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Reinvestigation of the Existence of Non-Uniqueness Solution of the Flow of Non-Newtonian Fluid over a Stretching Sheet

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ABSTRACT: The solution of the problem of existence of non-uniqueness of the flow of visco-elastic (Non-Newtonian) fluid past a stretching sheet is reinvestigated by simple analytical procedure by introducing four boundary conditions for fourth order highly non-linear ordinary differential equation of motion of visco-elastic fluid, Walter's liquid B'model. The uniqueness of the solution of the visco-elastic fluid past a stretching sheet is examined. The solution of the problem may not be necessarily unique.

Keywords: Non-Newtonian, non-uniqueness, stretching sheet, similarity solution reinvestigation

I. INTRODUCTION

The study of boundary layer behavior over a continuously moving flat wall finds wide applications in technological manufacturing process in industry. These includes aerodynamic extrusion of plastic sheets, rolling and extrusion in manufacturing process, the cooling of an infinite metallic plate in a cool bath, the boundary layer along a liquid film in condensation process and the controlled cooling system.

Rao *et al.* [1] have studied the momentum and heat transfer in a power-law fluid with arbitrary injection/suction at a moving wall. Sakiadis [2] has initiated the study of boundary layer flow over a continuously moving surface in a viscous fluid, which finds its application in the problem of polymer sheet extruded continuously from a dye. Markovitz and Coleman [3] have investigated the visco-elastic boundary layer flow over a stretching plastic sheet. Prasad et al. [4] analyzed the study of heat and mass transfer in a porous medium over a non-isothermal stretching sheet with the influence of momentum. Sonth *et al.* [5] have studied the effect of heat and mass transfer in a visco-elastic fluid flow over an accelerating surface with heat source. Gupta and Gupta [6] have examined the study of heat and mass transfer on stretching sheet with suction or blowing.. Kumaran and Ramanaiah [7] analyzed the boundary layer flow over a stretching sheet. Vajravelu and Nayfeh [8] have studied the convective heat and mass transfer in a electrically conducting stretching sheet through porous medium. Vajravelu [9] investigated in the convection of heat and mass transfer through a stretching sheet with suction or blowing. Hopwell [10] has studied momentum and heat transfer on a continuous moving surface in power law fluid. Chiam [11] has analysed the study of the magneto-hydrodynamic heat and mass transfer over a non- isothermal stretching sheet. Yih [12] has studied about the influence of non-Darcy forced convection flow over a flat plate with variable wall temperature in the porous medium. Acrivos *et al.* [13] made an analysis on momentum and heat transfer in laminar boundary-layer flows of non-newtonian fluids past external surfaces. Amkadni *et al.* [14] have discussed the exact solutions of laminar magneto hydro dynamic viscous flow over the stretching sheet. Troy *et al.* **[**15] analyzed the uniqueness of flow of a second order fluid past a stretching sheet in the presence of magnetic field. Cortell [16] has studied flow and heat transfer of a fluid through porous media over stretching surface with internal heat absorption and blowing.

Howell *et al.* [17] have analyzed Momentum and heat transfer on a continuous moving surface in power law fluid. Idress et al. [18] have studied visco-elastic flow past stretching sheet in a porous media. Naseem and Khan [19] have studied the boundary layer flow past a stretching plate with suction. Surmadevi *et al.* [20] have analyzed the boundary layer flow caused by a stretching/continuously moving sheet for different thermo-physical situations using variety of fluid models and boundary conditions. McCormack and Crane [21] have analyzed comprehensive analysis on boundary layer flow including the flow caused by stretching of flat surface and between two surfaces under different physical situations. Rajagopal, Na and Gupta [22] have studied the flow of a visco-elastic fluid over a stretching sheet. Wen-Dong Chang [23] studied that the solution of the problem is not necessarily unique. Beard and Walters [24] have studied the elastico-viscous boundary layer flows.

II. MATHEMATICAL FORMULATION

The non-dimensional form of momentum equation for the boundary layer model developed by using similarity solution principles for Walter' liquid B' model is of the type.

$$(g_\eta)^2 - g g_{\eta\eta} = g_{\eta\eta\eta} - k_1\{2 g_\eta g_{\eta\eta\eta} - (g_{\eta\eta})^2 - g g_{\eta\eta\eta\eta}\} \quad (1)$$

$$g_\eta(0) = 1,\ g(0) = 0,\ g_\eta(\infty) = 0 \quad \text{(2a, b, c)}$$

Where the non-dimensional physical quantity k_1 is positive constant and suffix η denotes the differentiation with respect to η first, second, third and fourth time. Equations (1) and (2) represent a two point fourth-order non-linear differential equation having only three boundary conditions. The fourth boundary condition is obtained by using (2a) and (2b) in equation (1) as

$$1 - 0 = g_{\eta\eta\eta}(0) - k_1\{2.1.g_{\eta\eta\eta}(0) - g_{\eta\eta}^2(0) - 0\} \qquad (1 - 2k_1) g_{\eta\eta\eta}(0) + k_1\{g_{\eta\eta}\}^2 = 1 \quad (3)$$

Again differentiating equation (1) with respect to η and applying the boundary conditions (2a) and (2b) we get

$$g_{\eta\eta}(0) = g_{\eta\eta\eta\eta}(0) - k_1\{g_{\eta\eta\eta\eta}(0)\} \quad (4)$$

$$g_{\eta\eta\eta\eta}(0) = \frac{g_{\eta\eta}(0)}{(1 - k_1)} \quad (5)$$

It is very clearly noted that $g_{\eta\eta\eta\eta}(0)$ in equation (5) becomes infinity for k_1=1.Thus the limit of the applicability of the solution of the problem with respect to the non-dimensional positive physical quantity k_1<1.

III. METHOD OF SOLUTION

To obtain the solution of momentum equation (1) it is assumed the solution is of the type

$$g = A - G \quad (6)$$

By substituting $\alpha e^{-H\eta}$ for G and differentiating. w.r.t η equation (1) converts to

$$k_1(A - \alpha)H^2 - H + (A - \alpha) = 0 \quad (7)$$

The non-zero roots of equation (7) are $H_{1,2} = -\dfrac{1 \pm \sqrt{1 - 4k_1 2(A - \alpha).}}{2(A - \alpha)k_1}$ (8)

Thus the solution of equation (1) can be designated as $g = A - \{D e^{H_1\eta} + E e^{H_2\eta}\}$ (9)

By using equation (8) in equations (2a) and (2b), the constants D and E are expressed in terms of A, H_1 and H_2

as $D = \dfrac{H_2(A - \alpha) + 1}{H_2 - H_1}$ $E = \dfrac{H_2(A - \alpha) + 1}{H_1 - H_2}$ (10a, b)

Using equation (7) in equation (10) the solutions for $g_{\eta\eta}(0)$ and $g_{\eta\eta\eta}(0)$ in terms of A can be obtained as

$$g_{\eta\eta}(0)=\frac{(A-\alpha)^2-1}{(A-\alpha)k_1} \qquad g_{\eta\eta\eta}(0)=\frac{1-((1+k_1)(A-\alpha))^2}{((A-\alpha)k_1)^2} \tag{11a, b}$$

Combining equation (3) and (11) we get

$$k_1(A-\alpha)^4-(1+k_1-k_1^2)(A-\alpha)^2+(1-k_1)=0 \tag{12}$$

and the roots of equation (12) are given by $(A-\alpha)^2=(1-k_1),\frac{1}{k_1}$ (13a,b)

$$(A-\alpha)=\sqrt{1-k_1} \qquad \& \qquad (A-\alpha)=\frac{1}{\sqrt{k_1}}$$

$$A=\alpha\pm\sqrt{1-k_1} \qquad A=\alpha\pm 1/\sqrt{k_1}$$

It is very interesting to note that for $k_1=\frac{1}{2}$, the non-dimensional velocity gradient at the wall $g_{\eta\eta}(0)$ from the additional boundary condition (3), is found to be $\pm\sqrt{2}$.This observation might give a clue for finding out the second closed-form solution for $k_1={}^1\!/_2$. Since there are two values for $(A-\alpha)^2$ two closed-form solutions are found here for all values of $k_1\in(0,1)$.

First solution, For $(A-\alpha)^2=(1-k_1)$ from equation (7) the values of H_1 and H_2 are derived as

$$H_1=1/A-\alpha \quad \text{and} \quad H_2=-\frac{(1-k_1)}{(A-\alpha)k_1} \tag{14a, b}$$

And from equation (11) the values of D & E are re designated as

$$D=\frac{A-\alpha}{1-2k_1} \qquad E=\frac{2(A-\alpha)k_1}{-(1-2k_1)} \tag{15a, b}$$

From equations (9), (14) and (15) the first solution of the problem is obtained as

$$g=B\{1-e^{-\eta/A}\}=\sqrt{(1-k_1)}\{1-e^{-\eta/B\sqrt{(1-k_1)}}\} \tag{16}$$

Where $B=(A-\alpha)$

In order to satisfy boundary condition (2c), the positive value of A from (14a, b) is considered in equation (16)

From second solution, $(A-\alpha)^2={}^1\!/_{k_1}$ equation (7) yields the roots H as

$$H_{1,2}=\frac{-1\pm\sqrt{3i}}{2Bk_1} \tag{17a, b}$$

And equation (10) yields the values of the constants D & E as follows

$$D=\frac{B}{2}\left\{1-\frac{(1-2k_1)}{\sqrt{3}}i\right\} \quad , \quad E=B/2\left\{1+\frac{(1-2k_1)}{\sqrt{3}}i\right\} \tag{18a, b}$$

From equations (8), (17) and (18), the second solution of the problem is found to be

$$g=B\left(1-e^{\xi}\left\{\cos(\zeta)+\frac{(1-2k_1)}{\sqrt{3}}\sin(\zeta)\right\}\right) \tag{19}$$

Where $\xi = \dfrac{\eta}{2Ak_1} = \dfrac{\eta}{2\sqrt{k_1}}$ $\zeta = \dfrac{\sqrt{3}\eta}{2Ak_1} = \dfrac{1}{2}\dfrac{\sqrt{3}}{\sqrt{k_1}}\eta$

In equation (19) the value of A is considered to be positive so as to satisfy the boundary conditions (2c).

These two solutions given in equations (16) and (19) are found to be quite different. It can be verified from equations (16) and (19) that for all $0 < k_1 < 1$ as $\eta \to \infty, g_{\eta\eta}(\eta) \to 0$, whereas at $\eta = 0$, $g_{\eta\eta}(0) = -\, 1\big/\sqrt{(1-k_1)} < 0$ from equation (16), and $g_{\eta\eta}(0) = (1-k_1)\big/k_1^{3/2} > 0$ from equation (19).

The values of $g_{\eta\eta}(0)$ can be obtained directly from equation (11) by substituting the values of A as $\alpha \pm \sqrt{(1-k_1)}$ and $\alpha \pm 1\big/\sqrt{k_1}$ respectively.

IV. VALIDITY OF THE SOLUTION

Equations (16) and (19) are representing two solutions for equation (1) with the boundary conditions (2) when $k_1 = 0$. Troy et al. [15] found the first solution of the problem as given in equation (16). Another solution of the problem for the case $k_1 = 1\big/2$, obtained by Wen-Dong Chang [23], can be found from equation (19). It has been proved simultaneously by McLeod and Rajagopal [22] and by Troy et al. [15] that equations (1) and (2) have unique solution.

$g(\eta) = 1 - e^{-\eta}$ for $k_1 = 0$, in which $g_{\eta\eta}(\eta) < 0$ for all $0 < \eta < \infty$.

Though for $k_1 \equiv 0$, two solutions exist for the present problem, the important constraint needed to get the realistic solution of the physical problem, which was missed in [23], is

$$g_{\eta\eta}(\eta) \le 0 \text{ for all } 0 < \eta < \infty$$

The requirement of $g_{\eta\eta}(\eta) < 0$ everywhere to get a realistic solution in the present study.

By physical intuition, one should expect that a slightly elastic fluid will produce a boundary layer only slightly altered in its dimensions from a viscous one. For a small value of k_1 (say 0.0001), the dimensionless velocity gradient at the wall from the first solution is $g_{\eta\eta}(0) = -1.00005$ and its value from the second solution is obtained as 999900 for $k_1 = 0, g_{\eta\eta}(0) = -1$ such a drastic change in the value of $g_{\eta\eta}(0)$ for small value of k_1 obtained from the second solution is not reasonable. Since the first solution gives insight into the boundary layer for weakly elastic fluids, in the sense that $k_1 \le 1$. It is realistic solution for $0 < k_1 < 1$.

Rajagopal et al. [22] used a perturbation analysis by expanding the solution in powers of k_1 and obtained numerical estimates on the behavior of the solution of equations by using the function suggested by Troy et al. [15] (which is nothing but the first solution of the $k_1(A-\alpha)H^2 - H + (A-\alpha) = 0$).

Since the first solution gives the boundary layer behavior for $0 < k_1 < 1$, the velocity gradient $g_{\eta\eta}(0) < 0$ everywhere for the present problem.

Beard and Walters [24] have extended the prandtl boundary layer theory for an idealized elastic -viscous liquid. The boundary layer equations are solved numerically for the case of two- dimensional flow near a stagnation point. It is demonstrated that the main effect of elasticity is to increase the velocity in the boundary layer and also to increase the stress on the solid boundary. It is noticed from the first solution that the magnitude of the velocity gradient at the wall increases with k_1. From the second solution, it is found that the velocity gradient at the wall decreases drastically with k_1.

Regarding the validity of small values of k_1, Surmadevi and Nath [20] pointed out that the second order fluid (i.e. visco-elastic fluid) governed by equation (1) represents the behavior of fluids with short memory and the characteristic time scale associated with the motion is large compared with time representing the memory of the fluid. Hence, the assumption of small values of k_1 is valid especially for dilute polymer solutions.

Based on the observations above, the first solution given in equation (16) represents a realistic solution for the present physical problem for all.

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