Statistical \(\lambda\) – Convergence for Double Sequences in Probabilistic Normed Spaces

Deepak Rawat, Reena, Meenakshi and Gursimran Kaur

Department of Mathematics, Chandigarh University, Mohali, India.

(Received 02 May 2019, Revised 10 July 2019, Accepted 08 August 2019)

ABSTRACT: Our main objective in this work is to describe new generalization of \(\lambda\)-convergence for double sequences in the probabilistic normed spaces. The notion of statistical \(\lambda\)-convergence and statistical \(\lambda\)-Cauchy of double sequences has been defined in this particular normed space. We have given example which demonstrates that this idea is more generalized than the usual convergence.

Keywords: Statistical convergence, Statistical \(\lambda\)-convergence, Probabilistic Normed Space (\(PN\)-space)

I. INTRODUCTION

Firstly, in 1951, statistical convergence was initially coined by Henry Fast [4]. In fact, Fast got this concept from Steinhaus [30]. Then Antoni Zygmund [32] was the first who proved statistical convergence for Fourier Series in his book “Trigonometric Series” [31] that was the first edition in 1935. But in that book, he was using the term ‘almost convergence’ rather than ‘statistical convergence’. The notion ‘almost convergence’ was already used by Lorentz [13] so Henry Fast [4] had to take some other name for his concept. So, he took ‘Statistical convergence’ as appropriate notion. After the papers of Friddy [5] and Salát [27], this concept became a dynamic area of research. After that, several authors contributed a vast literature for this concept. Several authors have given various extensions, generalizations, variants and applications about the notion [see 2, 6, 12, 15, 16, 18, 19, 22, 23, 24, 25, 28]. The term \(\lambda\)-statistical convergence for the sequences was introduced by Mursaleen [17]. He generalized this concept of statistical convergence using (V, \(\lambda\))-summability. Further, the idea of statistical convergence for double sequences came into the consideration after the work of Bromwich [1]. Double sequences also were of keen interest of many researchers like Hardy [8], Tripathy [31], Mursaleen and Edely [20], Mursaleen and Mohiuddine [21], Kostyrko et al., [3, 11] Savas and Patterson [29] etc. in the area of “statistical convergence”.

The generalized metric space named as statistical metric space was presented by Menger [14]. Now days, it is known as probabilistic metric space and converted in active area of research. It has so many applications in functional analysis. These types of sequences has motivated Karakus [9, 10] to define a new concept statistical convergence for double sequence in PN-Space.

In this paper, we describe and analyze the term statistical \(\lambda\)-convergence as well as statistical \(\lambda\)-Cauchy for double sequences in the PN-Space. First, we review some basic terms as follows:

A natural density of the set \(E\) (which is the subset of natural numbers \(\mathbb{N}\)) is characterized by

\[
\delta(E) = \frac{\left|\{a \in E : a \leq n\}\right|}{n}, \quad \text{when } n \to \infty
\]

where \(|\cdot|\) indicates the order of the enclosed set.

Definition 1.1 [4]: A sequence \(x = (x_m)\) converges statistically to some number \(L\) if for each \(\varepsilon > 0\),

\[
\delta(m \leq n : |x_m - L| \geq \varepsilon) = 0.
\]

Symbolically, \(\lim_{m \to \infty} x_m = L\) where \(\ell\) is the collection of all statistically convergent sequences.

Definition 1.2: The function \(\varphi: R \to R\) is known as a distribution function when it is a non-decreasing and left continuous with (i) \(\inf_{x \to a} \varphi(x) = 0\) and (ii) \(\sup_{x \to a} \varphi(x) = 1\). \(\ell_{\varphi}\) represents the collection of all the distribution functions.

Definition 1.3: A t-norm (triangular form) is a mapping \([0,1] \times [0,1] \to [0,1]\) which is continuous, non-decreasing, commutative and associative.

Definition 1.4: [7] Consider \(X\) is a linear space, \(*\) is a t-norm and and \(\ell_{\varphi}\) is the collection of distribution functions. Consider a map \(P: X \to \ell_{\varphi}\) with \(P_x = P(x)\) and \(P_x(t)\) is the value of \(P_x\) at \(t \in R\). Then \(P\) and \((X, P_\ast)\) is known as probabilistic norm and probabilistic normed space respectively, if it holds the next four axioms:

(i) \(P_x(0) = 0\),
(ii) \(P_x(t) = 1\), \(\forall t > 0\) iff \(x = 0\),
(iii) \(P_{x+y}(t) = P_x(t) \ast P_y(t)\) where \(\mu \neq 0\),
(iv) \(P_{x+y}(s + t) = P_x(s) \ast P_y(t), \forall y, x \in X\) and \(s, t \in \mathbb{R}_+ = [0, \infty)\),

Definition 1.5: [9] Let \((X, P_\ast)\) be a \(PN\)-space. A double sequence \(x = (x_{mh})\) is called convergent to some \(L\) with respect to \(P\) in \((X, P_\ast)\) if for any \(\varepsilon > 0\) and \(\varphi \in (0,1) \exists \text{ a integer } N > 0\) such that \(P_{x_n - L}(\varepsilon) > 1 - \varphi\), whenever \(g, h \geq N\).

Definition 1.6: [9] Let \((X, P_\ast)\) be a \(PN\)-space. A double sequence \(x = (x_{mh})\) is called Cauchy sequence with respect to \(P\) in \((X, P_\ast)\) if for any \(\varepsilon > 0\) and \(\varphi \in (0,1) \exists \text{ a integer } N > 0\) and \(M > 0\) such that \(P_{x_{(h+1)n} - x_{(h)n}}(\varepsilon) > 1 - \varphi\), whenever \(g, h \geq N\) and \(p, q \geq M\).

II. MAIN RESULTS

Before explaining the statistical \(\lambda\)-convergence for double sequences in \(PN\)-Space. First we mention \(\lambda\)-convergence for sequences which is defined by Mursaleen [26] as follows:

Suppose \(\lambda = (\lambda_i)_{i=0}^{\infty}\) is a real sequence of positive numbers that approaches to \(\infty\) with \(\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1\). Then, sequence \(x = (x_m)\) is termed as \(\lambda\)-convergent
to some $L$ when $\Lambda_m(x) \rightarrow L$ whenever $m \rightarrow \infty$, and $\Lambda_m(x) = \frac{1}{\lambda_m} \sum_{n=0}^{\infty} (\lambda_{m+1} - \lambda_m) n$, $(m \in \mathbb{N})$.

**Definition 2.1:** A double sequence $x = (x_{gh})$ is called statistically $\Lambda$-convergent to some $L$ if for any $\varepsilon > 0$,
\[
\frac{1}{mn} \left| \left\{ (g,h) : g \leq n, h \leq m : |Ax_{gh} - L| \geq \varepsilon \right\} \right| = 0,
\]
when $m, n \rightarrow \infty$.

We can denote it as $\text{St}_{\Lambda} x_{gh} = L$.

**Definition 2.2:** A double sequence $x = (x_{gh})$ is called statistically $\Lambda$-Cauchy if for any $\varepsilon > 0 \exists M > 0$ and $m > M$ such that
\[
\frac{1}{mn} \left| \left\{ (g,h) : g \leq n, h \leq m : |Ax_{gh} - Ax_{pq}| \geq \varepsilon \right\} \right| = 0,
\]
when $m, n \rightarrow \infty$.

**Definition 2.3:** Let $(X, P, +)$ be a $PN - \text{space}$. A double sequence $x = (x_{gh})$ is called statistically $\Lambda$-Cauchy with respect to $P$ in $(X, P, +)$ if for any $\varepsilon > 0$ and $\varphi \in (0,1)$, we have
\[
\frac{1}{mn} \left| \left\{ (g,h) : g \leq n, h \leq m : P_{Ax_{gh} - Ax_{pq}}(\varepsilon) \leq 1 - \varphi \right\} \right| = 0,
\]
when $m, n \rightarrow \infty$.

**Definition 4.2:** Let $(X, P, +)$ be a $PN - \text{space}$. A double sequence $x = (x_{gh})$ is called statistically $\Lambda$-convergent with respect to $P$ in $(X, P, +)$ if for any $\varepsilon > 0$ and $\varphi \in (0,1)$, we have
\[
\frac{1}{mn} \left| \left\{ (g,h) : g \leq n, h \leq m : P_{Ax_{gh} - Ax_{pq}}(\varepsilon) \geq 1 - \varphi \right\} \right| = 0,
\]
when $m, n \rightarrow \infty$.

**Theorem 2.6:** If a double sequence $x = (x_{gh})$ is $\Lambda$-Convergent in $PN - \text{space} (X, P, +)$ then it is also statistically $\Lambda$-convergent.

**Proof:** As sequence $x = (x_{gh})$ is $\Lambda$-convergent in $(X, P, +)$ then for $\varepsilon > 0$ and $\varphi (0,1) \exists n_0 \in \mathbb{N}$ such that $P_{Ax_{gh} - Ax_{pq}}(\varepsilon) > 1 - t$ for all $g \geq n_0$ and $h \geq n_0$. Thus, $\delta_{\Lambda} \left\{ (g,h) \in \mathbb{N} \times \mathbb{N} : P_{Ax_{gh} - Ax_{pq}}(\varepsilon) \leq 1 - \varphi \right\} = 0$, which establishes Theorem 2.6.

The next example can justify that the converse of above mentioned theorem may be not true.

**Example 2.1:** Consider $(R, |\cdot|)$ is a real normed space with $P_x(t) = \frac{t}{t+1}$, $t \geq 0$ and $x \in R$.

Here $(R, P, |\cdot|)$ is a $PN - \text{space}$. Define sequence $x = (x_{gh})$ as
\[
Ax_{gh} = \begin{cases} \sqrt{gh} & \text{if } g, h \text{ are squares} \\ 0 & \text{otherwise} \end{cases}
\]

Now, for every $\varepsilon > 0$ and $\varphi \in (0,1)$,
\[
N_m(x_{gh}) = \left\{ (g,h) : g \leq n, h \leq m : P_{Ax_{gh}}(\varepsilon) \leq 1 - \varphi \right\} = \left\{ (g,h) : g \leq n, h \leq m : \frac{t}{t + Ax_{gh}} \leq 1 - \varphi \right\} = \left\{ (g,h) : g \leq n, h \leq m : Ax_{gh} \geq 1 - \varphi t \right\} = \left\{ (g,h) : g \leq n, h \leq m : Ax_{gh} = \sqrt{gh} \right\} = \left\{ (g,h) : g \leq n, h \leq m : g \text{ and } h \text{ are squares} \right\}
\]

Then,
\[
\frac{1}{mn} \left| \left\{ (g,h) : g \leq n, h \leq m : Ax_{gh} = \sqrt{gh} \right\} \right| = \frac{1}{mn} \left| \left\{ (g,h) : g \leq n, h \leq m : Ax_{gh} = \sqrt{gh} \right\} \right| \leq \frac{\sqrt{mn}}{nm} = \frac{\sqrt{mn}}{nm}.
\]

Thus, above sequence is statistically $\Lambda$-convergent but not usually convergent in $(R, P, |\cdot|)$.

**Theorem 2.7:** Let $x = (x_{gh})$ and $y = (y_{gh})$ be two sequences in $PN - \text{space} (X, P, +)$. Then

(i) If $St_F^g x_{gh} = x_0$ and $St_F^a y_{gh} = y_0$, then $St_F^a (x_0 + y_0) = x_0 + y_0$.

(ii) If $St_F^g x_{gh} = x_0$ and $St_F^a c y_{gh} = y_0$, then $c x_0 = y_0$.

**Proof:** (i) Let $St_F^g x_{gh} = x_0$ and $St_F^a y_{gh} = y_0$. For $\varepsilon > 0$ and $\varphi \in (0,1)$, take $\theta \in (0,1)$ with $(1 - \theta) \ast (1 - \theta) = 1 - \varphi$.

Let $K_1(\theta, e) = \left\{ (g,h) \in \mathbb{N} \times \mathbb{N} : P_{Ax_{gh} - Ax_{pq}}(\varepsilon) \leq 1 - \theta \right\}$.

Let $K_2(\theta, e) = \left\{ (g,h) \in \mathbb{N} \times \mathbb{N} : P_{Ax_{gh} - Ax_{pq}}(\varepsilon) \leq 1 - \theta \right\}$.

As $St_F^g x_{gh} = x_0$, then we have $\delta_{\Lambda}(K_1(\theta, e)) = 0$ for all $\varepsilon > 0$.

Similarly, since $St_F^a y_{gh} = y_0$, we get $\delta_{\Lambda}(K_2(\theta, e)) = 0$ for all $\varepsilon > 0$.

Take $K(\theta, e) = K_1(\theta, e) \cup K_2(\theta, e)$, then we get $\delta_{\Lambda}(K(\theta, e)) = 0$ i.e. $\delta_{\Lambda}(\mathbb{N} \times \mathbb{N} - K(\theta, e)) = 1$.

If $(g,h) \in \mathbb{N} \times \mathbb{N} - K(\theta, e)$, then we have
\[
P_{Ax_{gh} - Ax_{pq}}(\varepsilon) = P_{Ax_{gh} - Ax_{pq}}(\varepsilon) \geq (1 - \theta) \ast (1 - \theta) \geq 1 - \varphi.
\]

We get $P_{Ax_{gh} - Ax_{pq}}(\varepsilon) = 1$ as $\varphi > 0$ is arbitrary, which gives $L_1 = L_2$.

Therefore, $St_F^a (x_0 + y_0) = x_0 + y_0$.
(ii) Let $S_{R}^{N} - x_{gh} = x_{q}$. 
First we take $c = 0$. For $\epsilon > 0$ and $\varphi \in (0,1)$
\[
P_{\alpha x_{gh} - x_{gh}}(\epsilon) = P_{\varphi}(\epsilon) = 1 > 1 - \varphi
\]
\[
P_{\alpha x_{gh} - 0} = 0.
\]
Then by Theorem 2.6, we have $S_{R}^{N} - lim 0 x_{gh} = 0$.
Now, take $c \in R$ but ($c \neq 0$).
Since $S_{R}^{N} - x_{gh} = x_{q}$, then for every $\epsilon > 0$ and $\varphi \in (0,1)$, we define the set
\[
A(\varphi, \epsilon) = \{(g, h) \in N \times N : P_{\alpha x_{gh} - x_{gh}}(\epsilon) \leq 1 - \varphi\}
\]
such that $\delta_{A}(A(\varphi, \epsilon)) = 0$.
In this case $\delta_{A}(A^{*}(\varphi, \epsilon)) = 1$.
If $(g, h) \in A^{*}(\varphi, \epsilon)$, then
\[
P_{\alpha x_{gh} - x_{gh}}(\epsilon) = P_{\alpha x_{gh} - x_{gh}}(\epsilon) = P_{\epsilon} \left( \frac{\epsilon}{|c|} - \epsilon \right)
\]
\[
= P_{\alpha x_{gh} - x_{gh}}(\epsilon) + 1
\]
\[
> 1 - \epsilon.
\]
For $c \in R (c \neq 0)$ this shows that
\[
\delta_{A}(\{(g, h) \in N \times N : P_{\alpha x_{gh} - x_{gh}}(\epsilon) \leq 1 - \epsilon\}) = 0.
\]
Hence $S_{R}^{N} - x_{gh} = x_{q}$.

**Theorem 2.8:** A double sequence $x = (x_{gh})$ in $PN - Space(X, P, \alpha)$ is statistically $\Lambda$-convergent to $L$ iff there exists a set $H = \{(g, h) \subseteq N \times N ; g, h = 1, 2, 3, \ldots \}$ with $\delta_{A}(H) = 1$ and $S_{R}^{N} - x_{gh} = L$.

**Proof:** Firstly we assume that $S_{R}^{N} - x_{gh} = L$, then for every $\epsilon > 0$ and $d \in N$.
Take $K(d, \epsilon) = \{(g, h) \in N \times N : P_{\alpha x_{gh} - x_{gh}}(\epsilon) \leq 1 - \frac{\epsilon}{d}\}$. Then $\delta_{K}(K(d, \epsilon)) = 0$ with $M(1, \epsilon) \supset M(2, \epsilon) \supset M(i, \epsilon) \supset M(i + 1, \epsilon) \ldots$ for $d = 1, 2, \ldots$ (a) and $\delta_{A}(M(d, \epsilon)) = 1, d = 1, 2, \ldots$ (b)
Next, we contrary prove the required result. Assume sequence $x = (x_{gh})$ is not statistically $\Lambda$-convergent to $L$. Then, for any $\epsilon > 0$ and $\varphi \in (0,1)$, we have has infinitely many terms in the set $\{(g, h) \in N \times N : P_{\alpha x_{gh} - x_{gh}}(\epsilon) \leq 1 - \varphi\}$. Take $M(\varphi, \epsilon) = \{(g, h) \in N \times N : P_{\alpha x_{gh} - x_{gh}}(\epsilon) \leq 1 - \varphi\}$ for $d \in N$.
Then $\delta_{A}(M(\varphi, \epsilon)) = 0$.
\[
\therefore \delta_{A}(M(d, \epsilon)) \subseteq M(\varphi, \epsilon). \quad \text{Hence} \quad \delta_{A}(M(d, \epsilon)) = 0 \quad \text{which is contradiction to (b)}.
\]
Therefore $x = (x_{gh})$ is statistically $\Lambda$-convergent to $L$.

Conversely, let there exists a set $H = \{(g, h) \in N \times N ; g, h = 1, 2, \ldots \}$ with $\delta_{A}(H) = 1$ and $S_{R}^{N} - x_{gh} = L$. Then, for any $\epsilon > 0$ and $\varphi \in (0,1)$ \forall $\epsilon \times N$ with $P_{\alpha x_{gh} - x_{gh}}(\epsilon) > 1 - \varphi, \forall g, h \in N$.
Now,
\[
M(\varphi, \epsilon) = \{(g, h) \in N \times N : P_{\alpha x_{gh} - x_{gh}}(\epsilon) > 1 - \varphi\}
\]
\[
\subseteq (N \times N) - \{(h_{k+1, h_{k+1}}, (g_{k+2, h_{k+2}}), \ldots)\}
\]
Then, $\delta_{A}(\{(g, h) \in N \times N : P_{\alpha x_{gh} - x_{gh}}(\epsilon) > 1 - \varphi\}) = 1 - 1 = 0$.
Hence, $S_{R}^{N} - x_{gh} = L$.

**Theorem 2.9:** A double sequence $x = (x_{gh})$ in $PN - Space(X, P, \alpha)$ is statistically $\Lambda$-convergent if it is statistically $\Lambda$-Cauchy in $PN - Space(X, P, \alpha)$.

**Proof:** Let $S_{R}^{N} - x_{gh} = L$.
For $\epsilon > 0$ and $\varphi \in (0,1)$, we have
\[
\delta_{A}(\{(g, h) \in N \times N : P_{\alpha x_{gh} - L}(\epsilon) \leq 1 - \varphi\}) = 0.
\]
Take two numbers $M$ and $N$ such that
\[
A(\varphi, \epsilon) = \{(g, h) \in N \times N : P_{\alpha x_{gh} - L}(\epsilon) \leq 1 - \varphi\}
\]
\[
B(\varphi, \epsilon) = \{(g, h) \in N \times N : P_{\alpha x_{gh} - L}(\epsilon) \leq 1 - \varphi\}
\]
\[
C(\varphi, \epsilon) = \{(g, h) \in N \times N : P_{\alpha x_{gh} - L}(\epsilon) \leq 1 - \varphi\}
\]
Then $A(\varphi, \epsilon) \subseteq B(\varphi, \epsilon) \cup C(\varphi, \epsilon)$ i.e. $\delta_{A}(A(\varphi, \epsilon)) \leq \delta_{A}(B(\varphi, \epsilon)) + \delta_{A}(C(\varphi, \epsilon)) = 0$.
Thus $x = (x_{gh})$ is statistically $\Lambda$-Cauchy;
Conversely, assume $x = (x_{gh})$ is statistically $\Lambda$-Cauchy, then
\[
\delta_{A}(A(\varphi, \epsilon)) = 0.
\]
We contrary prove that sequence statistically $\Lambda$-convergent. Assume sequence $x = (x_{gh})$ is not statistically $\Lambda$-convergent then $\delta_{A}(B(\varphi, \epsilon)) = 1$ i.e. $\delta_{A}(A(\varphi, \epsilon)) = 0$. We can write
\[
P_{\alpha x_{gh} - L}(\epsilon) \leq 2P_{\alpha x_{gh} - L}(\epsilon/2) < 1 - \varphi
\]
if $P_{\alpha x_{gh} - L}(\epsilon/2) < \frac{1 - \varphi}{2}$.
As $\delta_{A}(B(\varphi, \epsilon)) = 0 \Rightarrow \delta_{A}(A(\varphi, \epsilon)) = 0$ i.e $\delta_{A}(A(\varphi, \epsilon)) = 1$ which is a contradiction.

**III. CONCLUSIONS**
This paper presents the overview of statistical convergence in setup of $PN$-spaces by defining and studying the idea of statistical $\Lambda$-convergence as well as statistical $\Lambda$-Cauchy for double sequences. Here, we derived more generalized results than the analogous results for double sequences in $PN$-spaces.

**ACKNOWLEDGEMENT**
We would like to thank the anonymous referees of the paper for their valuable suggestions and constructive comments which will help in improving the quality of the paper.

**CONFLICT OF INTEREST**
Authors have no any conflict of interest.

**REFERENCES**


