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Modified CUIA Iteration for Asymptotically Pseudo Contractive Non -self Mapping

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ABSTRACT: Many researchers have been fascinated by various iterative processes to calculate the rate of convergence for self mappings, but only a few researchers have introduced iterative processes for non self mappings to check the rate of convergence. This paper is based on non-self mapping and is an extension of the result of Yuanheng Shi and Huimin Shi (Abstract and Applied Science, J, 2014) from two mappings to four mappings with help of CUIA iteration. Here, we also improve existing results on non-self-mappings.

Keywords: L-Lipschitz contraction, Asymptotically Pseudo Contractive Mapping, Modified CUIA, Retract map.

I. INTRODUCTION

Iterative method is a method in which we repeat the iteration again and again to find the solution of the equation $G(t) = t$. In the iterative fixed point procedure, the output varies like some of results are completely significant while the others are not. Many researches worked on different types of iterations with different types of mappings to find the rate of convergence. Mann [2] introduced new iterative method to find the solution of a fixed point equation for non-expansive mapping where as Picard's iterative method [1] failed to find the solution of fixed point equation for non –expansive mapping. Later, Ishikawa [4] introduced new iterative method for obtaining the convergence of a Lipschitzian pseudo-contractive operator while Mann's iterative method fails to apply on this mapping. Many authors [11-17] worked on pseudo contractive operators to find the convergence rate while some authors failed to find the convergence rate of pseudo contractive operators. Chaudhary [6], [10] worked on Hilbert space with help of Mann iteration to find the convergence rate of fixed point. There are many iterative schemes which are used to estimate the fixed point. The most important iterations are Mann iteration, Ishikawa iteration, Noor iteration, CUIA iteration. These iterative schemes also applicable in Physics, coding theory, statistical physics etc. In 1890, Picard [1] worked on iterative process for finding the fixed point and proved some convergence results by defining the iteration as follow:

$$i_{n+1} = G(i_n)$$

In 1953, Mann [2] gave the iteration as follow:

$$i_{n+1} = (1-\epsilon_n)i_n + \epsilon_n G(i_n)$$

where $\{\epsilon_n\}$ be a positive real numbers sequence in [0, 1]. In 1955, Krasnoselski [3] improved the Mann iteration by introducing a constant 'λ' instead of the sequence $\epsilon_n$and defined modified Krasnoselski iteration as:

$$i_{n+1} = (1-\lambda)i_n + \lambda\, G(i_n).$$

Where λ lies in closed interval [0, 1]. In 1978, Ishikawa [4] defined new iteration for finding the fixed point and define by:

$$i_{n+1} = (1-\epsilon_n)i_n + \epsilon_n G(i_n)$$

$$j_n = (1-\zeta_n)i_n + \zeta_n G(j_n)$$

where $\{\epsilon_n\}$ and $\{\zeta_n\}$ are the positive real numbers sequence in [0, 1). After this, in 2000, Noor [7] gave their iterative process with help of Mann, Agarwal, and Thianwan by defining:

$$i_{n+1} = (1-\epsilon_n)i_n + \epsilon_n G(i_n)$$

$$j_n = (1-\zeta_n)i_n + \zeta_n G(k_n)$$

$$k_n = (1-\eta_n)i_n + \eta_n G(i_n)$$

where $\{\epsilon_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$ are positive real numbers sequence in [0,1)
In 2012, CR [18] introduced new iterative process for finding the fixed point defining by

$$i_{n+1} = (1-\epsilon_n)j_n + \epsilon_n G(k_n)$$

$$j_n = (1-\zeta_n)G(j_n) + \zeta_n G(k_n)$$

$$k_n = (1-\eta_n)i_n + \eta_n G(i_n)$$

where $\{\epsilon_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$ are positive real numbers sequence in [0, 1). In 2017, Chauhan *et al.*, [22] did some changes and established a new iteration by

$$i_{n+1} = (1-\epsilon_n)j_n + \epsilon_n G(k_n)$$

$$j_n = (1-\zeta_n)G(l_n) + \zeta_n G(k_n)$$

$$k_n = (1-\eta_n)G(i_n) + \eta_n G(l_n)$$

$$l_n = (1-\theta_n)i_n + \theta_n G(i_n)$$

where $\{\epsilon_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$ and $\{\theta_n\}$ are positive real numbers sequence in [0, 1). Atsushib [5] worked on asymptotically non-expansive mapping with help of Mann iteration. Zegeye [8] and Berinde [9] worked on pseudo contractive operators to find the convergence rate with help of Ishikawa iteration and other iterative method respectively. Daman and Zegeye [19] and Tufa, Zegeye [21] worked on Pseudo-contractive non-self mapping to find the strong convergence rate. Yuanheng W and HuiminS[20] worked on asymptotically pseudo contractive mapping and modified the Ishikawa iteration and find the rate of convergence of fixed point.

Modified Ishikawa iteration:

$$\begin{aligned} i_{n+1} = Q[(1-\epsilon_n-\eta_n)i_n &+ \epsilon_n G_1(QG_1)^{n-1}[(1-\zeta_n)j_n \\ &+ \zeta_n G_1(QG_1)^{n-1}j_n] + \eta_n\mu_n] \\ j_n = Q[(1-\epsilon'_n-\eta'_n)i_n &+ \epsilon'_n G_2(QG_2)^{n-1}[(1-\zeta'_n)i_n \\ &+ \zeta'_n G_2(QG_2)^{n-1}i_n] + \eta'_n\nu_n] \end{aligned}$$

where,
$\{\epsilon_n\},\{\zeta_n\}.\{\eta_n\},\{\epsilon'_n\},\{\zeta'_n\},\{\eta'_n\}$ lie in [0,1] and the $\{\mu_n\},\{\nu_n\}$ are the bounded sequence of B. In this paper, we have modified CUIA iteration with help of retract map and used it for asymptotically pseudo contractive mapping which is as follow:

$$\begin{aligned} i_{n+1} = Q[(1-\epsilon_n-\eta_n)j_n &+ \epsilon_n G_1(QG_1)^{n-1}[(1-\zeta_n)j_n \\ &+ \zeta_n G_1(QT_1)^{n-1}j_n] + \eta_n\chi_n] \end{aligned}$$

$$\begin{aligned} j_n = Q[(1-\epsilon'_n-\eta'_n)G_2(QG_2)^{n-1}l_n &+ \epsilon'_n G_2(QG_2)^{n-1}[(1-\zeta'_n)k_n \\ &+ \zeta'_n G_2(QG_2)^{n-1}k_n] + \eta'_n\nu_n] \end{aligned}$$

$$\begin{aligned} k_n = Q[(1-\epsilon''_n-\eta''_n)G_3(QG_3)^{n-1}i_n &+ \alpha''_n G_3(QG_3)^{n-1}[(1-\zeta''_n)l_n \\ &+ \zeta''_n G_3(QG_3)^{n-1}l_n] + \eta''_n\tau_n] \end{aligned}$$

$$l_n = Q[(1-\epsilon'''_n-\gamma'''_n)i_n + \epsilon'''_n G_4(QG_4)^{n-1}[(1-\zeta'''_n)i_n + \zeta'''_n G_4(QG_4)^{n-1}i_n] + \eta'''_n o_n] \quad (1)$$

where
$\{\epsilon_n\},\{\zeta_n\},\{\eta_n\},\{\epsilon'_n\},\{\zeta'_n\},\{\eta'_n\},\{\epsilon''_n\},\{\zeta''_n\},\{\eta''_n\},\{\epsilon'''_n\},\{\zeta'''_n\},\{\eta'''_n\}$ are the sequence of positive real numbers lie in $[0,1]$ and the sequences $\{\chi_n\},\{\nu_n\},\{\tau_n\},\{o_n\} \subset B$

If $G_1 = G_2 = G_3 = G_4 = G$ and $Q = I$ identical mapping and $\eta_n = \eta'_n = \eta''_n = \eta'''_n = 0$ and also $\zeta_n = \zeta'_n = \zeta''_n = \zeta'''_n = 0$ then (1) will be modified general CUIA iteration with iterative sequence:

$$i_{n+1} = (1-\epsilon_n)j_n + \epsilon_n G^n(j_n)$$
$$j_n = (1-\zeta_n)G(l_n) + \zeta_n G^n(k_n)$$

$$k_n = (1-\eta_n)G(i_n) + \eta_n G^n(l_n)$$

$$l_n = (1-\theta_n)i_n + \theta_n G^n(i_n)$$

Definition 1.1 [20] Let B be non – empty set closed convex subset of K and let $Q: K \to B$ be non-expansive mapping then $\forall y \in B\ Q(t) = t$ then the mapping Q is said to be non-expansive retract map.

Definition 1.2 [20] Let K is a real Banach space with norm $\|.\|$ and B is a non-empty subset of K. let G:B→K be a non-self mapping then it will be uniformly - Lipschtiz with retract map (Q) if $\exists H > 0$ such that

$$\|G(QG)^{n-1}(i) - G(QG)^{n-1}(j)\| \le H\|i-j\| \forall i,j \in B$$

Definition 1.3 [20] Let us take an real Banach space K and let B be non-empty closed convex subset of K and let $U: K \to 2^{K^*}$ be a mapping and this mapping is said to be normalized duality mapping if it is defined by: $U(i) = \{i^* \in K^*: \langle i|i^*\rangle = \|i\|.\|i^*\|, \|i\| = \|i\|\}, i \in K$ (2)

Definition 1.4 [20] Let $G: B \to K$ be a non-self mapping and let ∃ a sequence $q_n \subset [1,\infty)$ which satisfied the condition $q_n \to 1$ as n→∞ and $u(i-j) \in U(i-j)$ such that

$$\langle G^n(i) - G^n(j)|u(i-j)\rangle \le q_n\|i-j\|^2 \quad \forall i,j \in B$$

Lemma 1.1 [20]: Let K be a real Banach space, then it satisfied following inequality:

$$\|i+j\|^2 \le \|i\|^2 + 2\langle i|u(i+j)\rangle \forall i,j \in K$$

and $u(i+j) \in U(i+j)$

Lemma 1.2 [20]: Let ∃ three sequences $\{m_n\},\{n_n\},\{o_n\}$ of non-negative numbers and let it satisfied the following condition:

$$m_{n+1} \le (1+n_n)m_n + o_n \qquad \forall p \ge p_0$$

where p_0 is an non-negative integer. If $\sum_{n=1}^{\infty} n_n < \infty$ and $\sum_{n=1}^{\infty} o_n < \infty$ then $\lim_{n\to\infty} m_n$ will exist.

Lemma 1.3 [20]: Let ∃ a function φ:[0,+∞)→[0,∞) be a strictly increasing function with satisfy $\varphi(0) = 0$ let there are four sequence $\{m_n\},\{n_n\},\{o_n\}$, and $\{\lambda_n\}$ lie in$(0 \le \lambda_n \le 1)$ satisfy the inequality:-

$$m_{n+1} \le (1+n_n)m_n - \lambda_n\varphi(m_{n+1}) + o_n \qquad \forall n \ge n_0$$

and If $\sum_{n=1}^{\infty} n_n < \infty$, $\sum_{n=1}^{\infty} 0_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$ hold, then $\lim_{n\to\infty} m_n = 0$

Lemma 1.4 [20]

Let ∃ a strictly increasing function φ:[0,+∞)→[0,∞) with satisfy φ(0)=0 let there are five sequence $\{m_n\},\{n_n\},\{o_n\},\{\varepsilon_n\}$ and $\{\lambda_n\}$ lie in$(0 \le \lambda_n \le 1)$ satisfy the inequality:

$$m_{n+1} \le (1+n_n)m_n - \lambda_n\varphi(m_{n+1}) + o_n + \lambda_n\varepsilon_n \qquad \forall n \ge n_0$$

and if $\sum_{n=1}^{\infty} n_n < \infty$, $\sum_{n=1}^{\infty} 0_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$ hold, then $\lim_{n\to\infty} m_n = 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$

The motive of this paper is to use the modified mixed CUIA iteration for asymptotically pseudo contractive mapping to prove the strong convergence result for common fixed point of four mappings.

II. MAIN RESULTS

In this section, one fixed point theorem related to four mappings has been proved.

2.1 Theorem: Let B be a non-expansive retract mapping (with Q) of a real Banach space K. let us assume that ∃ four uniformly H-Lipschitzian non-self mappings (with Q) $G_1, G_2, G_3, G_4: B \to K$ and let G_1 is an asymptotically pseudo-contractive mapping with co-efficient numbers $\{q_n\} \subset [1,\infty): q_n \to 1$ satisfying $S = S(G_1) \cap S(G_2) \cap S(G_3) \cap S(G_4) \neq \emptyset$ let us assume that there are four bounded sequences $\{\chi_n\}, \{\nu_n\}, \{\tau_n\}, \{o_n\} \subset C$ and let there are twelve sequences $\{\epsilon_n\}, \{\zeta_n\}, \{\eta_n\}, \{\epsilon'_n\}, \{\zeta'_n\}, \{\eta'_n\}, \{\epsilon''_n\}, \{\zeta''_n\}, \{\eta''_n\}, \{\epsilon'''_n\}, \{\zeta'''_n\}, \{\eta'''_n\} \subset [0,1]$ satisfying the following condition:

1) $\sum_{n=1}^{\infty} \epsilon_n = +\infty, \sum_{n=1}^{\infty} \epsilon_n^2 < +\infty, \sum_{n=1}^{\infty} \epsilon_n (q_n - 1) < +\infty$
2) $\epsilon_n + \eta_n \leq 1, \epsilon'_n + \eta'_n \leq 1, \epsilon''_n + \eta''_n \leq 1, \epsilon'''_n + \eta'''_n \leq 1, \sum_{n=1}^{\infty} \eta_n < \infty$
3) $\sum_{n=1}^{\infty} \epsilon_n \zeta_n < +\infty, \sum_{n=1}^{\infty} \epsilon_n \epsilon'_n < +\infty, \sum_{n=1}^{\infty} \epsilon_n \eta'_n < +\infty$

Let $i_1 \in B$ be an arbitrary number then the iterative sequence $\{i_n\}$ defined by (1) will be strongly converge at a fixed point $i^* \in S$ iff ∃ a strictly increasing function $\varphi: [0,+\infty) \to [0,+\infty)$ with $\varphi(0) = 0$ such that

$$lim_{n\to\infty} \sup inf_{u(i_{n+1}-i^*) \in U(i_{n+1}-i^*)} [\langle G_1(QG_1)^{n-1} i_{n+1} - i^* | u(i_{n+1} - i^*)\rangle - q_n \|i_{n+1} - i^*\|^2 + \varphi(\|i_{n+1} - i^*\|)] \leq 0 \quad (3)$$

Proof: Let i^* is fixed point of $S$and let it satisfied the condition:

Let

$\varepsilon'_n = inf_{u(i_{n+1}-i^*) \in U(i_{n+1}-i^*)} [\langle G_1(QG_1)^{n-1} i_{n+1} - i^* | u(i_{n+1} - i^*)\rangle - q_n \|i_{n+1} - i^*\|^2 + \varphi(\|i_{n+1} - i^*\|$

and let $\varepsilon_n = max\{\varepsilon'_n, 0\} + \frac{1}{n}$ (4)

and

let $\exists u(i_{n+1} - i^*) \in U(i_{n+1} - i^*)$ such that it satisfied $\langle G_1(QG_1)^{n-1} i_{n+1} - i^* | u(i_{n+1} - i^*)\rangle - q_n \|i_{n+1} - i^*\|^2 + \varphi(\|i_{n+1} - i^*\| \leq \varepsilon_n$ (5)

then Eqn. (3) will be

$$\lim_{n\to\infty} sup\, \varepsilon'_n \leq 0$$

From Eqn. (4)

$$\lim_{n\to\infty} \varepsilon_n = 0$$

From Eqn. (1) let

$$\sigma_n = (1-\zeta_n) j_n + \zeta_n G_1(QT_1)^{n-1} j_n \quad (6)$$

$$\delta_n = (1-\zeta'_n) k_n + \zeta'_n G_2(QG_2)^{n-1} k_n \quad (7)$$

$$\rho_n = (1-\zeta''_n) l_n + \zeta''_n G_3(QG_3)^{n-1} l_n \quad (8)$$

$$\mu_n = (1-\zeta'''_n) i_n + \zeta'''_n G_4(QG_4)^{n-1} i_n \quad (9)$$

and

let $N = max_{p\geq 1}(\|\chi_n - i^*\|, \|\nu_n - i^*\|, \|\tau_n - i^*\|, \|o_n - i^*\|$ (10)

Firstly, we find the value of $\|\mu_n - i^*\|$

So ,

$$\|\mu_n - i^*\| = \|(1-\zeta'''_n) i_n + \zeta'''_n G_4(QG_4)^{n-1} i_n - i^*\|$$
$$= \|(1-\zeta'''_n) i_n + \zeta'''_n G_4(QG_4)^{n-1} i_n - i^* - \zeta'''_n i^* + \zeta'''_n i^*\|$$
$$\leq \zeta'''_n \|G_4(QG_4)^{n-1} i_n - i^*\| + (1-\zeta'''_n)\|i_n - i^*\|$$

by definition(1.2)

$$\leq \zeta'''_n H\|i_n - i^*\| + \|i_n - i^*\|$$
$$\|\mu_n - i^*\| \leq (1 + \zeta'''_n H)\|i_n - i^*\| \quad (11)$$

Now, we will find the value of $\|l_n - i\|$

So,

$$\|l_n - i^*\| = \|Q[(1-\epsilon'''_n - \eta'''_n) i_n + \epsilon'''_n G_4(QG_4)^{n-1}[(1-\zeta'''_n) i_n + \zeta'''_n G_4(QG_4)^{n-1} i_n] + \eta'''_n o_n] - i^*\|$$

From Eqn. (9) and definition(1.1)

$$= \|(1-\epsilon'''_n - \eta'''_n) i_n + \epsilon'''_n G_4(QG_4)^{n-1}\mu_n + \eta'''_n o_n - i^*\|$$
$$= \|(1-\epsilon'''_n - \eta'''_n) i_n + \epsilon'''_n G_4(QG_4)^{n-1}\mu_n + \eta'''_n o_n - i^* + \alpha'''_n i^* - \alpha'''_n i^* + \eta'''_n i^* - \eta'''_n i^*\|$$

$\leq \epsilon'''_n \|G_4(QG_4)^{n-1}\mu_n - i^*\| + (1-\epsilon'''_n - \eta'''_n)\|i_n - i^*\| + \eta'''_n \|o_n - i^*\|$ by (10) and definition (1.2)

$$\leq \epsilon'''_n H\|\mu_n - i^*\| + \|i_n - i^*\| + \eta'''_n N \quad (12)$$

Now put value of Eqn. (11) in (12),we get

$$\leq \epsilon'''_n H(1 + \zeta'''_n H)\|i_n - i^*\| + \|i_n - i^*\| + \eta'''_n N$$
$$\leq (1 + \epsilon'''_n H + \epsilon'''_n \zeta'''_n H^2)\|i_n - i^*\| + \eta'''_n N$$
$$\|l_n - i^*\| \leq (1 + H + H^2)\|i_n - i^*\| + N \quad (13)$$

Now, $\|\rho_n - i^*\|$

$$\|\rho_n - i^*\| = \|(1-\zeta''_n) l_n + \zeta''_n G_3(QG_3)^{n-1} l_n - i^*\|$$
$$= \|(1-\zeta''_n) l_n + \zeta''_n G_3(QG_3)^{n-1} l_n - i^* + \zeta''_n i^* - \zeta''_n i^*\|$$
$$\leq \zeta''_n \|G_3(QG_3)^{n-1} l_n - i^*\| + (1-\zeta''_n)\|i_n - i^*\|$$
$$\leq \zeta''_n H\|l_n - i^*\| + \|l_n - i^*\|$$
$$\leq H\|l_n - i^*\| + \|l_n - i^*\|$$
$$\|\rho_n - i^*\| \leq (1 + H)\|l_n - i^*\|$$

Now ,from Eqn. (13) we get,

$$\leq (1 + H)[(1 + H + H^2)\|i_n - i^*\| + N]$$
$$\|\rho_n - i^*\| \leq (1 + H)(1 + H + H^2)\|i_n - i^*\| + (1 + H)N \quad (14)$$

Now ,

$$\|k_n - i^*\| = \|Q[(1-\epsilon''_n - \eta''_n) G_3(QG_3)^{n-1} i_n + \epsilon''_n G_3(QG_3)^{n-1}[(1-\zeta''_n) l_n + \zeta''_n G_3(QG_3)^{n-1} l_n] + \eta''_n \tau_n] - i^*\|$$

From Eqn. (8) and definition 1.1

$$= \|(1-\epsilon''_n - \eta''_n) G_3(QG_3)^{n-1} i_n + \epsilon''_n G_3(QG_3)^{n-1}\rho_n + \eta''_n \tau_n - i^*\|$$
$$= \|(1-\epsilon''_n - \eta''_n) G_3(QG_3)^{n-1} i_n + \epsilon''_n G_3(QG_3)^{n-1}\rho_n + \eta''_n \tau_n - i^* + \epsilon''_n i^* - \epsilon''_n i^* + \eta''_n i^* - \eta''_n i^*\|$$

$$\leq (1-\epsilon''_n - \eta''_n)\|G_3(QG_3)^{n-1} i_n - i^*\| + \epsilon''_n \|G_3(QG_3)^{n-1}\rho_n - i^*\| + \eta''_n \|\tau_n - i^*\|\|k_n - i^*\|$$

$$\leq H\|i_n - i^*\| + \epsilon''_n H\|\rho_n - i^*\| + \eta''_n \|\tau_n - i^*\|$$

By Eqn. (10) and definition (1.2)

$$\leq H\|i_n - i^*\| + H\|\rho_n - i^*\| + \eta''_n N$$

$$\|k_n - i^*\| \leq H\|i_n - i^*\| + H\|\rho_n - i^*\| + N$$

By Eqn. (14) we get,

$$\leq H\|i_n - i^*\| + H[(1 + H)(1 + H + H^2)\|i_n - i^*\| + (1 + H)N] + N$$

$\|k_n - i^*\| \leq [H + H(1+H)(1+H+H^2)]\|i_n - i^*\| + [1 + H(1+H)]N$ (15)

Now, $\|\delta_n - i^*\| = \|(1-\zeta_n')k_n + \zeta_n' G_2(QG_2)^{n-1}k_n - i^*\|$

$$= \|(1-\zeta_n')k_n + \zeta_n' G_2(QG_2)^{n-1}k_n - i^* + \zeta_n' i^* - \zeta_n' i^*\|$$
$$\leq \zeta_n'\|G_2(QG_2)^{n-1}k_n - i^*\| + (1-\zeta_n')\|k_n - i^*\|$$

By definition (1.2)

$$\leq \zeta_n' H\|k_n - i^*\| + \|k_n - i^*\|$$
$$\leq H\|k_n - i^*\| + \|k_n - i^*\|$$
$$\leq (1+H)\|k_n - i^*\|$$

From Eqn. (15) we get,

$$\leq (1+H)[[H + H(1+H)(1+H+H^2)]\|i_n - i^*\| + [1+H(1+H)]N$$

$$\|\delta_n - i^*\| \leq (1+H)[H + H(1+H)(1+H+H^2)]\|i_n - i^*\|] + (1+H)[1+H(1+H)]N \quad (16)$$

Now we will find value of $\|j_n - i^*\|$

So,

$$\|j_n - i^*\| = \|Q[(1-\epsilon_n' - \eta_n')G_2(QG_2)^{n-1}l_n + \epsilon_n' G_2(QG_2)^{n-1}[(1-\zeta_n')k_n + \zeta_n' G_2(QG_2)^{n-1}k_n] + \eta_n'\nu_n] - i^*\|$$

By definition 1.1 and Eqn. (7), we get

$$= \|(1-\epsilon_n' - \eta_n')G_2(QG_2)^{n-1}l_n + \epsilon_n' G_2(QG_2)^{n-1}\delta_n + \eta_n'\nu_n - i^*\|$$
$$= \|(1-\epsilon_n' - \eta_n')G_2(QG_2)^{n-1}l_n + \epsilon_n' G_2(QG_2)^{n-1}\delta_n + \eta_n'\nu_n - i^* + \epsilon_n' i^* - \epsilon_n' i^* + \eta_n' i^* - \eta_n' i^*\|$$
$$\leq (1-\epsilon_n' - \eta_n')\|G_2(QG_2)^{n-1}l_n - i^*\| + \epsilon_n'\|G_2(QG_2)^{n-1}\delta_n - i^*\| + \eta_n'\|\nu_n - i^*\|$$
$$\leq (1-\epsilon_n' - \eta_n')H\|l_n - i^*\| + \epsilon_n' H\|\delta_n - i^*\| + \eta_n' N$$
$$\leq H\|l_n - i^*\| + H\|\delta_n - i^*\| + N$$

From Eqns. (13) and (16) we get,

$$\leq H[(1+H+H^2)\|i_n - i^*\| + N] + H[(1+H)[H + H(1+H)(1+H+H^2)]\|i_n - i^*\|] + (1+H)[1+H(1+H)]N] + N$$

$$\|j_n - i^*\| \leq \left[H(1+H+H^2) + H(1+H)[H + H(1+H)(1+H+H^2)]\right]\|i_n - i^*\| + \left[H + H(1+H)[1+H(1+H)+1]\right]N \quad (17)$$

Let $H(1+H+H^2) + H(1+H)[H + H(1+H)(1+H+H^2)] = s_n$

and let $H + H(1+H)[1+H(1+H)+1] = t_n$

Then Eqn. (17) will be

$$\|j_n - i^*\| \leq s_n\|i_n - i^*\| + t_n N \quad (18)$$

Now,

$$\|\sigma_n - i^*\| = \|(1-\zeta_n)j_n + \zeta_n G_1(QG_1)^{n-1}j_n - i^*\|$$
$$= \|(1-\zeta_n)j_n + \zeta_n G_1(QG_1)^{n-1}j_n - i^* + \zeta_n i^* - \zeta_n i^*\|$$
$$\leq \zeta_n\|G_1(QG_1)^{n-1}j_n - i^*\| + (1-\zeta_n)\|j_n - i^*\|$$
$$\leq \zeta_n H\|j_n - i^*\| + \|j_n - i^*\|$$
$$\leq (1+H)\|j_n - i^*\|$$

From Eqn. (18)

$$\|\sigma_n - i^*\| \leq (1+H)[s_n\|i_n - i^*\| + t_n N] \quad (19)$$

$$\|j_n - i_{n+1}\| = \|(1-\epsilon_n' - \eta_n')G_2(QG_2)^{n-1}l_n + \epsilon_n' G_2(QG_2)^{n-1}\delta_n + \eta_n'\nu_n - [(1-\epsilon_n - \eta_n)j_n + \epsilon_n G_1(QG_1)^{n-1}\sigma_n + \eta_n\chi_n]\|$$

$$= \|(1-\epsilon_n' - \gamma_n')G_2(QG_2)^{n-1}l_n + \epsilon_n' G_2(QG_2)^{n-1}\delta_n + \eta_n'\nu_n - [(1-\epsilon_n - \gamma_n)j_n + \epsilon_n G_1(QG_1)^{n-1}\sigma_n + \eta_n\chi_n] + \epsilon_n' i^* - \epsilon_n' i^* + \eta_n' i^* - \eta_n' i^* + \epsilon_n i^* - \epsilon_n i^* + \eta_n i^* - \eta_n i^*\|$$

By Eqn. (10) and definition (1.2)

$$\leq H\|l_n - i^*\| + \epsilon_n' H\|\delta_n - i^*\| + \|j_n - i^*\| + \epsilon_n H\|\sigma_n - i^*\| + \eta_n N + \eta_n' N$$

Now from Eqns. (13), (16), (18) and (19) we get

$$\leq H[(1+H+H^2)\|i_n - i^*\| + N] + \epsilon_n' H[(1+H)[H + H(1+H)(1+H+HG^2)]\|i_n - i^*\|] + (1+H)[1+H(1+H)]N] + s_n\|i_n - i^*\| + t_n N + \epsilon_n H[(1+H)[s_n\|i_n - i^*\| + t_n N]] + (\eta_n + \eta_n')N$$
$$\leq [H(1+H+H^2) + \epsilon_n' H[(1+H)[H + H(1+H)(1+H+H^2)] + s_n + \epsilon_n H[(1+H)s_n]]\|i_n - i^*\| + [H + t_n + \epsilon_n H t_n + (\eta_n + \eta_n')]N \quad (20)$$

Let$[H(1+H+H^2) + \epsilon_n' H[(1+H)[H + H(1+H)(1+H+H^2)] + s_n + \epsilon_n H[(1+H)s_n] = r_n$

Let $[H + t_n + \epsilon_n H t_n + (\eta_n + \eta_n')]N = h_n$

So Eqn. (18) will reduce

$$\|j_n - i_{n+1}\| \leq r_n\|i_n - i^*\| + h_n \quad (21)$$

Now, $\|\sigma_n - i_{n+1}\| = \|(1-\zeta_n)j_n + \zeta_n G_1(QG_1)^{n-1}j_n - i_{n+1}\|$

$$\leq \|j_n - i_{n+1}\| + \zeta_n\|\zeta_n G_1(QG_1)^{n-1}j_n - j_n\|$$

From Eqn. (21) we get

$$\|\sigma_n - i_{n+1}\| \leq r_n\|i_n - i^*\| + h_n \quad (22)$$

Now, take

$$2\epsilon_n\langle G_1(QG_1)^{n-1}\sigma_n - G_1(QG_1)^{n-1}i_{n+1}|u(i_{n+1} - i^*)\rangle \leq 2\epsilon_n M\|i_{n+1} - i^*\|\|\sigma_n - i_{n+1}\|$$

[By definition (1.2) and (1.3)] (23)

From Eqn. (22) the Eqn. (23) will be reduce as $2\epsilon_n\langle G_1(QG_1)^{n-1}\sigma_n - G_1(QG_1)^{n-1}i_{n+1}|u(i_{n+1} - i^*)\rangle \leq 2\epsilon_n M\|i_{n+1} - i^*\|(r_n\|i_n - i^*\| + h_n)$ (24)

Now take value of $\|i_{n+1} - i^*\|^2$

By lemma (1.1) and (1)

$$\|i_{n+1} - i^*\|^2 \leq (1-\epsilon_n - \eta_n)^2\|j_n - i^*\|^2 + 2\epsilon_n\langle G_1(QG_1)^{n-1}\sigma_n - i^*|u(i_{n+1} - i^*)\rangle + 2\eta_n\langle\chi_n - i^*|u(i_{n+1} - i^*)\rangle$$

$$\|i_{n+1} - i^*\|^2 \leq (1-\epsilon_n - \eta_n)^2\|j_n - i^*\|^2 + 2\epsilon_n\left\langle G_1(QG_1)^{n-1}\sigma_n - i^* + G_1(QG_1)^{n-1}i_{n+1} - G_1(QG_1)^{n-1}i_{n+1} \middle| u(i_{n+1} - i^*)\right\rangle + 2\eta_n\langle\chi_n - i^*|u(i_{n+1} - i^*)\rangle$$

By definition (1.3) and Eqn. (10)

$$\|i_{n+1} - i^*\|^2 \leq (1-\epsilon_n - \eta_n)^2[s_n\|i_n - i^*\| + t_n N]^2 + 2\epsilon_n\langle G_1(QG_1)^{n-1}\sigma_n - G_1(QG_1)^{n-1}i_{n+1}|u(i_{n+1} - i^*)\rangle + 2\epsilon_n\langle G_1(QG_1)^{n-1}i_{n+1} - i^*|u(i_{n+1} - i^*)\rangle + 2\eta_n M\|i_{n+1} - i^*\| \quad (25)$$

$$2\epsilon_n\langle T_1(PT_1)^{n-1}i_{n+1} - i^*|u(i_{n+1} - i^*)\rangle = 2\epsilon_n\langle G_1(QG_1)^{n-1}i_{n+1} - i^*|u(i_{n+1} - i^*)\rangle + 2\epsilon_n q_n\|i_{n+1} - i^*\|^2 - 2\epsilon_n q_n\|i_{n+1} - i^*\|^2 + 2\epsilon_n\varphi\|i_{n+1} - i^*\| - 2\epsilon_n\varphi\|i_{n+1} - i^*\| \quad (26)$$

Let $f_n = \langle G_1(QG_1)^{n-1}i_{n+1} - i^*|u(i_{n+1} - i^*)\rangle - q_n\|i_{n+1} - i^*\|^2 + \varphi\|i_{n+1} - i^*\| \leq \varepsilon_n$ by (5)

Then Eqn. (26) will reduce an

$$2\epsilon_n\langle G_1(QT_1)^{n-1}i_{n+1}-i^*|u(i_{n+1}-i^*)\rangle \le 2\epsilon_n f_n + 2\epsilon_n[q_n\|i_{n+1}-i^*\|^2 - \varphi\|i_{n+1}-i^*\|]$$

$$2\epsilon_n\langle G_1(QG_1)^{n-1}i_{n+1}-i^*|u(i_{n+1}-i^*)\rangle \le 2\epsilon_n\varepsilon_n + 2\epsilon_n[q_n\|i_{n+1}-i^*\|^2 - \varphi\|i_{n+1}-i^*\|] \quad (27)$$

Now substitute (27) and (24) in (25)

$$\|i_{n+1}-i^*\|^2 \le (1-\epsilon_n-\eta_n)^2 s_n^2\|i_n-i^*\|^2 + (1-\epsilon_n-\eta_n)^2(t_nN)^2 + 2s_n\|i_n-i^*\|t_nN + 2\epsilon_n\varepsilon_n + 2\epsilon_n[q_n\|i_{n+1}-i^*\|^2-\varphi\|i_{n+1}-i^*\|] + 2\epsilon_n H\|i_{n+1}-i^*\|\|\sigma_n - i_{n+1}\| + 2\eta_n N\|i_{n+1}-i^*\|$$

$$\|i_{n+1}-i^*\|^2 \le (1-\epsilon_n-\eta_n)^2 s_n^2\|i_n-i^*\|^2 + (1-\epsilon_n-\eta_n)^2(t_nN)^2 + 2s_n\|i_n-i^*\|t_nN + 2\epsilon_n\varepsilon_n + 2\epsilon_n[q_n\|i_{n+1}-i^*\|^2-\varphi\|i_{n+1}-i^*\|] + 2\epsilon_n H\|i_{n+1}-i^*\|[r_n\|i_n-i^*\|+h_n]$$

$$\|i_{n+1}-i^*\|^2 \le (1-\epsilon_n)^2 s_n^2\|i_n-i^*\|^2 + (t_nN)^2 + 2s_n\|i_n-i^*\|t_nN + 2\epsilon_n\varepsilon_n + 2\epsilon_n q_n\|i_{n+1}-i^*\|^2 - 2\epsilon_n\varphi\|i_{n+1}-i^*\| + 2\epsilon_n H\|i_{n+1}-i^*\|r_n\|i_n-i^*\| + 2\epsilon_n Hh_n\|i_{n+1}-i^*\| + 2\eta_n N\|i_{n+1}-i^*\| \quad (28)$$

Let $\|i_n-i^*\|^2 = m_n, 2\varphi(\sqrt{t}) = \phi(t)$

$\xi_n = \epsilon_n Hr_n$ and $\epsilon_n Hh_n + \eta_n N = \mu_n$ and $s_nt_nN = \omega_n$ and $(t_nN)^2 = d_n$ (29)

Then Eqn. (28) will reduce

$$m_{n+1} \le (1-\epsilon_n)^2 s_n^2 m_n + d_n + 2\omega_n\|i_n-i^*\| + 2\epsilon_n\varepsilon_n + 2\epsilon_n q_n m_{n+1} - \epsilon_n\varphi(m_{n+1}) + 2\xi_n\|i_{n+1}-i^*\|\|i_n-i^*\| + 2\mu_n\|i_{n+1}-i^*\|$$

Now use the identity

$$2mn \le m^2+n^2$$

$$m_{n+1} \le (1-\epsilon_n)^2 s_n^2 m_n + d_n + \omega_n(1+m_n) + 2\epsilon_n\varepsilon_n + 2\epsilon_n q_n m_{n+1} - \epsilon_n\varphi(m_{n+1}) + \xi_n(m_{n+1}+n_n) + \mu_n(1+m_{n+1})$$

$$\epsilon_{n+1} \le [(1+\epsilon_n^2-2\epsilon_n)s_n^2+\omega_n+\xi_n]m_n + (2\epsilon_n q_n+\xi_n+\mu_n)m_{n+1} - \epsilon_n\varphi(m_{n+1}) + d_n+\omega_n+\mu_n+2\epsilon_n\varepsilon_n$$

From given condition

$$\sum_{n=1}^{\infty}\epsilon_n^2 < +\infty$$

From Eqn. (29)

$\sum_{n=1}^{\infty}\xi_n < +\infty$, $\sum_{n=1}^{\infty}\mu_n < +\infty, \sum_{n=1}^{\infty}d_n < +\infty, \sum_{n=1}^{\infty}\omega_n < +\infty, \sum_{n=1}^{\infty}s_n < +\infty$

i.e. $\lim_{n\to\infty}(2\epsilon_n q_n+\xi_n+\mu_n) = 0\ \exists n_0$ such that $\forall p \ge p_0(2\epsilon_n q_n+\xi_n+\mu_n) \le \frac{1}{2}$

Let, $n_n = \frac{(1+\epsilon_n^2-2\epsilon_n)s_n^2+\omega_n+\xi_n}{1-(2\epsilon_n q_n+\xi_n+\mu_n)} - 1 = \frac{2\epsilon_n(q_n-s_n^2)+s_n^2(1+\epsilon_n^2)+2\xi_n-1+\omega_n+\mu_n}{1-(2\epsilon_n q_n+\xi_n+\mu_n)}$ (30)

$o_n = \frac{d_n+\omega_n+\mu_n}{1-(2\alpha_n k_n+\xi_n+\mu_n)}$ we get,

$$0 \le n_n \le 2(2\epsilon_n(q_n-s_n^2)+s_n^2(1+\epsilon_n^2)+2\xi_n-1+\omega_n+\mu_n)$$

$$0 \le o_n < 2(d_n+\omega_n+\mu_n)$$

From given condition and (30) we get $\sum_{n=1}^{\infty}n_n < +\infty$ and $\sum_{n=1}^{\infty}o_n < +\infty$

$$m_{n+1} \le (1+n_n)m_n - \epsilon_n\phi(m_{n+1}) + 2\epsilon_n\varepsilon_n + o_n$$

So by lemma (1.3)

$$\lim_{n\to\infty}m_n = \lim_{n\to\infty}\|i_n-i^*\|^2 = 0$$

$$\Rightarrow \lim_{n\to\infty}i_n = i^* \in S = S(G_1)\cap S(G_2)\cap S(G_3)\cap S(G_4)$$

the sequence $\{i_n\}$ generated will converge at fixed point

Necessary condition: Let $\lim_{n\to\infty}i_n = i^* \in G$ and let ∃ a function $\varphi:[0,+\infty)\to[0,+\infty)$ with $\varphi(0)=0$ such that $\varphi(t)=t$ and $\lim_{n\to\infty}\varphi(\|i_{n+1}-i^*\|^2)=0$

As G_1 is a pseudo asymptotically with retract map (Q) so for $q\in S(G_1)\supseteq S$

So we have,

$$\lim_{n\to\infty}\ supsup_{i\in B}\lim inf_{u(i-i^*)\in U(i-i^*)}[\langle G(QG)^{n-1}i-q|u(i-i^*)\rangle - q_n\|i-i^*\|^2 \le 0$$

So,

$\lim_{n\to\infty} supinf_{u(i_{n+1}-i^*)\in U(i_{n+1}-i^*)}[\langle G_1(QG_1)^{n-1}i_{n+1}-i^*|u(i_{n+1}-i^*)\rangle - q_n\|i_{n+1}-i^*\|^2 + \varphi(\|i_{n+1}-i^*\|)] = \lim_{n\to\infty} supinf_{u(i_{n+1}-i^*)\in U(i_{n+1}-i^*)}[\langle G_1(QG_1)^{n-1}i_{n+1}-i^*|u(i_{n+1}-i^*)\rangle - q_n\|i_{n+1}-i^*\|^2] + \lim_{n\to\infty}\varphi(\|i_{n+1}-i^*\|) \le 0+0 = 0$

This proofs the result.

III. CONCLUSION

Here, we conclude that modified CUIA iteration for non-self asymptotically pseudo contractive mapping converge to fixed point.

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