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Fixed Point Theorems in Complete Metric Spaces for Weakly Compatible Mappings

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ABSTRACT: Present paper, has two fixed point theorems in complete metric space. For proving theorems; we use concept of weakly compatible mappings for four mappings using α property.
Mathematics Subject Classification (MSC): 47H10, 54H25s.

Keywords: Cauchy sequence, weakly compatible mapping, α property. Complete metric space.

I. INTRODUCTION

With the celebration of Banach's fixed point theorem [1] in 1922, Researchers find a new direction to prove fixed point in different metric spaces. Banach's fixed point theorem define as 'let $(X;d)$ be a complete metric space. If T satisfies $d(Tx;Ty) \leq kd(x;y)\ \forall\ x;y \in X\ \&\ 0 \leq k \leq 1$; then T has a unique fixed point in *X*'. This theorem has many applications, but suffers from one drawback; definition requires that *T* be continuous throughout *X*. In 1962, Edelstein [2] established a new concept of contractive mapping in place of contraction mapping. Contractive mappings are more general than contraction mapping in 1968, R. Kannan [3] proved an important result which does not require the continuity of T. In 1988, Gerald Jungck [4] proved a common fixed points for commuting and compatible maps on compacta.in 1996, Gerald Jungck [5] proved Common fixed points for non-continuous non-self maps on non-metric spaces, Jungck and B Rhoades,[6] Fixed points for set valued functions without continuity. There then follows a flood of papers involving contractive definition that do not require the continuity of T. This result was further generalized and extended in various ways by many authors. On the other hand Sessa [7] defined weak commutativity and proved common fixed point theorem for weakly commuting mapping. Further Jungck also introduced commutivity, the generalization of weakly compatibility, which is more general than that of weak commutativity. Since then various fixed point theorems, for compatible mappings satisfying contractive type conditions and assuming continuity of at least one of the mappings, have been obtained by many authors. It has been known from the paper of Kannan [3] that there exists mapping that have a discontinuity in the domain but which have fixed points, moreover, the mappings involved in every case were continuous at the fixed point. In 1998, Jungck and Rhoades [6] introduced the notion of weakly compatible and showed that compatible mappings are weakly compatible but converse need not be true. Suzuki, [8] proved new type of fixed point theorem in metric spaces. Suzuki [9] also generalized banach contraction principal. Altun and Erduran [10], Doric and Lazovic [11] proved szuki type fixed point theorems in complete metric space. Karapınar [12] wrote a remark on szuki type fixed point theorems. Ciri´c and Ume [13] proved Some common fixed point theorems for weakly compatible mappings.Sessa and Kaneko. Sessa and Kaneko [14] Proved Fixed point theorems for compatible multi valued and single valued mapping. Ahmed, [15], Chugh and Kumar, [16], Sedghi, [17], popa [18] proved fixed point theorem for four weakly compatible mappings.

II. PRELIMINARIES

In the present paper, we introduce a binary operation which is a modification of the definition of ordinary metric. We give some properties about this operation metric and we prove two common fixed point theorems for four weakly compatible maps in complete metric spaces satisfying a new general contractive type condition by assuming **N** is the set of all natural numbers and R^+ is the set of all positive real numbers.

Definition 2.1. A mapping $T: X \rightarrow X$ defined on metric space $(X;d)$ is called contraction mapping if
$d(Tx;Yy) \leq K.d(x;y) \forall\ x;y \in X\ \&\ 0 < k < 1$

Definition2.2; A mapping $T: X \rightarrow X$ defined on metric space $(X;d)$ is called contractive mapping if
$d(Tx;Ty) \leq d(x;y) \forall\ x;y \in X$

Definition 2.3. Let $\diamond: \mathrm{R}^+ \times \mathrm{R}^+ \rightarrow \mathrm{R}^+$ be a binary operation satisfying the following axioms:
(i) is associative and commutative,
(ii) is continuous.
Here are typical examples:
$1.\ \mathrm{a}\diamond\mathrm{b} = \alpha\ \mathrm{Max}(\mathrm{a};\mathrm{b});\ 2.\ \mathrm{a}\diamond\mathrm{b} = \mathrm{a}+\mathrm{b};\ 3.\ \mathrm{a}\diamond\mathrm{b} = \mathrm{ab}+\mathrm{a}+\mathrm{b};$

$4.\ a\diamond b = ab;\ 5.\ a\diamond b = \dfrac{ab}{\mathrm{Max}(a;b;1)}\ \ \forall a,b \in R^{+}$

$6.\ (a\diamond b)\diamond c = \alpha^2 \mathrm{Max}(a;b;c)\ \ \forall a,b,c \in R^{+}$

Definition 2.4. The binary operation is said to satisfy α-property if there exists a positive real number α such that
$a\diamond b = \alpha\,\mathrm{Max}(a;b) \forall a,b \in R^{+}$

Definition 2.5. (i) If $a\diamond b = a + b\ \forall a,b \in R^{+}$; then for
$\alpha \geq 2\ \Rightarrow\ a\diamond b = \alpha\,\mathrm{Max}(a;b)$
(ii) If . $a\diamond b = \frac{ab}{\mathrm{Max}(a;b;1)}\ \forall a,b \in R^{+}$; then for $\alpha \geq 1 \Rightarrow\ a\diamond b = \alpha\,\mathrm{Max}(a;b)$

Definition 2.6. Let $F\ \&\ G$ be two self mappings for a complete metric space$(X;\ d)$. Then mappings are said to be compatible if
$\lim_{n\to\infty} d(FGx_n; GFx_n; t) = 0\ \ \forall\, x\ \in X$
whenever a sequence $\{x_n\} \in X$ such that
$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = x \in X$

Definition 2.7. Let $F\ \&\ G$ be two self mappings for a complete metric space$(X;\ d)$. Then mappings are said to be non-compatible if there is at least one sequence $\{x_n\} \in X$ such that
$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Gx_n = x \in X$ but $\lim_{n\to\infty} d(FGx_n; GFx_n; t) \neq 0$ for at least one $x > 0$

Definition 2.8. Let $F\ \&\ G$ be two self mappings for a complete metric space$(X;\ d)$. Then mappings are said to be weak compatible if they commute at their point of coincident; that is$Fx = Gx\ \Rightarrow FGx = GFx$.

Remark 2.9. Every pair of compatible self mappings *F* and *G* of a complete metric space $(X;\ d)$ is weak compatible. But the converse is not true.

MAIN RESULTS

In present paper, we prove two fixed point theorems for four mappings applying α property.

Theorem 3.1: Let $(X;d)$ be a complete metric space such that $\diamond$ satisfies α- property with $\alpha > 0$ and mappings $A;B;S;\ T \in X$ satisfying the following conditions:

[i]] $A(x) \subset S(X)$ or $B(x) \subset T(X)$;

[ii] the pair $(A;S)$ or $(B;T)$ are weakly compatible;

[iii] $\forall\, x;y \in X; \Delta_1;\Delta_2;\Delta_3;\ \Delta_4;\Delta_5 > 0$ and $0 < \alpha(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 < 1$;

$$\begin{aligned} d(Ax;By) &\leq \Delta_1.d(Sx;Ty)\diamond d(Sx;Ty) + \Delta_2 d(By;Ty)\diamond\ d(Ax;By) \\ &+\Delta_3.d(Ax;Sx)\diamond d(Ax;Sx) + \Delta_4 d(Ax;Sx)\diamond d(Ax;By) + \\ &\Delta_5 d(By;Ty)\diamond\ d(By;Ty) \end{aligned} \tag{3.11}$$

Then;$A;B;S$ and T have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point of X. therefore there exists two sequences $\{x_n\};\{y_n\} \in X$ such that $y_{2n} = Ax_{2n} = Sx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Tx_{2n+2}$ for $n = 0;1;2;....$

Step 1: Now we assume sequence$\{y_n\}$ is a Cauchy sequence. If not; then using (3.11)

$$\begin{aligned} &d(y_{2n};y_{2n+1}) = d(Ax_{2n};Bx_{2n+1}) \\ &\leq \Delta_1.d(Sx_{2n};Tx_{2n+1})\diamond d(Sx_{2n};Tx_{2n+1}) + \Delta_2 d(Bx_{2n+1};Tx_{2n+1})\diamond\ d(Ax_{2n};Bx_{2n+1}) \\ &+\Delta_3.d(Ax_{2n};Sx_{2n})\diamond d(Ax_{2n};Sx_{2n}) + \Delta_4 d(Ax_{2n};Sx_{2n})\diamond d(Ax_{2n};Bx_{2n+1}) \\ &+\Delta_5 d(Bx_{2n+1};Tx_{2n+1})\diamond\ d(Bx_{2n+1};Tx_{2n+1}) \\ &\Rightarrow d(y_{2n};y_{2n+1}) \\ &\qquad\leq \Delta_1.d(y_{2n-1};y_{2n})\diamond d(y_{2n-1};y_{2n}) \\ &\qquad+\Delta_2 d(y_{2n+1};y_{2n})\diamond\ d(y_{2n};y_{2n+1}) \\ &+\Delta_3.d(y_{2n};y_{2n-1})\diamond d(y_{2n};y_{2n-1}) + \Delta_4(y_{2n};y_{2n-1})\diamond d(y_{2n};y_{2n+1}) \\ &\qquad+\ \Delta_5 d(y_{2n+1};y_{2n})\diamond\ d(y_{2n+1};y_{2n}) \end{aligned}$$

Letting $d_n = d(y_n; y_{n+1})$; we have

$$d_{2n} \leq \Delta_1(d_{2n-1} \diamond d_{2n-1}) + \Delta_2(d_{2n} \diamond d_{2n}) + \Delta_3(d_{2n-1} \diamond d_{2n-1}) + \Delta_4(d_{2n-1} \diamond d_{2n}) + \Delta_5(d_{2n} \diamond d_{2n})$$

$$d_{2n} \leq \Delta_1.\alpha.d_{2n-1} + \Delta_2\alpha.d_{2n} + \Delta_3.\alpha.d_{2n-1} + \Delta_4\alpha.\max\{d_{2n-1}; d_{2n}) + \Delta_5\alpha.d_{2n} \quad (3.12)$$

If $d_{2n} > d_{2n-1}$; then we have

$$d_{2n} \leq \alpha(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5)d_{2n} < d_{2n}$$

A contradiction. Hence $d_{2n+1} \leq d_{2n} \Rightarrow d_n \leq d_{n-1} \; \forall \; n = 0; 1; 2;$ ....Thus from inequality (3.12); we have

$$d_n \leq \alpha(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5)d_{n-1} = \Delta \, d_{n-1}$$

Assuming$\alpha \, (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5) = \Delta < 1$. By successive Iterations; we have

$$d_n \leq \Delta.d_{n-1} \leq \Delta^2 d_{n-2} \leq \Delta^3 d_{n-3} \dots \leq \Delta^n d_0$$

$$d_n \leq \Delta^n d_0 \Rightarrow d(y_n; y_{n+1}) \leq \Delta^n d(y_0; y_1)$$

Now for $m; n \in N$ where $m > n$; we have

$$d(y_n; y_m) \leq d(y_n; y_{n+1}) + d(y_{n+1}; y_{n+2}) + d(y_{n+2}; y_{n+3}) + \dots + d(y_{m-1}; y_m)$$

$$d(y_n; y_m) \leq \Delta^n d(y_0; y_1) + \Delta^{n+1} d(y_0; y_1) + \Delta^{n+2} d(y_0; y_1) + \dots . + \Delta^{m-1} d(y_0; y_1)$$

$$d(y_n; y_m) \leq \frac{\Delta^n}{1 - \Delta} d(y_0; y_1) \to 0 \text{ as } n \to \infty$$

⇒ Sequence $\{y_n\} \in X$ is a Cauchy sequence. For the completeness of X; $\{y_n\}$ converses to $y \in X$ such that

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Tx_{2n+2} = y$$

Step2: Suppose $T(x) \subset X$; Then there exists $u \in X$ such that $d(Tu; y) = 0$. Assert $d(Bu; y) = 0$; If not; then from (3.11);

$$d(Ax_{2n}; Bu) \leq \Delta_1.d(Sx_{2n}; Tu) \diamond d(Sx_{2n}; Tu) + \Delta_2 d(Bu; Tu) \diamond d(Ax_{2n}; Bu) + \Delta_3.d(Ax_{2n}; Sx_{2n}) \diamond d(Ax_{2n}; Sx_{2n}) + \Delta_4 d(Ax_{2n}; Sx_{2n}) \diamond d(Sx_{2n}; Bu) + \Delta_5 d(Bu; Tu) \diamond d(Bu; Tu)$$

When $n \to \infty$; we have,

$$d(y; Bu) \leq \Delta_1\alpha.\text{Max}\{d(y; y); d(y; y)\} + \Delta_2\alpha.\text{Max}\{d(y; Bu); d(y; Bu)\} + \Delta_3.\alpha.\text{Max}\{d(y; y); d(y; y)\} + \Delta_4\alpha.\text{Max}\{d(y; y); d(y; Bu)\} + \Delta_5\alpha.\text{Max}\{d(Bu; y); d(By; y)\}$$

$$d(y; Bu) < d(Bu; y)$$

$\Rightarrow d(y; Bu) = 0 \Rightarrow Bu = y = Tu$; Since B and T are weakly compatible$\Rightarrow BTu = TBu \Rightarrow By = Ty$. Assert $d(By; y) = 0$; If not; then using (3.11)

$$\lim_{n\to\infty} d(Ax_{2n}; Bu) \leq \lim_{n\to\infty} [\Delta_1 d(Sx_{2n}; Tu) \diamond d(Sx_{2n}; Tu) + \Delta_2 d(Bu; Tu) \diamond d(Ax_{2n}; Bu) \Delta_3 d(Ax_{2n}; Sx_{2n}) \diamond d(Ax_{2n}; Sx_{2n}) + \Delta_4 d(Ax_{2n}; Sx_{2n}) \diamond d(Sx_{2n}; Bu) + \Delta_5 d(Bu; Tu) \diamond d(Bu; Tu)]$$

$$d(y; By) \leq \Delta_1.d(y; Ty) \diamond d(y; Ty) + \Delta_2 d(By; Ty) \diamond d(y; By) + \Delta_3.d(y; y) \diamond d(y; y) + \Delta_4 d(y; y) \diamond d(y; By) + \Delta_5 d(By; Ty) \diamond d(By; Ty)$$

$$d(y; By) \leq \Delta_1\alpha.\text{Max}\{d(y; By); d(y; By)\} + \Delta_2\alpha.\text{Max}\{d(By; By); d(y; By)\} + \Delta_3.\alpha.\text{Max}\{d(y; y); d(y; y)\} + \Delta_4\alpha.\text{Max}\{d(y; y); d(y; By)\} + \Delta_5\alpha.\text{Max}\{d(By; y); d(By; y)\}$$

$$d(y; By) < d(y; By) \Rightarrow d(By; y) = 0 \Rightarrow By = y = Ty \quad (3.12)$$

Suppose $S(x) \subset X$; Then there exists $v \in X$ such that $(Sv; y) = 0$. Assert $d(Av; y) = 0$; If not; using (3.11)

$$d(Av; By) \leq \Delta_1.d(Sv; Ty) \diamond d(Sv; Ty) + \Delta_2 d(By; Ty) \diamond d(Av; By) + \Delta_3.d(Av; Sv) \diamond d(Av; Sv) + \Delta_4 d(Av; Sv) \diamond d(Sv; By) + \Delta_5 d(By; Ty) \diamond d(By; Ty)$$

When $n \to \infty$

$$d(Av; y) \leq \Delta_1\alpha.\text{Max}\{d(Sv; y); d(Sv; y)\} + \Delta_2\alpha.\text{Max}\{d(y; y); d(Av; y)\} + \Delta_3.\alpha.\text{Max}\{d(Av; Sv); d(Sv; y)\} + \Delta_4\alpha.\text{Max}\{d(y; y); d(y; y)\} + \Delta_5\alpha.\text{Max}\{d(y; y); d(y; y)\} \Rightarrow d(Av; y) < d(Av; y)$$

$\Rightarrow d(Av; y) = 0 \Rightarrow Av = y = Sv$; Since A and S are weakly compatible$\Rightarrow ASv = SAv \Rightarrow Ay = Sy$. Assert $d(Ay; y) = 0$; if not; from (3.11); we have

$$d(Ay;y) = d(Ay;By) \leq \Delta_1.d(Sy;Ty)\Diamond d(Sy;Ty) + \Delta_2 d(By;Ty)\Diamond d(Ay;By)$$
$$+\Delta_3.d(Ay;Sy)\Diamond d(Ay;Sy) + \Delta_4 d(Ay;Sy)\Diamond d(Sy;By) + \Delta_5 d(By;Ty)\Diamond d(By;Ty)$$

$$d(Ay;y) \leq \Delta_1\alpha.\text{Max}\{d(Ay;y);d(Ay;y)\} + \Delta_2\alpha.\text{Max}\{d(y;y);d(Ay;y)\} +$$
$$\Delta_3.\alpha.\text{Max}\{d(Ay;Ay);d(Ay;Ay)\} + \Delta_4\alpha.\text{Max}\{d(Ay;Ay);d(Ay;y)\} +$$
$$\Delta_5\alpha.\text{Max}\{d(y;y);d(y;y)\}$$
$$d(Ay;y) \leq \Delta_1.\alpha d(Ay;y) + \Delta_2.\alpha d(Ay;y) + \Delta_4.\alpha d(Ay;y)\}$$
$$d(Ay;y) < d(Ay;y) \Rightarrow d(Ay;y) = 0 \Rightarrow By = y = Ty \quad (3.13)$$

Now from (3.12) and (3.13); $Ay = By = Sy = Ty = y$. Hence y is a common fixed point for $A;B;S;\ T$.

Step3: for uniqueness; suppose z is another common point. From (2.11); we have

$$d(Ay;Bz) \leq \Delta_1.d(Sy;Tz)\Diamond d(Sy;Tz) + \Delta_2 d(Bz;Tz)\Diamond d(Ay;Bz)$$
$$+\Delta_3.d(Ay;Sy)\Diamond d(Ay;Sy) + \Delta_4 d(Ay;Sy)\Diamond d(Ay;Bz)$$
$$+\ \Delta_5 d(Bz;Tz)\Diamond d(Bz;Tz)$$
$$d(y;z) \leq \Delta_1.d(y;z)\Diamond d(y;z) + \Delta_2 d(z;z)\Diamond d(y;z)$$
$$+\Delta_3.d(y;y)\Diamond d(y;y) + \Delta_4 d(y;y)\Diamond d(y;z) + \Delta_5 d(z;z)\Diamond d(z;z)$$
$$d(y;z) \leq \Delta_1.d(y;z) + \Delta_2\, d(y;z) + \Delta_4 d(y;z) \Rightarrow d(y;z) < d(y;z) \Rightarrow y = z$$

Hence y is a unique common fixed point for $A;B;S\ \&\ T$.

Theorem 3.2: Let $(X;d)$ be a complete metric space such that $\Diamond$ satisfies α- property with $\alpha > 0$ and mappings $A;B;S;\ T \in X$ satisfying the following conditions:

(I) $A(x) \subset S(X)$ or $B(x) \subset T(X)$;

(II) the pair $(A;S)$ or $(B;T)$ are weakly compatible;

(III) $\forall\ x;y \in X$

$$d(Ax;By) \leq \Delta_1\{(d(Ax;By)\Diamond d(Ax;By))\Diamond d(Sx;Ty)\}$$
$$+\Delta_2\{(d(Ax;By)\Diamond d(Ax;Sx))\Diamond d(By;Ty)\} + \Delta_3\{(d(Sx;Ty)\Diamond d(Sx;Ty))\Diamond d(Ax;Ty)\}$$
$$+\Delta_4\{(d(Ax;Ty)\Diamond d(Ax;Ty))\Diamond d(Sx;Ty)\}$$

Where $A;B;S$ and T have a common fixed point in $X;\Delta_1;\Delta_2;\Delta_3;\ \Delta_4 > 0$ and $0 < \alpha^2(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) < 1;A;B;S$ and T have a unique common fixed point in X

Proof: Let x_0 be an arbitrary point of X. therefore there exists two sequences $\{x_n\};\{y_n\} \in X$ such that $y_{2n} = Ax_{2n} = Sx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Tx_{2n+2}$ for $n = 0;1;2;\ldots$.

Step 1: Now we assume sequence$\{y_n\}$ is a Cauchy sequence. If not; using (2.11)

$$d(y_{2n};y_{2n+1}) = d(Ax_{2n};Bx_{2n+1})$$
$$\leq \Delta_1\{(d(Ax_{2n};Bx_{2n+1})\Diamond d(Ax_{2n};Bx_{2n+1}))\Diamond d(Sx_{2n};Tx_{2n+1})\}$$
$$+\Delta_2\{(d(Ax_{2n};Bx_{2n+1})\Diamond d(Ax_{2n};Sx_{2n}))\Diamond d(Bx_{2n+1};Tx_{2n+1})\}$$
$$+\Delta_3\{(d(Sx_{2n};Tx_{2n+1})\Diamond d(Sx_{2n};Tx_{2n+1}))\Diamond d(Ax_{2n};Tx_{2n+1})\}$$
$$+\Delta_4\{(d(Ax_{2n};Tx_{2n+1})\Diamond d(Ax_{2n};Tx_{2n+1}))\Diamond d(Sx_{2n};Tx_{2n+1})\}$$
$$d(y_{2n};y_{2n+1}) \leq \Delta_1\{(d(y_{2n};y_{2n+1})\Diamond d(y_{2n};y_{2n+1}))\Diamond d(y_{2n-1};y_{2n})\}$$
$$+\Delta_2\left\{\left((d(y_{2n};y_{2n+1}))\Diamond d(y_{2n};y_{2n})\right)\Diamond d(y_{2n+1};y_{2n})\right\}$$
$$+\Delta_3\{(d(y_{2n-1};y_{2n})\Diamond d(y_{2n-1};y_{2n}))\Diamond d(y_{2n};y_{2n})\}$$
$$+\Delta_4\{(d(y_{2n};y_{2n})\Diamond d(y_{2n};y_{2n}))\Diamond d(y_{2n-1};y_{2n})\}$$

Letting$d_n = d(y_n;y_{n+1})$; we have

$$d_{2n} \leq \Delta_1.\{((d_{2n}\Diamond d_{2n}))\Diamond d_{2n-1}\} + \Delta_2\{(((d_{2n}\Diamond d_{2n})))\Diamond d_{2n}\}$$
$$+\Delta_3.\{((d_{2n-1}\Diamond d_{2n-1}))\Diamond d_{2n}\} + \Delta_4\{(d_{2n}\Diamond d_{2n})\Diamond d_{2n-1}\}$$
$$d_{2n} \leq \Delta_1.\{(\alpha\max(d_{2n};d_{2n}))\Diamond d_{2n-1}\} + \Delta_2\{(\alpha\max(d_{2n};d_{2n}))\Diamond d_{2n}\}$$
$$+\Delta_3.\{((\alpha\max(d_{2n-1};d_{2n-1}))\Diamond d_{2n}\}$$
$$+\Delta_4\{((\alpha\max(d_{2n};d_{2n}))\Diamond d_{2n-1}\}$$

$d_{2n} \leq \Delta_1 . \alpha\{d_{2n} \diamond d_{2n-1}\} + \Delta_2 \alpha\{d_{2n} \diamond d_{2n}\} + \Delta_3 . \alpha\{d_{2n-1} \diamond d_{2n}\} +$
$\Delta_4 \alpha\{(d_{2n} \diamond d_{2n-1}\}$

$$d_{2n} \leq \Delta_1 . \alpha^2(\max(d_{2n}; d_{2n-1})) + \Delta_2 \alpha^2(\max(d_{2n}; d_{2n})) + \Delta_3 . \alpha^2(\max(d_{2n}; d_{2n-1})) + \Delta_4 \alpha^2(\max(d_{2n}; d_{2n-1})) \quad (2.22)$$

If $d_{2n} > d_{2n-1}$; then

$d_{2n} \leq \alpha^2(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) d_{2n} < d_{2n}$

A contradiction. Hence $d_{2n} \leq d_{2n-1}$; similarly $d_{2n+1} \leq d_{2n} \Rightarrow d_n \leq d_{n-1} \ \forall \ n = 0; 1; 2; ....$
Thus from inequality (2.22); we have

$d_n \leq \alpha^2(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) d_{n-1} = \Delta . d_{n-1}$

Assuming $\alpha^2(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 +) = \Delta < 1$. By successive iteration;

$d_n \leq \Delta . d_{n-1} \leq \Delta^2 d_{n-2} \leq \Delta^3 d_{n-3} \ ..... \leq \Delta^n d_0$

$d_n \leq \Delta^n d_0 \Rightarrow d(y_n; y_{n+1}) \leq \Delta^n d(y_0; y_1) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ since} \Delta < 1$

Now for $m; n \ \in N$ where $m > n$; we have

$d(y_n; y_m) \leq d(y_n; y_{n+1}) + d(y_{n+1}; y_{n+2}) + d(y_{n+2}; y_{n+3}) + \cdots + d(y_{m-1}; y_m)$

$d(y_n; y_m) \leq \Delta^n d(y_0; y_1) + \Delta^{n+1} d(y_0; y_1) + \Delta^{n+2} d(y_0; y_1) + \cdots . + \Delta^{m-1} d(y_0; y_1)$

$$d(y_n; y_m) \leq \frac{\Delta^n}{1 - \Delta} d(y_0; y_1) \rightarrow 0 \text{ as } n \rightarrow \infty$$

⇒ Sequence $\{y_n\} \in X$ is a Cauchy sequence. For the completeness of X; $\{y_n\}$ converses to $y \in X$ such that

$$\lim_{n\to\infty} y = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Tx_{2n+2} = y$$

Step2: Suppose $(x) \subset X$; Then there exists $u \in X$ such that $Tu = y$. Assert $d(y; Bu) = 0$; If not then from (2.21);

$$d(Ax_{2n}; Bu) \leq \Delta_1\{(d(Ax_{2n}; Bu) \diamond d(Ax_{2n}; Bu)) \diamond d(Sx_{2n}; Tu)\}$$
$$+\Delta_2\{(d(Ax_{2n}; Bu) \diamond d(Ax_{2n}; Sx_{2n})) \diamond d(Bu; Tu)\}$$
$$+\Delta_3\{(d(Ax_{2n}; Tu) \diamond d(Ax_{2n}; Tu)) \diamond d(Ax_{2n}; Tu)\}$$
$$+\Delta_4\{(d(Ax_{2n}; Tu) \diamond d(Ax_{2n}; Tu)) \diamond d(Sx_{2n}; Tu)\}$$

$$\lim_{n\to\infty} d(Ax_{2n}; Bu) \leq \lim_{n\to\infty} \Delta_1\{(d(Ax_{2n}; Bu) \diamond d(Ax_{2n}; Bu)) \diamond d(Sx_{2n}; Tu)\}$$
$$+ \lim_{n\to\infty} \Delta_2\{(d(Ax_{2n}; Bu) \diamond d(Ax_{2n}; Sx_{2n})) \diamond d(Bu; Tu)\}$$
$$+ \lim_{n\to\infty} \Delta_3\{(d(Ax_{2n}; Tu) \diamond d(Ax_{2n}; Tu)) \diamond d(Ax_{2n}; Tu)\}$$
$$+ \lim_{n\to\infty} \Delta_4\{(d(Ax_{2n}; Tu) \diamond d(Ax_{2n}; Tu)) \diamond d(Sx_{2n}; Tu)\}$$

$$d(y; Bu) \leq \Delta_1 \alpha^2 . \text{Max}\{d(y; Bu); d(y; Bu)\} + \Delta_2 \alpha^2 . \text{Max}\{d(y; Bu); d(y; Bu)\}$$
$$+\Delta_3 . \alpha^2 . \text{Max}\{d(y; Bu); d(y; Bu)\} + \Delta_4 \alpha^2 . \text{Max}\{d(y; Bu); d(y; Bu)\}$$

$d(y; Bu) \leq \alpha^2(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) . d(y; Bu)$

$d(y; Bu) < d(Bu; y)$

$\Rightarrow Bu = y = Tu$; since B and T are weakly compatible $\Rightarrow BTu = TBu \Rightarrow By = Ty$. Assert $d(y; By) = 0$; If not; then using (2.21)

$$d(Ax_{2n}; Bu) \leq \Delta_1\{(d(Ax_{2n}; Bu) \diamond d(Ax_{2n}; Bu)) \diamond d(Sx_{2n}; Tu)\}$$
$$+\Delta_2\{(d(Ax_{2n}; Tu) \diamond d(Ax_{2n}; Sx_{2n})) \diamond d(Bu; Tu)\}$$
$$+\Delta_3\{(d(Sx_{2n}; Tu) \diamond d(Sx_{2n}; Tu)) \diamond d(Ax_{2n}; Tu)\}$$
$$+\Delta_4\{(d(Ax_{2n}; Tu) \diamond d(Ax_{2n}; Tu)) \diamond d(Sx_{2n}; Tu)\}$$

$$\lim_{n\to\infty} d(Ax_{2n}; Bu) \leq \lim_{n\to\infty} \Delta_1\{(d(Ax_{2n}; Bu) \diamond d(Ax_{2n}; Bu)) \diamond d(Sx_{2n}; Tu)\}$$
$$+ \lim_{n\to\infty} \Delta_2\{(d(Ax_{2n}; Tu) \diamond d(Ax_{2n}; Sx_{2n})) \diamond d(Bu; Tu)\}$$
$$+ \lim_{n\to\infty} \Delta_3\{(d(Sx_{2n}; Tu) \diamond d(Sx_{2n}; Tu)) \diamond d(Ax_{2n}; Tu)\}$$
$$+ \lim_{n\to\infty} \Delta_4\{(d(Ax_{2n}; Tu) \diamond d(Ax_{2n}; Tu)) \diamond d(Sx_{2n}; Tu)\}$$

$$d(y; By) \leq \Delta_1\{(d(y; By) \diamond d(y; By)) \diamond d(y; By)\}$$

$+\Delta_2\{(d(y;By)\diamond d(y;y))\diamond d(By;By)\}$
$+\Delta_3(d(y;By)\diamond d(y;By))\diamond d(y;By) + \Delta_4\{(d(y;By)\diamond d(y;By))\diamond d(y;By)\}$
Since $a\diamond b = \alpha.\mathrm{Max}\ (a;b)$
$d(y;By) \le \Delta_1\{(\alpha.d(y;By)\diamond d(y;By))\} + \Delta_2\{(\alpha.d(y;By)\diamond d(By;By))\}$
$+\Delta_3\{(\alpha.d(y;By)\diamond d(y;By))\} + \Delta_2\{(\alpha.d(y;By)\diamond (By;By))\}$
$d(y;By) \le \Delta_1\{(\alpha.d(y;By)\diamond d(y;By))\} + \Delta_2\{(\alpha.d(y;By)\diamond d(By;By))\}$
$+\Delta_3\{(\alpha.d(y;By)\diamond d(y;By))\} + \Delta_2\{(\alpha.d(y;By)\diamond (By;By))\}$
$d(y;By) \le \alpha^2(\Delta_1+\Delta_2+\Delta_3+\Delta_4)\ d(y;By)$
$d(y;By) \le d(y;By)$ since $0 < \alpha^2.(\Delta_1+\Delta_2+\Delta_3+\Delta_4) < 1$
$\Rightarrow By = y = Ty$; since B and T are weakly compatible $\Rightarrow BTy = TBy \Rightarrow By = Ty = y$.
Suppose $S(x) \subset X$; Then there exists $v \in X$; such that $d(Sv;y)$. Assert $d(Av;y) = 0$; if not; using (2.21)
$d(Av;By) \le \Delta_1\{(d(Av;By)\diamond d(Av;By))\diamond d(Sv;Ty)\}$
$+ \Delta_2\{(d(Av;Ty)\diamond d(Av;Sv))\diamond d(By;Ty)\}$
$+\Delta_3\{(d(Sv;Ty)\diamond d(Sv;Ty))\diamond d(Av;Ty)\} +$
$\Delta_4\{(d(Av;Ty)\diamond d(Av;Ty))\diamond d(Sv;Ty)\}$
$d(Av;By) \le \Delta_1\{\alpha(d(Av;By)\diamond d(Sv;Ty))\}\Delta_1\{\alpha(d(Av;Ty)\diamond d(By;Ty))\}$
$+\Delta_3\{\alpha(d(Av;Ty)\diamond d(Av;Ty))\} + \Delta_4\{(\alpha(d(Av;Ty))\diamond d(Sv;Ty))\}$
Since $a\diamond b = \alpha\mathrm{Max}\ (a;b)$
When $n \to \infty$;
$d(Ay;y) \le \Delta_1\{\alpha^2 d(Ay;y)\} + \Delta_2\{\alpha^2 d(Ay;y)\} + \Delta_3\{\alpha^2 d(Ay;y)\} + \Delta_4\{\alpha^2 d(Ay;y)\}$
$d(Ay;y) \le \alpha^2(\Delta_1+\Delta_2+\Delta_3++\Delta_4)d(Ay;y);$ since $0 < \alpha^2(\Delta_1+\Delta_2+\Delta_3++\Delta_4) < 1$
$\Rightarrow Av = y = Sv$; Since A and S are weakly compatible $\Rightarrow ASv = SAv \Rightarrow Ay = Sy$. Assert $d(Ay;y) = 0$; if not; then from (2.21);
$d(Ay;By) \le \Delta_1\{(d(Ay;By)\diamond d(Ay;By))\diamond d(Sy;Ty)\}$
$+ \Delta_2\{(d(Ay;Ty)\diamond d(Ay;Sv))\diamond d(By;Ty)\}$
$+\Delta_3\{(d(Sy;Ty)\diamond d(Sy;Ty))\diamond d(Ay;Ty)\} +$
$\Delta_4\{(d(Ay;Ty)\diamond d(Ay;Ty))\diamond d(Sy;Ty)\}$
$d(Av;y) \le \Delta_1\{\alpha(d(Ay;y)\diamond d(Ay;y))\}\Delta_1\{\alpha(d(Ay;y)\diamond d(y;y))\}$
$+\Delta_3\{\alpha(d(Ay;y)\diamond d(Ay;y))\} + \Delta_4\{(\alpha(d(Ay;y))\diamond d(Ay;y))\}$
Since $a\diamond b = \alpha\mathrm{Max}\ (a;b)$
When $n \to \infty$;
$d(Ay;y) \le \Delta_1\{\alpha^2 d(Ay;y)\} + \Delta_2\{\alpha^2 d(Ay;y)\} + \Delta_3\{\alpha^2 d(Ay;y)\} + \Delta_4\{\alpha^2 d(Ay;y)\}$
$d(Ay;y) \le \alpha^2(\Delta_1+\Delta_2+\Delta_3++\Delta_4)\ d(Ay;y);$ since $0 < \alpha^2(\Delta_1+\Delta_2+\Delta_3++\Delta_4) < 1$

$$d(Ay;y) < d(Ay;y) \Rightarrow By = y = Ty \quad (2.14)$$

Now from (2.13) and (2.14); $Ay = By = Sy = Ty = y$. Hence y is a common fixed point for $A;B;S;\ T$.

Step3: for uniqueness; suppose z is another common point. From (2.11); we have
$d(Ay;Bz) \le \Delta_1\{(d(Ay;Bz)\diamond d(Ay;Bz))\diamond d(Sy;Tz)\}$
$+\Delta_2\{(d(Ay;Bz)\diamond d(Ay;Sy))\diamond d(Bz;Tz)\} + \Delta_3\{(d(Sy;Tz)\diamond d(Sy;Tz))\diamond d(Ay;Tz)\}$
$+\Delta_4\{(d(Ay;Tz)\diamond d(Ay;Tz))\diamond d(Sy;Tz)\}$
$d(y;z) \le \alpha^2(\Delta_1+\Delta_2+\Delta_3+\Delta_4)d(y;z)$
$\Rightarrow d(y;z) < d(y;z) \Rightarrow y = z$
Hence y is a unique common fixed point for $A;B;S\ \&\ T$.

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