  $\{2n+6+j, j+n+3\} = \{\alpha+j, 2\alpha\}$ . Now since  $\alpha < 2n+6$ , it follows that  $2n+6+j \neq \alpha+j$ . Thus  $2n+6+j = 2\alpha$  and  $j+n+3 = \alpha+j$ . Hence  $j = 0 > n+3$  which is impossible.

**Theorem 2.3:** For every two integers  $m \geq 4$  and  $n \geq 1$ , the forest  $F \cong P_m \cup K_{1,n}$  is supermagic.

**Proof:** Let  $V(F) = \{u_i / 1 \leq i \leq m\} \cup \{v_i / 1 \leq i \leq n\} \cup \{w\}$  and

$E(F) = \{u_i u_{i+1} / 1 \leq i \leq m-1\} \cup \{v_i w / 1 \leq i \leq n\}$ . We consider four cases for the vertex labeling  $f: V(F) \rightarrow \{m+n, m+n-1, \dots, 2m+2n\}$ .

**Case I:** For  $m \equiv 0 \pmod{4}$ , let

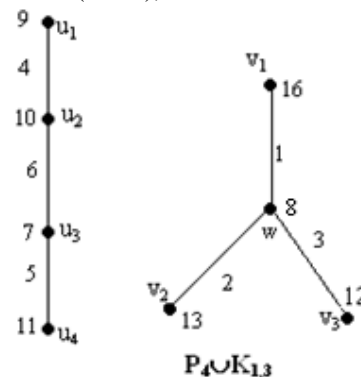


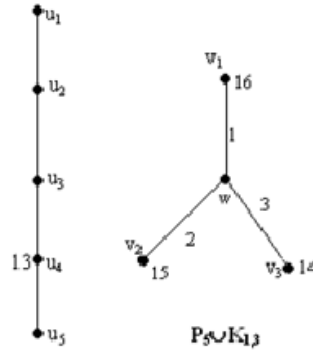
Fig. 1.

$$f(u_j) = \begin{cases} \frac{3m+2n}{2} & \text{if } j = 1 \\ \frac{3m+2n-4}{2} & \text{if } j = 3 \\ 2m+n-2i+2 & \text{if } j = 4i \text{ and } 1 \leq i \leq \frac{m}{4} \\ \frac{3m+2n-4i-4}{2} & \text{if } j = 4i+1 \text{ and } 1 \leq i \leq \frac{(m-4)}{4} \\ 2m+n-2i-1 & \text{if } j = 4i+2 \text{ and } 0 \leq i \leq \frac{m-4}{4} \\ \frac{3m+2n-4i-2}{2} & \text{if } j = 4i+3 \text{ and } 1 \leq i \leq \frac{m-4}{4} \end{cases}$$

$$f(v_i) = 2m+2n-i+1 \quad \text{if } 1 \leq i \leq n \text{ and}$$

$$f(w) = \frac{3m+2n-2}{2}.$$

**Case II:** For  $m \equiv 1 \pmod{4}$ , let



**Fig. 2.**

$$f(u_j) = \begin{cases} 2m+n-2i+2 & \text{if } j = 4i \text{ and } 1 \leq i \leq \frac{m-1}{4} \\ \frac{3m+2n-4i-4}{2} & \text{if } j = 4i+1 \text{ and } 0 \leq i \leq \frac{(m-1)}{4} \\ 2m+n-2i-1 & \text{if } j = 4i+2 \text{ and } 0 \leq i \leq \frac{m-5}{4} \\ \frac{3m+2n-4i-5}{2} & \text{if } j = 4i+3 \text{ and } 0 \leq i \leq \frac{m-5}{4} \end{cases}$$

$$f(v_i) = 2m+2n-i+1 \text{ if } 1 \leq i \leq n, \quad f(w) = \frac{3m+2n-1}{2}.$$

**Case III:** For  $m \equiv 2 \pmod{4}$ , let

$$f(u_j) = \begin{cases} m+n & \text{if } j = 1 \\ m+n+2 & \text{if } j = 3 \\ \frac{3m+2n+4i}{2} & \text{if } j = 4i \text{ and } 1 \leq i \leq \frac{m-2}{4} \\ m+n+2i+2 & \text{if } j = 4i+1 \text{ and } 1 \leq i \leq \frac{m-2}{4} \\ \frac{3m+2n+4i+6}{2} & \text{if } j = 4i+2 \text{ and } 0 \leq i \leq \frac{m-6}{4} \\ m+n+2i+1 & \text{if } j = 4i+3 \text{ and } 1 \leq i \leq \frac{m-6}{4} \\ m+n+1 & \text{if } j = m \end{cases}$$

$$f(v_i) = 2m + 2n - i + 1, \quad \text{if } 1 \leq i \leq n \quad \text{and} \quad f(w) = \frac{3m + 2n}{2}$$

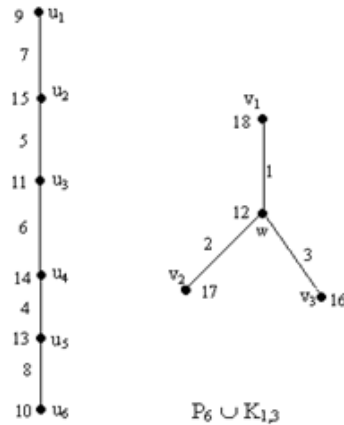


Fig. 3.

Case IV: For  $m \equiv 3 \pmod{4}$ , let

$$f(u_j) = \begin{cases} \frac{3m + 2n + 1}{2} & \text{if } j = 1 \\ \frac{3m + 2n - 3}{2} & \text{if } j = 3 \\ 2m + n - 2i + 2 & \text{if } j = 4i \text{ and } 1 \leq i \leq \frac{m-3}{4} \\ \frac{3m + 2n - 4i - 3}{2} & \text{if } j = 4i + 1 \text{ and } 1 \leq i \leq \frac{m-3}{4} \\ 2m + n - 2i - 1 & \text{if } j = 4i + 2 \text{ and } 0 \leq i \leq \frac{m-7}{4} \\ \frac{3m + 2n - 4i - 1}{2} & \text{if } j = 4i + 3 \text{ and } 1 \leq i \leq \frac{m-3}{4} \\ \frac{3m + 2n + 3}{2} & \text{if } j = m - 1 \end{cases}$$

$$f(v_i) = 2m + 2n - i + 1, \quad \text{if } 1 \leq i \leq n \quad \text{and} \quad f(w) = \frac{3m+2n-1}{2}$$

Therefore, by Lemma 1.1,  $f$  extends to a supermagic labeling of  $F$  with valence

$$c_f = \begin{cases} \frac{7m + 6n + 2}{2} & \text{if } m \equiv 2 \pmod{4} \\ 4m + 3n - \lfloor \frac{m}{2} \rfloor & \text{otherwise} \end{cases}$$

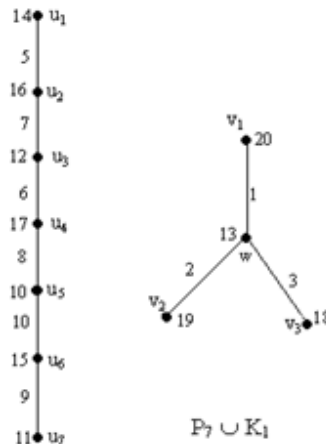


Fig. 4.

The next class of forest that we study is  $2P_n$ . Before that, we have proved that the forest  $K_{1,2} \cup K_{1,n}$  is a supermagic if and only if  $n$  is a multiple of 3, (Theorem 2.2). Hence the forest  $2P_3$  is not supermagic. However, it is complimentary edge-magic by labeling the vertices of one  $P_3$  with  $10 - 2 - 9$  and one of the other  $P_3$  with  $8 - 7 - 6$ , and letting the valency be 16. Finally, notice that D G Akka and Nanda Warad [1] proved that the forest  $nP_2$  is super edge-magic if and only if  $n$  is odd.

**Theorem 2.4:** *The forest  $F \cong 2P_n (n > 1)$  is supermagic if and only if  $n \neq 2$  or 3.*

**Proof:** Suppose that  $n \geq 4$  and define the forest  $F \cong 2P_n$  with

$$V(F) = \{u_i / 1 \leq i \leq n\} \cup \{v_i / 1 \leq i \leq n\} \text{ and}$$

$$E(F) = \{u_i u_{i+1} / 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} / 1 \leq i \leq n-1\}$$

We consider cases according to the possible values of the integer  $n$ .

**Case I:** For  $n = 9$ , Define  $f: V(F) \rightarrow \{17, 18, \dots, 34\}$  be the vertex labeling such that  $(f(u_i))_{i=1}^9 = \{25, 18, 28, 21, 31, 22, 19, 19, 26\}$  and

$$(f(v_i))_{i=1}^9 = \{27, 17, 30, 20, 34, 24, 33, 23, 32\}$$

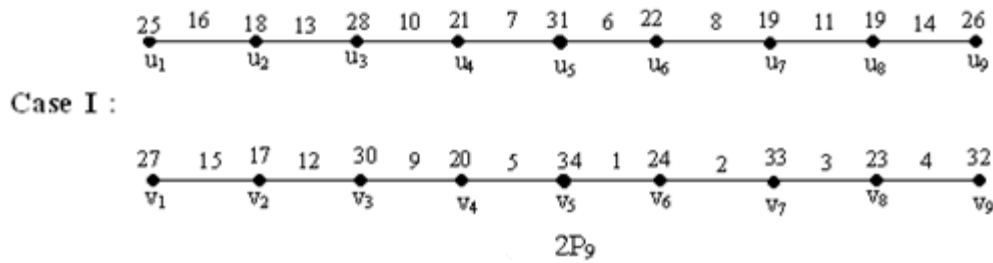
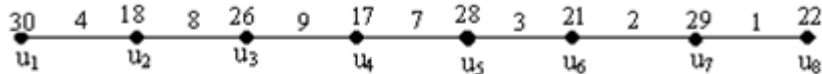


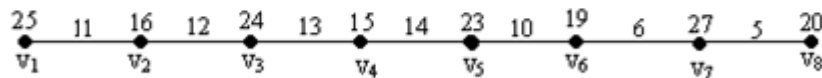
Fig. 5.(a)

**Case II:** For  $n = 4k$ , where  $k$  is a positive integer, let  $f: V(F) \rightarrow \{8k-1, 8k, \dots, 16k-2\}$  be the vertex labeling such that

$$f(u_j) = \begin{cases} 15k & \text{if } j = 1 \\ 13k - i + 2 & \text{if } j = 2i - 1 \text{ and } 2 \leq i \leq k \\ 13k + i - 1 & \text{if } j = 2i - 1 \text{ and } k + 1 \leq i \leq 2k \\ 9k - i + 1 & \text{if } j = 2i \text{ and } 1 \leq i \leq k \\ 9k + i & \text{if } j = 2i \text{ and } k + 1 \leq i \leq 2k \end{cases}$$



Case II :



2P8

Fig. 5(b)

$$f(v_j) = \begin{cases} 12k - i + 2 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq k + 1 \\ 12k + i - 1 & \text{if } j = 2i - 1 \text{ and } k + 2 \leq i \leq 2k \\ 8k - i + 1 & \text{if } j = 2i \text{ and } 1 \leq i \leq k \\ 8k + i & \text{if } j = 2i \text{ and } k + 1 \leq i \leq 2k \end{cases}$$

**Case III:** For  $n = 12k - 7$  where  $k$  is a positive integer, let  $f: V(F) \rightarrow \{24k-15, 24k-14, \dots, 48k - 30\}$  and  $f: E(F) \rightarrow \{1, 2, \dots, 24k - 16\}$  be the vertex labeling and edge labeling such that

$$f(u_j) = \begin{cases} 36k + 3i - 25 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k - 1 \\ 24k - 3i & \text{if } j = 2i - 1 \text{ and } 3k \leq i \leq 6k - 3 \\ 24k + 3i - 17 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k - 2 \\ 42k - 3i - 25 & \text{if } j = 2i \text{ and } 3k - 1 \leq i \leq 6k - 4 \end{cases}$$

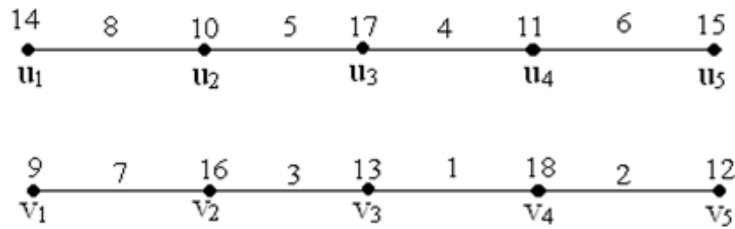


Fig. 6. (2P<sub>5</sub>).

$$f(v_j) = \begin{cases} 24k + 3i - 18 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k - 2 \\ 36k + 3i - 23 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k - 2 \\ 33k + 3i - 23 & \text{if } j = 6k + 6i - 9 \text{ and } 1 \leq i \leq k \\ 45k + 3i - 30 & \text{if } j = 6k + 6i - 8 \text{ and } 1 \leq i \leq k \\ 33k + 3i - 24 & \text{if } j = 6k + 6i - 7 \text{ and } 1 \leq i \leq k \\ 45k + 3i - 28 & \text{if } j = 6k + 6i - 6 \text{ and } 1 \leq i \leq k - 1 \\ 33k + 3i - 22 & \text{if } j = 6k + 6i - 5 \text{ and } 1 \leq i \leq k - 1 \\ 45k + 3i - 29 & \text{if } j = 6k + 6i - 4 \text{ and } 1 \leq i \leq k - 1 \end{cases}$$

**Case IV:** For  $n = 12k - 6$  where  $k$  is a positive integer, let  $f:V(F) \rightarrow \{24k-13, 24k-12, \dots, 48k - 26\}$  and  $f:E(F) \rightarrow \{1,2, \dots, 24k - 14\}$  be the vertex and edge labelings such that

$$f(u_j) = \begin{cases} 36k + 3i - 22 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k - 1 \\ 54k - 3i - 27 & \text{if } j = 2i - 1 \text{ and } 3k \leq i \leq 6k - 3 \\ 24k + 3i - 14 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k - 2 \\ 42k - 3i - 22 & \text{if } j = 2i \text{ and } 3k - 1 \leq i \leq 6k - 3 \end{cases}$$

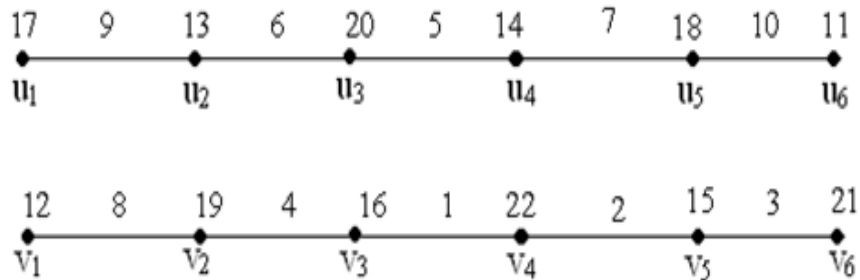


Fig. 7. (2P<sub>6</sub>).

$$f(v_j) = \begin{cases} 24k + 3i - 15 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k - 2 \\ 36k + 3i - 20 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k - 2 \\ 33k + 3i - 20 & \text{if } j = 6k + 6i - 9 \text{ and } 1 \leq i \leq k \\ 45k + 3i - 27 & \text{if } j = 6k + 6i - 8 \text{ and } 1 \leq i \leq k - 1 \\ 33k + 3i - 21 & \text{if } j = 6k + 6i - 7 \text{ and } 1 \leq i \leq k \\ 45k + 3i - 25 & \text{if } j = 6k + 6i - 6 \text{ and } 1 \leq i \leq k - 1 \\ 33k + 3i - 19 & \text{if } j = 6k + 6i - 5 \text{ and } 1 \leq i \leq k - 1 \\ 46k + 3i - 26 & \text{if } j = 6k + 6i - 4 \text{ and } 1 \leq i \leq k - 1 \\ 48k - i - 25 & \text{if } j = 12k + 2i - 10 \text{ and } 1 \leq i \leq 2 \end{cases}$$

**Case V:** For  $n = 12k - 5$  where  $k$  is a positive integer, let  $f:V(F) \rightarrow \{24k-11, 24k-10, \dots, 48k - 22\}$  be the vertex labeling such that

$$f(u_j) = \begin{cases} 24k + 3i - 24 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k - 1 \\ 42k - 3i - 26 & \text{if } j = 2i - 1 \text{ and } 3k \leq i \leq 6k - 2 \\ 36k + 3i - 28 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k - 1 \\ 42k - 3i - 33 & \text{if } j = 2i \text{ and } 3k \leq i \leq 6k - 3 \end{cases}$$

$$f(v_j) = \begin{cases} 36k + 3i - 29 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k - 1 \\ 24k + 3i - 22 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k - 2 \\ 33k + 3i - 27 & \text{if } j = 6k + 6i - 8 \text{ and } 1 \leq i \leq k \\ 45k + 3i - 33 & \text{if } j = 6k + 6i - 7 \text{ and } 1 \leq i \leq k - 1 \\ 33k + 3i - 28 & \text{if } j = 6k + 6i - 6 \text{ and } 1 \leq i \leq k \\ 45k + 3i - 31 & \text{if } j = 6k + 6i - 5 \text{ and } 1 \leq i \leq k - 1 \\ 33k + 3i - 26 & \text{if } j = 6k + 6i - 4 \text{ and } 1 \leq i \leq k - 1 \\ 45k + 3i - 32 & \text{if } j = 6k + 6i - 3 \text{ and } 1 \leq i \leq k - 1 \\ 48k - i - 31 & \text{if } j = 12k + 2i - 9 \text{ and } 1 \leq i \leq 2 \end{cases}$$

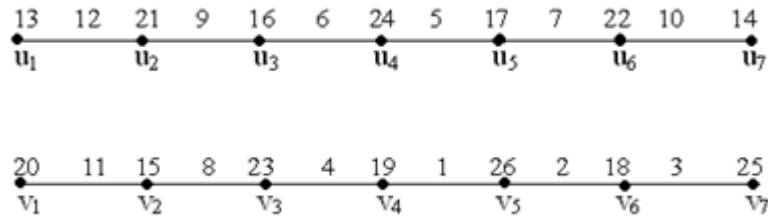


Fig. 8. (2P<sub>7</sub>).

**Case VI:** For  $n = 12k - 2$  where  $k$  is a positive integer, let  $f:V(F) \rightarrow \{24k-5, 24k-4, \dots, 48k - 10\}$  be the vertex labeling such that

$$f(u_j) = \begin{cases} 36k + 3i - 10 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k \\ 54k - 3i - 8 & \text{if } j = 2i - 1 \text{ and } 3k + 1 \leq i \leq 6k - 1 \\ 24k + 3i - 7 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k - 1 \\ 42k - 3i - 8 & \text{if } j = 2i \text{ and } 3k \leq i \leq 6k - 1 \end{cases}$$

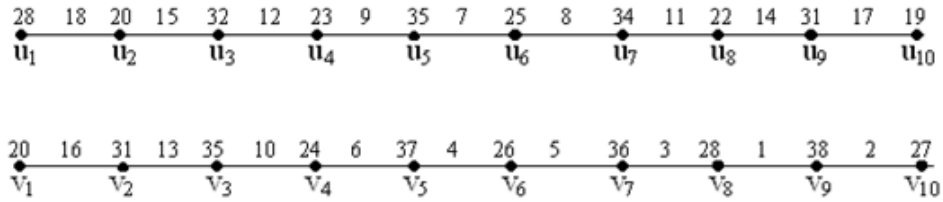


Fig. 9. (2P<sub>10</sub>).

$$f(v_j) = \begin{cases} 38k + 3i - 9 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k - 1 \\ 24k + 3i - 6 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k - 1 \\ 45k + 3i - 11 & \text{if } j = 6k + 6i - 7 \text{ and } 1 \leq i \leq k \\ 33k + 3i - 8 & \text{if } j = 6k + 6i - 6 \text{ and } 1 \leq i \leq k \\ 45k + 3i - 12 & \text{if } j = 6k + 6i - 5 \text{ and } 1 \leq i \leq k \\ 33k + 3i - 8 & \text{if } j = 6k + 6i - 4 \text{ and } 1 \leq i \leq k \\ 45k + 3i - 10 & \text{if } j = 6k + 6i - 3 \text{ and } 1 \leq i \leq k \\ 33k + 3i - 9 & \text{if } j = 6k + 6i - 2 \text{ and } 1 \leq i \leq k \end{cases}$$

**Case VII:** For  $n = 12k - 1$  where  $k$  is a positive integer, let  $f:V(F) \rightarrow \{24k-3, 24k-2, \dots, 48k - 6\}$  be the vertex labeling such that

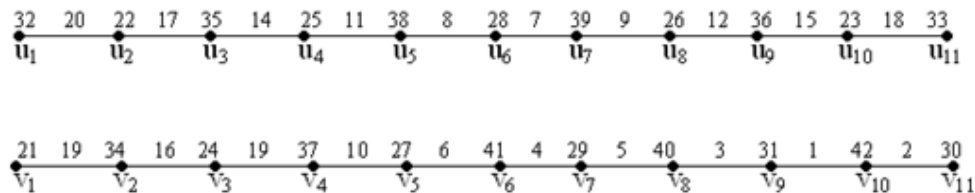


Fig. 10. (2P<sub>11</sub>).

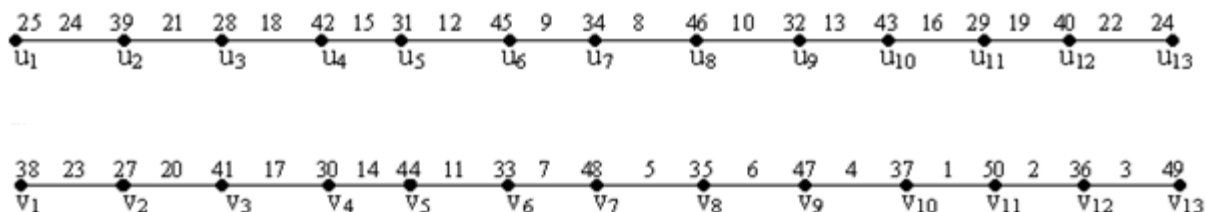
$$f(u_j) = \begin{cases} 36k + 3i - 7 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k \\ 54k - 3i - 3 & \text{if } j = 2i - 1 \text{ and } 3k + 1 \leq i \leq 6k \\ 24k + 3i - 5 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k \\ 42k - 3i - 4 & \text{if } j = 2i \text{ and } 3k + 1 \leq i \leq 6k - 1 \end{cases}$$

$$f(v_j) = \begin{cases} 24k + 3i - 6 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k \\ 36k + 3i - 5 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k - 1 \\ 45k - 4 & \text{if } j = 6k \\ 33k + 3i - 7 & \text{if } j = 6k + 6i - 5 \text{ and } 1 \leq i \leq k \\ 45k + 3i - 8 & \text{if } j = 6k + 6i - 4 \text{ and } 1 \leq i \leq k \\ 33k + 3i - 5 & \text{if } j = 6k + 6i - 3 \text{ and } 1 \leq i \leq k \\ 45k + 3i - 6 & \text{if } j = 6k + 6i - 2 \text{ and } 1 \leq i \leq k \\ 33k + 3i - 6 & \text{if } j = 6k + 6i - 1 \text{ and } 1 \leq i \leq k \\ 45k + 3i - 4 & \text{if } j = 6k + 6i \text{ and } 1 \leq i \leq k - 1 \end{cases}$$

**Case VIII:** For  $n = 12k + 1$  where  $k$  is a positive integer, let  $f:V(F) \rightarrow \{24k+1, 24k+2, \dots, 48k+2\}$  be the vertex labeling such that

$$f(u_j) = \begin{cases} 24k + 3i - 2 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k + 1 \\ 42k - 3i + 4 & \text{if } j = 2i - 1 \text{ and } 3k + 2 \leq i \leq 6k + 1 \\ 36k + 3i - 1 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k \\ 54k - 3i + 3 & \text{if } j = 2i \text{ and } 3k + 1 \leq i \leq 6k \end{cases}$$

$$f(v_j) = \begin{cases} 36k + 3i - 2 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k \\ 24k + 3i - 1 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k \\ 45k + 3i - 1 & \text{if } j = 6k + 6i - 5 \text{ and } 1 \leq i \leq k \\ 33k + 3i - 2 & \text{if } j = 6k + 6i - 4 \text{ and } 1 \leq i \leq k \\ 45k + 3i - 2 & \text{if } j = 6k + 6i - 3 \text{ and } 1 \leq i \leq k \\ 33k + 3i & \text{if } j = 6k + 6i - 2 \text{ and } 1 \leq i \leq k \\ 45k + 3i & \text{if } j = 6k + 6i - 1 \text{ and } 1 \leq i \leq k - 1 \\ 33k + 3i - 1 & \text{if } j = 6k + 6i \text{ and } 1 \leq i \leq k \\ 48k - i + 2 & \text{if } j = 12k + 2i - 3 \text{ and } 1 \leq i \leq 2 \end{cases}$$



**Fig. 11.**  $(2P_{13})$ .

**Case IX:** For  $n = 12k + 2$  where  $k$  is a positive integer, let  $f:V(F) \rightarrow \{24k+3, 24k+4, \dots, 48k+6\}$  be the vertex labeling such that

$$f(u_j) = \begin{cases} 36k + 3i - 2 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k + 1 \\ 42k - 3i + 10 & \text{if } j = 2i - 1 \text{ and } 3k + 2 \leq i \leq 6k + 1 \\ 24k + 3i + 1 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k \\ 42k - 3i + 6 & \text{if } j = 2i \text{ and } 3k + 1 \leq i \leq 6k + 1 \end{cases}$$



$$f(v_j) = \begin{cases} 36k + 3i + 3 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k \\ 24k + 3i + 2 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k + 1 \\ 45k - i + 8 & \text{if } j = 6k + 2i - 1 \text{ and } 1 \leq i \leq 2 \\ 33k + 3i + 1 & \text{if } j = 6k + 6i - 2 \text{ and } 1 \leq i \leq k \\ 45k + 3i + 6 & \text{if } j = 6k + 6i - 1 \text{ and } 1 \leq i \leq k \\ 33k + 3i + 3 & \text{if } j = 6k + 6i \text{ and } 1 \leq i \leq k - 1 \\ 45k + 3i + 5 & \text{if } j = 6k + 6i + 1 \text{ and } 1 \leq i \leq k \\ 33k + 3i + 5 & \text{if } j = 6k + 6i + 2 \text{ and } 1 \leq i \leq k - 1 \\ 45k + 3i + 7 & \text{if } j = 6k + 6i + 3 \text{ and } 1 \leq i \leq k - 1 \\ 36k - i + 5 & \text{if } j = 12k + 2i - 2 \text{ and } 1 \leq i \leq 2. \end{cases}$$

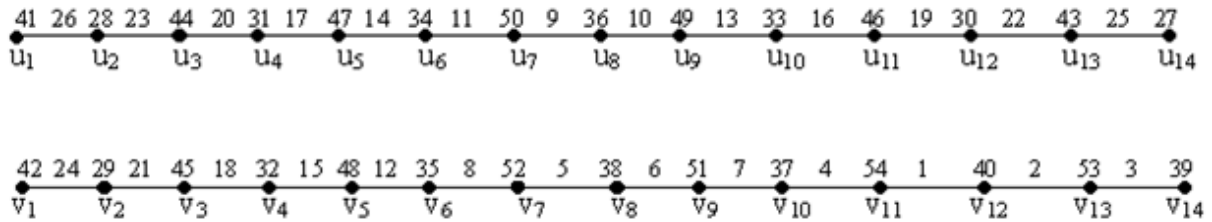


Fig. 12. (2P<sub>14</sub>).

**Case X:** For  $n = 12k + 3$  where  $k$  is a positive integer, let  $f:V(F) \rightarrow \{24k+5, 24k+6, \dots, 48k+10\}$  be the vertex labeling such that

$$f(u_j) = \begin{cases} 36k + 3i + 5 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k + 1 \\ 54k - 3i + 15 & \text{if } j = 2i - 1 \text{ and } 3k + 2 \leq i \leq 6k + 2 \\ 24k + 3i + 3 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k + 1 \\ 48k - 3i + 10 & \text{if } j = 2i \text{ and } 3k + 2 \leq i \leq 6k + 1 \end{cases}$$

$$f(v_j) = \begin{cases} 24k + 3i + 2 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k + 2 \\ 36k + 3i + 7 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k \\ 45k - i + 12 & \text{if } j = 6k + 2i \text{ and } 1 \leq i \leq 2 \\ 43k + 3i + 4 & \text{if } j = 6k + 6i - 1 \text{ and } 1 \leq i \leq k \\ 45k + 3i + 10 & \text{if } j = 6k + 6i \text{ and } 1 \leq i \leq k \\ 33k + 3i + 6 & \text{if } j = 6k + 6i + 1 \text{ and } 1 \leq i \leq k - 1 \\ 45k + 3i + 9 & \text{if } j = 6k + 6i + 2 \text{ and } 1 \leq i \leq k \\ 33k + 3i + 8 & \text{if } j = 6k + 6i + 3 \text{ and } 1 \leq i \leq k - 1 \\ 45k + 3i + 11 & \text{if } j = 6k + 6i + 4 \text{ and } 1 \leq i \leq k - 1 \\ 36k - i + 8 & \text{if } j = 12k + 2i - 1 \text{ and } 1 \leq i \leq 2. \end{cases}$$

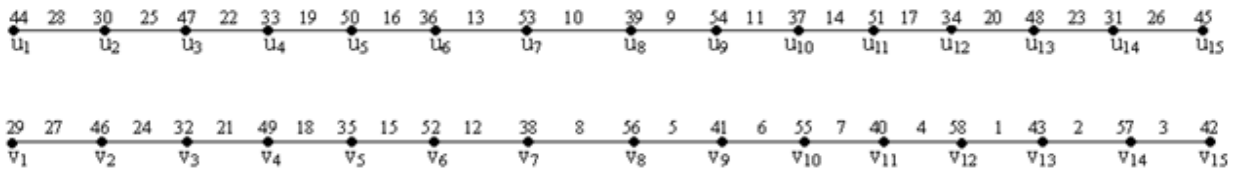


Fig. 13. (2P<sub>15</sub>).

**Case XI:** For  $n = 12k + 9$  where  $k$  is a positive integer, let  $f:V(F) \rightarrow \{24k+17, 24k+18, \dots, 48k+34\}$ .

$$f(u_j) = \begin{cases} 36k + 3i + 23 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k + 3 \\ 54k - 3i + 42 & \text{if } j = 2i - 1 \text{ and } 3k + 4 \leq i \leq 6k + 5 \\ 24k + 3i + 15 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k + 2 \\ 42k - 3i + 31 & \text{if } j = 2i \text{ and } 3k + 2 \leq i \leq 6k + 4 \end{cases}$$

$$f(v_j) = \begin{cases} 24k + 3i + 14 & \text{if } j = 2i - 1 \text{ and } 1 \leq i \leq 3k + 2 \\ 36k + 3i + 25 & \text{if } j = 2i \text{ and } 1 \leq i \leq 3k + 3 \\ 33k - i + 25 & \text{if } j = 6k + 2i + 3 \text{ and } 1 \leq i \leq 2 \\ 45k + 3i + 30 & \text{if } j = 6k + 6i + 2 \text{ and } 1 \leq i \leq k + 1 \\ 33k + 3i + 23 & \text{if } j = 6k + 6i + 3 \text{ and } 1 \leq i \leq k \\ 45k + 3i + 32 & \text{if } j = 6k + 6i + 4 \text{ and } 1 \leq i \leq k \\ 33k + 3i + 22 & \text{if } j = 6k + 6i + 2 \text{ and } 1 \leq i \leq k \\ 45k + 3i + 36 & \text{if } j = 6k + 6i + 6 \text{ and } 1 \leq i \leq k \\ 33k + 3i + 24 & \text{if } j = 6k + 6i + 7 \text{ and } 1 \leq i \leq k - 1 \\ 36k - i + 26 & \text{if } j = 12k + 2i + 5 \text{ and } 1 \leq i \leq 2 \end{cases}$$

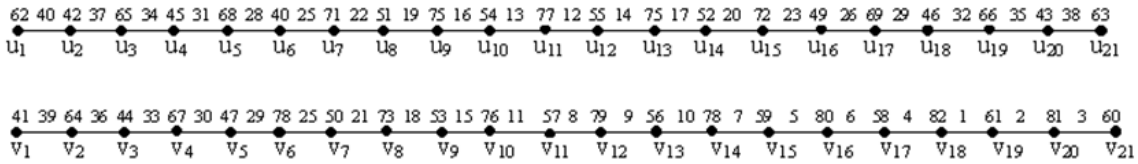


Fig. 14. (2P<sub>21</sub>).

Therefore, by Lemma 1.1, f extends to a supermagic labeling of F with valence 7n - 1 when n = 4k and 7n otherwise.

**Theorem 2.5:** The forest F ≅ K<sub>1,m</sub> ∪ 2nP<sub>2</sub> where m and n are positive integers, is supermagic. Furthermore if m + 2n and 2n + 3 are relatively prime then only the valence 2m + 9n + 4 and 3m + 9n + 1 are attained by the supermagic labeling of F.

**Proof:** Let F ≅ K<sub>1,m</sub> ∪ 2nP<sub>2</sub> be a (p, q) forest such that

$$V(F) = \{u\} \cup \{v_i / 1 \leq i \leq m\} \cup \{w_i / 1 \leq i \leq 4n\}$$

$$\text{and } E(F) = \{uv_i / 1 \leq i \leq m\} \cup \{w_i w_{2n+i} / 1 \leq i \leq 2n\}$$

Then f, g: V(F) → {m + 2n + 1, …, 2m + 6n + 1} be the vertex labelings of F with

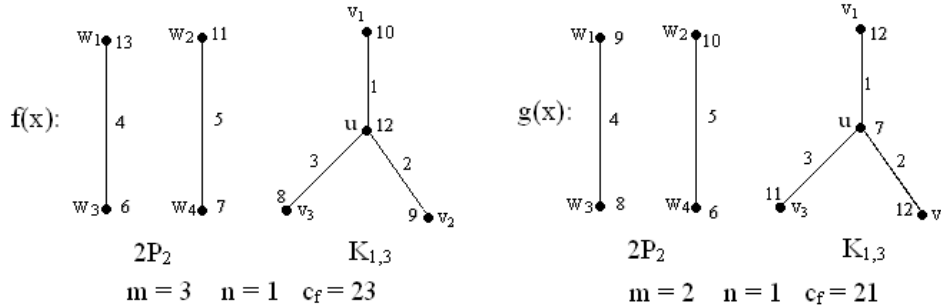


Fig. 15.

$$f(x) = \begin{cases} 2m + 5n + 1 & \text{if } x = u \\ 2m + 4n - i + 1 & \text{if } x = v_i \text{ and } 1 \leq i \leq m \\ 2m + 6n + 2 - i & \text{if } x = w_i \text{ and } 1 \leq i \leq n \\ 2m + 6n - i + 1 & \text{if } x = w_i \text{ and } n + 1 \leq i \leq 2n \\ m + 5n - i + 1 & \text{if } x = w_i \text{ and } 2n + 1 \leq i \leq 3n \\ m + 7n - i + 1 & \text{if } x = w_i \text{ and } 3n + 1 \leq i \leq 4n \end{cases}$$

$$g(x) = \begin{cases} m + 3n + 1 & \text{if } x = u \\ 2m + 6n - i + 2 & \text{if } x = v_i \text{ and } 1 \leq i \leq m \\ m + 6n - 2i + 2 & \text{if } x = w_i \text{ and } 1 \leq i \leq n \\ m + 8n - 2i + 3 & \text{if } x = w_i \text{ and } n + 1 \leq i \leq 2n \\ m + n + i + 1 & \text{if } x = w_i \text{ and } 2n + 1 \leq i \leq 3n \\ m - n + i & \text{if } x = w_i \text{ and } 3n + 1 \leq i \leq 4n \end{cases}$$

Thus, by Lemma 1.1, f and g extends to supermagic labelings of F with valences 4m+9n+2 and 3m+9n+3 respectively.

We have made it clear that the above two valences are the only possible ones when  $m + 2n$  and  $2n+1$  are relatively prime, let  $\lambda$  be the valence of supermagic labeling  $h$  of  $F$ . Then

$$\begin{aligned}\lambda &= \frac{(m-1)[q+h(u)] + \sum_{i=1}^{p+q} i}{q} \\ &= 3m + 10n + 3 + h(u) + \frac{(2n+1)[n+1-m-2n-h(u)]}{m+2n}\end{aligned}$$

This gives that there exists an integer  $\alpha$  such that

$\alpha(m+2n) = 1 + n - m - 2n - h(u)$ . Now since  $1 \leq h(w) \leq p$  it follows that  $\alpha$  is -1 or -2 values that lead to the valences  $4m + 9n + 2$  and  $3m + 9n + 3$  respectively.

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