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# Comparison between Bayesian and Maximum Likelihood Estimation of Scale Parameter in Generalized Gamma Type Distribution with Known Shape Parameters under Different Loss Functions

***Gaurav Shukla, Vinod Kumar, Shivani Shukla and Mohit Giri Goswami***

*Department of Management, Invertis University, Bareilly*
*Department of MSCH, CBSH, G.B. Pant Univ. of AG & Tech. Pantnagar*
*Department of AS&H, Invertis University, Bareilly*
*Department of CSE, AITS Haldwani*

**ABSTRACT: Weibull distribution and exponential distributions are widely used in modeling and analyzing of lifetime data. In spite of these, some other distributions are also useful for analyzing of lifetime data. The present paper considers the estimation of the scale parameter of three parameter generalized gamma type distribution with known shape parameters. Maximum likelihood estimation is also discussed. Bayes estimator is obtained by using Jeffreys' prior under Linex loss function, Asymmetric Precautionary Loss function and Square error loss function. Relative efficiencies of the estimators are also calculated for the data set and it is observed that Bayes estimator fairs better than MLE in all cases.**

**Keywords:** MLE, Bayes Estimator, Loss Function, Prior, Posterier

## I. INTRODUCTION

Weibull distribution and exponential distributions are widely used as a lifetime distribution in analyzing of lifetime data. Besides these, gamma, log normal, inverse gamma and the generalized gamma are also suggested as a lifetime distributions. Here we use a three parameters generalized gamma type model whose probability density function is given as

$$f(x)=\frac{p}{\theta^k \Gamma k} e^{-x^p/\theta} x^{pk-1} \qquad x>0,\ p,k,\theta>0 \qquad \text{.... (1)}$$

Where θ is a scale parameter and k and p are shape parameters. This model includes exponential (p = k =1), Weibull (k=1), gamma (p=1) and model proposed by Stacy ( $\theta=\alpha^p$ ) as special cases. Its utility as a lifetime model has been examined by Shukla and Kumar (2006) by applying this model on various sets of lifetime data. Pandey and Rao (2006) have obtained Bayes estimators of scale parameter by using precautionary loss function whereas Shukla and Kumar (2008) have derived Bayes estimators of scale parameter for different priors by using Lindley approach for this distribution. They (2009) have also derived Bayes estimators of shape parameters for different priors under the assumption that scale parameter is known. The maximum likelihood (ML) method of estimation is quite efficient and very popular. In Bayesian approach, a prior distribution for the parameter is considered and then the posterior distribution is obtained by conditioning on the data and after that the inference is done based on the posterior.

Ahmed et al.(2010) have considered ML and Bayesian estimation of the scale parameter of Weibull distribution with known shape and compared their performance under squared error loss. Pandey *et.al.* (2011) have made the comparison between Bayesian and maximum likelihood estimation of scale parameter in Weibull distribution under linex loss function. Saima *et.al.* (2015) have shown Bayesian estimators of unknown parameters of a three parameter generalized gamma distribution, based on different priors using different loss functions. They also presented posterior means and variances for α under different priors by using different loss functions.

In this paper, ML estimator and Bayes estimator of the scale parameter of the three parameters generalized gamma distribution is considered under linex loss function, asymmetric precautionary loss function and square error loss function with the assumption that the shape parameters are known.

## II. MAXIMUM LIKELIHOOD FUNCTION

Let $x_1, x_2, x_3, \_\_\_\_\_\_ \; x_n$ be a random sample of size n from the proposed life testing model, whose p.d.f. is given by (1), then Likelihood function (L) is given by

$$L = \left[ \frac{p}{\theta^k \overline{)k}} \right]^n e^{-\sum_{i=1}^{n} \frac{x_i^p}{\theta}} \prod_{i=1}^{n} x_i^{pk-1} \qquad \dots (2)$$

Logarithm of this likelihood function is

$$\log_e L = n \log_e p - nk \log_e \theta - n \log_e \overline{)k} - \sum_{i=1}^{n} \frac{x_i^p}{\theta} + \sum_{i=1}^{n} \log_e x_i^{pk-1}$$

Or,

$$\log_e L = n \log_e p - nk \log_e \theta - n \log_e \overline{)k} - \sum_{i=1}^{n} \frac{x_i^p}{\theta} + (pk-1) \sum_{i=1}^{n} \log_e x_i \qquad \dots (3)$$

The MLE 0f $\theta$ can be obtained as $\theta = \frac{1}{nk} \sum_{i=1}^{n} x_i^p$ … (4)

Where p and k are known.

## III. BAYESIAN ESTIMATION PRIOR DISTRIBUTION

In Bayesian inference, the prior distribution or a prior probability distribution, often called simply the prior, is a key part and represents the information about an uncertain parameter θ that is combined with the probability distribution of new data to yield the posterior distribution, which in turn is used for future inferences and decisions involving. The derivation of the prior distribution based on information other than the current data is impossible or rather difficult because the likelihood function and the prior provide quite difficult posterior forms which are impossible to analyze analytically and are even very challenging from the usual numerical perspective. Moreover, the statistician may be required to employ as little subjective input as possible, so that the conclusion may appear solely based on sampling model and the current data.

Jeffreys proposed a formal rule for obtaining a non-informative prior. It is proportional to the square root of the determinant of the Fisher information:

$$g(\theta) \propto \sqrt{|I(\theta)|} \qquad \dots (5)$$

Where θ is k-vector valued parameter and I(θ) is the Fisher's information matrix of order k ×k. In particular, if θ is a scalar parameter, Jeffreys non-informative prior for θ is $g(\theta) \propto \sqrt{I(\theta)}$ . Thus in our problem, we consider

$$g(\theta) \propto \frac{1}{\theta} \quad \Rightarrow \quad g(\theta) = c\frac{1}{\theta} \qquad \dots (6)$$

Where c is a constant.

## IV. POSTERIOR DISTRIBUTION

The posterior distribution of θ given the random sample when p and k are fixed is given as

$$\pi(\theta / x_1, x_2, ...., x_n) = \frac{L(x_1, x_2, ........, x_n | \theta) g(\theta)}{\int_{\Theta} L(x_1, x_2, \_\_\_\_, x_n | \theta) g(\theta) d\theta} \quad \text{... (7)}$$

$$\pi(\theta / x_1, x_2, ...., x_n) = \frac{\frac{1}{\theta^{nk}} e^{-\frac{\sum x_i^p}{\theta}} g(\theta)}{\int \frac{1}{\theta^{nk}} e^{-\frac{\sum x_i^p}{\theta}} g(\theta) d(\theta)} \quad \text{... (8)}$$

Where g (θ) is a prior of θ. A distinctive feature of the Bayesian approach is the introduction of a prior density to represent prior information about the possible values of the parameters of the model. There are three distinct Bayesian approaches for selection of prior distribution [Diaconis and Ylvisker (1985)]. The choice of a convenient prior distribution which combines easily with the likelihood function has recently been simplified by the construction of conjugate family. The concept of conjugate family was introduced first by Bernard (1954) and fully explained by Raiffa and Schlaifer (1961). The restriction to the conjugate family is not necessary, but it has the advantage that the posterior distribution belongs to the same family.

*If* $g(\theta) = c\frac{1}{\theta}$, then

$$\pi(\theta / x_1, x_2, ...., x_n) = \frac{\frac{1}{\theta^{nk+1}} e^{-\frac{\sum x_i^p}{\theta}}}{\int \frac{1}{\theta^{nk+1}} e^{-\frac{\sum x_i^p}{\theta}} d(\theta)} \quad \text{... (9)}$$

$$\pi(\theta / x_1, x_2, ...., x_n) = \frac{t^{nk}}{\Gamma nk} \frac{e^{\frac{-t}{\theta}}}{\theta^{nk+1}} \quad \text{... (10)}$$

*Where* $t = \sum x_i^p$

**V. LOSS FUNCTION**

The loss function plays an important role in Bayesian inference. A loss function is a function that maps an event or values of one or more variables onto a real number intuitively representing some "cost" associated with the event. An optimization problem seeks to minimize a loss function. The estimator having the least expected loss is usually preferable compare to the others. Most authors use the simple quadratic (symmetric) loss function and obtain the posterior mean as the Bayesian estimate. However, in practice, the real loss function is often not symmetric. Varian (1975) introduced LINEX (Linear-Exponential) loss function, which is the simple generalization of squared error (SE) loss function and can be used in almost every situation.

**VI. LINEX LOSS FUNCTION**

The LINEX loss function is defined as follows:

$$L(\delta) = \exp(a\delta) - a\delta - 1 \quad \text{... (11)}$$

Where $\delta = \frac{\hat{\theta}}{\theta} - 1 \quad and \quad a \neq 0$

## VII. ESTIMATION UNDER LINEX LOSS

To obtain the Bayes estimator, we minimize the posterior expected loss which is given as

$$\rho=\int_0^\infty [\exp(a\delta)-a\delta-1]\,\pi(\theta/x_1,x_2.....,x_n)\,d\theta \quad \ldots (12)$$

$$Where \quad \delta=\frac{\hat{\theta}}{\theta}-1$$

Integrating, we have

$$\rho=\frac{e^{-a}t^{nk}}{\left(t-a\hat{\theta}\right)^{nk}}-\frac{a\hat{\theta}nk}{t}+(a-1) \quad \ldots (13)$$

$$Where \quad t=\sum x_i^p$$

$$Solving \ \frac{\partial\rho}{\partial\hat{\theta}}=0,$$

We obtain Bayes Estimator as

$$\hat{\theta}_b=\frac{t}{a}\left[1-\exp\left(-\frac{a}{nk+1}\right)\right]$$

$$where \quad t=\sum x_i^p \quad \ldots (14)$$

## VIII. ASYMMETRIC PRECAUTIONARY LOSS FUNCTION (APLF)

A very useful and simple asymmetric precautionary loss function is

$$L(\hat{\theta},\theta)=\frac{(\hat{\theta}-\theta)^2}{\hat{\theta}} \quad \ldots (15)$$

## IX. ESTIMATION UNDER ASYMMETRIC PRECAUTIONARY LOSS

To obtain the Bayes estimator, we minimize the posterior expected loss which is given as

$$\rho=\int_0^\infty\left(\frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}\right)\pi(\theta/x_1,x_2,......,x_n)\,d\theta \quad \ldots (16)$$

Integrating, we have

$$\rho=\hat{\theta}+\frac{t^2}{\hat{\theta}(nk-1)(nk-2)}-\frac{2k}{(nk-1)} \ Where \ t=\sum x_i^p \quad \ldots (17)$$

$$Solving \quad \frac{\partial}{\partial\hat{\theta}}\rho=0$$

We obtain Bayes Estimator as

$$\hat{\theta}_b = \left[\frac{t^2}{(nk-1)(nk-2)}\right]^{\frac{1}{2}}$$

$$where \;\; t = \sum x_i^p \qquad \text{… (18)}$$

**X. SQUARED ERROR LOSS FUNCTION (SELF)**

A commonly used loss function is the squared error loss function (SELF) which is given as

$$L(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2$$

19)

**XI. ESTIMATION UNDER SQUARE ERROR LOSS**

To obtain the Bayes estimator, we minimize the posterior expected loss which is given as

$$\rho = \int_0^\infty (\hat{\theta}-\theta)^2 \pi(\theta / x_1, x_2, ......, x_n)\, d\theta \qquad \text{… (20)}$$

Integrating, we have

$$\rho = \hat{\theta}^2 + \frac{t^2}{(nk-1)(nk-2)} - \frac{2\hat{\theta} t}{(nk-1)} \qquad \text{… (21)}$$

$Solving \; \frac{\partial \rho}{\partial \hat{\theta}} = 0,$ We obtain Bayes Estimator as

$$\hat{\theta}_b = \frac{t}{nk-1}$$

$$where \;\; t = \sum x_i^p$$

… (22)

**XII. RELATIVE EFFICIENCY**

The relative efficiency of the Bayes estimator with respect to the ML estimator is given by

$$RE = \frac{Risk\,(\widehat{\theta}_{mle})}{Risk\,(\widehat{\theta}_{Bayes})} \qquad \text{… (23)}$$

**XIII. NUMERICAL ILLUSTRATION**

In this study, we have generated random samples of size 100 from generalized gamma type distribution by using R-3.1.2 software (p=1, k= 1 and theta = 2) and compared the performance of ML and Bayes estimator based on them. In Table: 1, we present MLE, Bayes estimators of scale parameter by using different loss functions for Jeffrey's Prior, their corresponding Risk functions and relative efficiencies.

**Table: 1**

| | | MLE | Bayes | R MLE(Linex) | R Bayes (Linex) | RE |
|---|---|---|---|---|---|---|
| N=100 | a= -3 | 1.792790943 | 1.801665 | 0.045107526 | 0.044998884 | 1.002414328 |
| | a= -2 | 1.792790943 | 1.792732 | 0.019933338 | 0.019933336 | 1.000000108 |
| | a= -1 | 1.792790943 | 1.783857 | 0.00497927 | 0.004966874 | 1.002495818 |
| | a= 1 | 1.792790943 | 1.766282 | 0.00504600 | 0.00493400 | 1.02271500 |
| | a= 2 | 1.792790943 | 1.757581 | 0.020477578 | 0.019671919 | 1.040954813 |
| | a= 3 | 1.792790943 | 1.748938 | 0.046991432 | 0.044116579 | 1.065164922 |
| | | **MLE** | **Bayes** | **R MLE(APLF)** | **R Bayes (APLF)** | **RE** |
| N=100 | --------- | 1.792790943 | 1.820115778 | 0.018848142 | 0.018431671 | 1.022595 |
| | | **MLE** | **Bayes** | **R MLE(SELF)** | **R Bayes (SELF)** | **RE** |
| N = 100 | ……….. | 1.792790943 | 1.810899943 | 0.033790779 | 0.033462843 | 1.009800 |

## XIV. CONCLUSION

The present table explores ML and Bayesian estimation of scale parameter in generalized gamma type distribution under linex loss, asymmetric precautionary loss and square error lossand demonstrates that the Bayes estimator performs better than the MLE in all cases. Relative efficiency is minimum in case of linex loss function when a = -2 and maximum when a =3.

## REFERENCES


[1]. Ahmed, A. O. M., Al-Kutubi, H.S., and Ibrahim, N. A. (2010). Comparison of the Bayesian and maximum likelihood estimation for Weibull distribution. *Journal of Mathematics and Statistics* **6**, 2,100-104.

[2]. Cohen, A. C. (1965). Maximum likelihood estimation in Weibull distribution based on complete and on censored samples. *Technometrics,* **7**, (579-588).

[3]. Jeffreys, H. (1946). An invariant form for the prior probability in estimation problems. *Proceedings of the Royal Society of London, Series -A* 186, 453-461.

[4]. Lawless, J. F. (1982). Statistical Models and Methods for Life Time Data. John Wiley & Sons, New York.

[5]. Mann, N. R., Schaffer, R. E., and Singpurwala, N.D. (1974). Methods for Statistical Analysis of Reliability and Life Data. John Wiley & Sons, New York.

[6]. Martz, H. F., and Waller, R. A. (1982). Bayesian Reliability Analysis: John Wiley & Sons, New Yark.

[7]. Naqash, S., Ahamad, S.P. and Ahamad, A. (2015). Bayesian Analysis of Generalized Gamma Distribution, *Journal of Statistics Applications & Probability*, **4**, 3, 499-512.

[8]. Pandey, B.N. (1997). Testimator of scale parameter of exponential distribution using linex loss function. *Communication in Statistics- Theory and Methods,* **26**, 9, 2191-2202.

[9]. Pandey, B.N., Mishra, G.C., and Srivastava, A.K. (2004). Invariant version of linex loss function and its applications in exponential type II censored data. *The Aligarh Journal of Statistics* **24**, 1-22.

[10]. Pandey, H. and Rao, A. K. (2006). Bayesian estimation of scale parameter of generalized gamma distribution using precautionary loss function, *Indian Journal of Applied Statistics,* **10** (2006), 21-27.

[11]. Shukla, G. and Kumar, V. (2008). "Bayes Estimators of the Scale Parameter of A Generalized Gamma Type Model", *Journal of Reliability and Statistical Studies*, **1**: 1, 54-60.

[12]. Soland, R.M. (1968). Bayesian analysis of the Weibull process with unknown scale parameter and its application to acceptance sampling. *IEEE Transactions on Reliability* **17**, 84-90.

[13]. Varian, H.A. (1975). A Bayesian approach to real estate assessment. In Studies in Bayesian Econometrics and Statistics in Honorof Leonard J. Savage, S. E. Fienberg and A. Zellner, Eds. North-Holland, Amsterdam, pp.195-208.

[14]. Zellner, A. (1986). Bayesian estimation and prediction using asymmetric loss functions. *Journal of American Statistical Association* **81**, 394, 446-451.