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# Some Investigation on Fractal Interpolation Function

***K. Katiyar and S. Sharma***
*Department of Mathematics, Chandigarh University, Punjab, India.*

*(Corresponding author: S. Sharma)*
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**ABSTRACT: Fractal interpolation is a modern interpolation technique in which we find an attractor of iterated function system, consisting a set of contraction maps which is defined on the data set $(x_i, y_i) \in I \times R$; $i = 0,1,2,3,\ldots,N$, where $I = [x_0, x_n] \in R$. The intent of this paper is to define a $C^2$ rational quintic fractal interpolation function. We have also find restrictions over shape parameters and vertical scaling factor to preserve positivity condition.**

**Keywords:** Fractal interpolation, Shape parameters, positivity, vertical scaling factors.

## I. INTRODUCTION

The word "fractal" was first introduce in 1975 by American mathematician Benoit B Mandelbrot [14], to define the object that are too irregular, have some kind of self-similarity and cannot be measured accurately by Euclidian geometry. According to Pickover, when the philosopher and mathematician "Gottfried Leibniz" considered recursive self -similarity by then the science behind fractals started taking shape near the seventeenth century. In spite of the fact that he wrongly thought that self - similarity is just possessed by the straight line in this sense. After this couple of mathematicians handled the issues, and crafted by the individuals who did remained darkened to a great extent in view of protection from such new rising ideas. In July 18, 1872 the definition of a function with graph is presented by the Karl Weierstrass [23] which would today be viewed as a fractal. After sometime, in 1883 "Georg Cantor" had attended the lecture given by the Weierstrass, gave an example known as "cantor set" which is actually a subsets of real line. On that particular time, this cantor set had uncommon properties and does not satisfy the laws of Euclidean geometry. In 1904, when "Helge von Koch" broadening thoughts given by Poincare and was not satisfied with the ideas of Weierstrass, gave a definition which was geometrically stronger than earlier and had also included pictures (hand - drawn) of the similar function, which is currently known as Koch snowflake [13]. The thought of "self-similar curves" was further modified by the "Paul Levy" who was about to describe the fractal-curves which are known as the "Levy C curves".

Interpolation methods have importance due to their significate role in various problems arising in scientific computation, computer aided design, computer graphics, data visualization, information sciences and etc. There are various interpolation methods available in the classical numerical analysis to construct the smooth interpolants for a given data set. These interpolation methods also deals with various shape preserving properties like positivity and monotonicity. Construction of shape preserving interpolation functions was initiated by Schweikert [21]. Fritsch and Carlson [8] developed piecewise cubic polynomials for monotonicity preserving interpolation [4, 19]. Recently Hussain *et al*., [9] introduced the "$C^2$ rational quintic function", which was compared with many other interpolant functions introduced by Duan *et al*., [7], Sarfraz *et al*., [19, 20], Butt and Brodlie [4] and some others. Hussain *et al*., [9] use three parameters for preserving the shape of data, presented by applying constraints on the first-order derivative at the knot. But Hussain [10] had also included second order derivative to achieve the $C^2$ RQF (rational quintic function), so hence this scheme is applicable to the normal data and the data which include derivative.

However, in many practical situations, we come across problems which are full of complexities. To deal these cases, Barnsley [1] introduced the concept of fractal interpolation. Fractal interpolation function introduced by Barnsley [1-3] was quite flexible to interpolate the data set which are complex in nature. The fractal interpolation function is widely used in the PC designs, power predictions and weather forecasting etc. Most of the fractal interpolation functions are studied in by researcher in form of polynomials and rational polynomials [17, 18]. Chand *et al*., [11] used rational functions to construct fractal interpolation functions and studied shape preserving analysis. Prasad *et al*., [16] used quadratic trigonometric fractal interpolation function for approximating the given data set. Thus, the method of approximation by smooth FIFs and their various extensions in the form of fractal interpolation surfaces, coalescence hidden variable FIFs etc. have been extensively studied in the literature [5, 11, 12, 15, 23].

Motivated by recent work of Chand and Tyada [6] and Hussain *et al*., [10], we introduced the $C^2$ rational quintic fractal interpolation function with parameters and obtained the sufficient conditions by restricting the scaling factors and shape parameters.

## II. PRELIMINARIES

In this section, we present some basic concepts and definitions required to introduce the quintic fractal interpolation function.

**Definition 2.1** [2] Let $(X, d)$ be a metric space. A transformation $w: X \to X$ is said to be Lipschitz with Lipschitz constant $s \in R$ iff $d(w(x), w(y)) \le s\, d(x, y)$ for all $x, y \in X$. A transformation $w: X \to X$ is called contraction iff it is Lipschitz with Lipschitz constant $s \in [0, 1)$. A Lipschitz constant $s \in [0, 1)$ is also called a contraction factor.

**Definition 2.2** [2] A hyperbolic iterated function system (IFS) consists of a complete metric space $(X, d)$ together with a finite set of contraction mappings $w_i: X \to X$, with respective contractivity factors $s_i$ for $i = 1, 2, ..., n$. This IFS is represented by $\{X; w_i: i = 1, 2, ..., n\}$ with contractivity factor $s = \max\{s_i: i = 1, 2, ..., n\}$.
We now state a Lemma of Barsnley [2] which guarantees a contraction map in $\{H(X), h\}$ out of a contraction map on $(X, d)$.

**Lemma 2.1** [2] Let $w: X \to X$ be a contraction on a metric space $(X, d)$ with contractivity factor $s$. Then $w: H(X) \to H(X)$ defined by

$$w(B) = \{w(x) : x \in B\} \quad \forall B \in H(X)$$

is a contraction on $\{H(x), h\}$ with contractivity factor $s$.
The following theorem of [4] ensures the existence of an attractor of an IFS.

**Theorem 2.1** [2] Let $\{X; w_i, i = 1, 2, 3, \ldots, n\}$ be a hyperbolic iterated function system with contractivity factor $s$. Then the transformation $W: H(X) \to H(X)$ defined by

$$W(B) = \bigcup_{i=1}^{n} w_i(B) \text{ for all } B \in H(X)$$

is a contraction mapping on the complete metric space $(H(X), h)$ with contractivity factor $s$. That is, $h(W(B), W(C)) \le s\, h(B, C)$ for all $B, C \in H(X)$. Its unique fixed point (or an attractor), $A \in H(X)$ obeys $A = W(A) = \bigcup_{i=1}^{n} w_i(A)$ and is given by $A = \lim_{i \to \infty} W^i(B)$ for any $B \in H(X)$, where $W^i(B)$ represents $i^{th}$ iterate of $W$.

### Fractal Interpolation Function

Let us consider a function $g(x)$, the graph of the function is an attractor of the hyperbolic iterated function system $\{R^2, w_i : i = 1,2,3,\ldots,n\}$, $n > 1$ (a positive integer), corresponding to the data set $\{(x_i, y_i); i = 1,2,3,\ldots,n\}$ with $x_1 < x_2 < \ldots < x_n$, where $I = [x_1, x_n]$ and $D$ a compact set in $R$. Assume $I_i = [x_i, x_{i+1}]$ and let $l_i : I \to I_i$, $i = 1 : n-1$ be a contraction map given by

$L_i(x) = a_i x + b_i$

satisfying

$L_i(x_1) = x_i,\ L_i(x_n) = x_{i+1}$

and $|L_i(c_1) - L_i(c_2)| \le \alpha |c_1 - c_2|$,

where $c_1, c_2 \in I,\ i = 1 : n-1$ and $0 \le \alpha < 1$.

Therefore, we have $a_i = (x_{i+1} - x_i)/(x_n - x_1)$

and $b_i = (x_n x_i - x_1 x_{i+1})/(x_n - x_1)$.

Let $F_i: C \to D$, $i = 1 : n$-1 be defined by

$F_i(x, y) = \alpha_i y + q_i(x)$

satisfying

$F_i(x_1, y_1) = y_i,\ F_i(x_n, y_n) = y_{i+1},$

where $|\alpha_i| < 1$, for $i = 1 : n$ -1 and $q_i(x)$ is an appropriate continuous function on $I$. Define the functions $w_i: C \to I_i \times D$, $i = 1 : n$ -1 as follows

$w_i(x, y) = (L_i(x), F_i(x, y))$ for every $(x, y) \in C$.

Then $\{C, w_i(x, y), i = 1: n\text{-}1\}$ is called an IFS for given interpolating data set. Thus by Banach contraction principle, this IFS has a unique fixed point $g$ which is the graph of continuous function $f : I \to D$ satisfying $f(x_i) = y_i$ for $n = 1 : n$-1 [1].
If $q_i(x)$ is replaced by an appropriate trigonometric function for $i = 1: n$-1, then the IFS $\{C, w_i(x, y), i = 1: n\text{-}1\}$ is called trigonometric iterated function system (TIFS) for the given data set.
The FIFs of TIFS for varying values of $\alpha_i$ are illustrated in the following example.

**Example:** Consider the data set

$$\{(0, 1), (1, 40), (3, 55)\}.$$

Then, FIF for various selection of vertical scaling factors (VSF) are shown in Figs. 1 and 2.

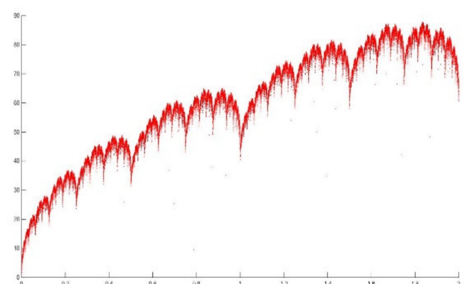

**Fig. 1.** FIF with VSF: 0.76204, 0.86084, 0.9.

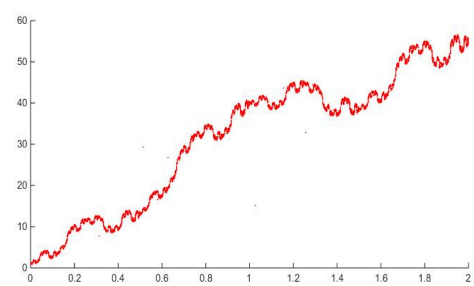

**Fig. 2.** FIF with VSF: -0.6204, -0.76084, -0.89.

## III. CONSTRUCTION OF $C^2$ RATIONAL QUINTIC FRACTAL INTERPOLATION FUNCTION

In this section, we have considered $C^2$ rational quintic fractal interpolation function having three free parameters to analyze its positivity. $C^2$ Rational quintic fractal interpolation functions are some extra added values or points used for the preservation of shape of the data.

Let $\mathcal{F} = \{F \in C^1[x_1, x_N] : F(x_1) = f_1, F(x_N) = f_N\}$, then $(\mathcal{F}, \rho)$ is a complete metric space with metric $\rho$ induced by $C^2$ norm on $C^2[x_1, x_N]$.

Define the *R-B* operator $\Im : \mathcal{F} \to \mathcal{F}$ as

$$\Im F(x) = \delta_i F(l_i^{-1}(x)) + s_i(l_i^{-1}(x)).$$

Since $a_i = \frac{x_{i+1} - x_i}{x_N - x_1} < 1$, the condition $|\delta_i| < a_i < 1$ ensures that $\Im$ is a contraction operator on $(\mathcal{F}, \rho)$. Then the fixed point $\psi \in C^1[x_1, x_N]$ of $\Im$ is a fractal function that satisfies the functional relation

$$\psi(l_i(x)) = \delta_i \psi(x) + s_i(x), \quad x \in I \text{ for } i = 1, 2, ..., n-1. \quad (1)$$

where $\psi(l_i(x)$ is the rational quintic fractal interpolation function with vertical scaling factor $\delta_i$, where $s_i(x)$ is the rational quintic function defined as:

$$s_i(x) = \frac{P_i(\theta)}{Q_i(\theta)}; Q_i(\theta) \neq 0$$

$$P_i(\theta) = \sum_{i=0}^{5} (1-\theta)^{5-i} \theta^i A_i \quad (2)$$

$$P_i(\theta) = (1-\theta)^5 A_0 + (1-\theta)^4 \theta A_1 + (1-\theta)^3 \theta^2 A_2 + (1-\theta)^2 \theta^3 A_3 + (1-\theta)^1 \theta^4 A_4 + \theta^5 A_5$$

$$Q_i(\theta) = \alpha_i (1-\theta)^2 + \beta_i (1-\theta)^1 \theta + \gamma_i \theta^2 \quad (3)$$

$$\theta = \frac{x - x_1}{x_n - x_1}; x_n - x_1 = h_i \quad (4)$$

Here $\theta$ is an interpolation function characterize above has three positive genuine parameters that are $\alpha_i, \beta_i, \gamma_i$ all are greater than zero. These parameters can create a group of curve for given information by changing the estimation of free parameters.

We have some conditions which rational quintic fractal interpolation function follows:

1. $\psi(x_i) = f_i$
2. In case of first derivative

   $\psi'(x_i) = d_i$
3. And in case of second derivative

   $\psi''(x_i) = D_i$

Now, we calculate the shape parameters satisfying the conditions 1-3.

Put $x = x_1$ in Eqn. $(4)$ this

$$\Rightarrow \theta = \frac{x_1 - x_1}{x_n - x_1} \Rightarrow \theta = 0.$$

Put $x = x_1$ ting these values from Eqn. (1), we will get,

$$\psi(L_i(x_1)) = \delta_i \psi(x_1) + s_i(x_1)$$

$$\Rightarrow f_i = \delta_i f_1 + \frac{P_i(0)}{Q_i(0)}$$

$$\Rightarrow f_i = \delta_i f_1 + \frac{A_0}{\alpha_i}$$

$$\Rightarrow A_0 = \alpha_i (f_i - \delta_i f_1) \quad (5)$$

Now put $x = x_n$ in Eqn. $(4)$ this $\Rightarrow \theta = \frac{x_n - x_1}{x_n - x_1} \Rightarrow \theta = 1$

.

Put $x = x_n$ in Eqn. (1), we will get

$$\psi(L_i(x_n)) = \delta_i \psi(x_n) + s_i(x_n)$$

$$\Rightarrow f_{i+1} = \delta_i f_n + \frac{P_i(1)}{Q_i(1)}$$

$$\Rightarrow f_{i+1} = \delta_i f_n + \frac{A_5}{\gamma_i}$$

$$\Rightarrow A_5 = \gamma_i (f_{i+1} - \delta_i f_n) \quad (6)$$

Differentiate $(1)$ with respect to "x"

$$\psi'(L_i(x)) = \delta_i \psi'(x) + s_i'(x) \quad (7)$$

After differentiating $s_i(x)$ we have,

$$s_i'(x) = \frac{1}{h_i} \frac{P_i'(\theta) Q_i(\theta) - Q_i'(0) P_i(\theta)}{[Q_i(\theta)]^2} \quad (8)$$

Differentiate Eqn. $(2)$ with respect to $\theta$, we implies

$$s(x_1) = f_1, s(x_n) = f_n$$

$$P'(\theta) = -5(1-\theta)^4 A_0 - 4(1-\theta)^3 \theta A_1 + (1-\theta)^4 A_1$$
$$-3(1-\theta)^2 \theta^2 A_2 + 2(1-\theta)^3 \theta A_2 + 3(1-\theta)^2 \theta^2 A_3 \quad (9)$$
$$-2(1-\theta)\theta^3 A_3 + 4(1-\theta)\theta^3 A_4 - \theta^4 A_4 + 5\theta^4 A_5.$$

Also differentiate Eqn. (3) with respect to $\theta$ and we will get,

$$Q'(\theta) = -2\alpha_i (1-\theta) + \beta_i (1-\theta) - \beta_i \theta + 2\gamma_i \theta. \quad (10)$$

Put $x = x_1$ in Eqn. (4) this $\Rightarrow \theta = \frac{x_1 - x_1}{x_n - x_1} \Rightarrow \theta = 0$ put the value of $\theta$ in Eqns. (2),(3),(7),(8) and (9) we will get

$$\psi'(L_i(x_1)) = \delta_i \psi'(x_1) + s'(x_1)$$

By using the second condition:

$$d_i = \delta_i d_1 + \frac{1}{h_i} \frac{P_i'(0) Q_i(0) - Q_i'(0) P_i(0)}{[Q_i(0)]^2}$$

$$d_i = \delta_i d_1 + \frac{1}{h_i} \frac{(-5A_0 + A_1)(\alpha_i) - (-2\alpha_i + \beta_i)(A_0)}{\alpha_i^2}.$$

Multiplying whole equation by $h_i$

$$\Rightarrow d_i h_i = \delta_i d_1 h_i + \frac{-5\alpha_i A_0 + \alpha_i A_1 + 2\alpha_i A_0 - \beta_i A_0}{\alpha_i^2}$$

Multiplying the above equation with $\alpha_i^2$

$$\Rightarrow \alpha_i^2 d_i h_i = \alpha_i^2 \delta_i d_1 h_i - 3\alpha_i A_0 + \alpha_i A_1 - \beta_i A_0$$

$$\Rightarrow \alpha_i^2 d_i h_i = \alpha_i^2 \delta_i d_1 h_i - 3\alpha_i(\alpha_i f_i - \alpha_i \delta_i f_1) + \alpha_i A_1 - \alpha_i \beta_i f_i + \alpha_i \beta_i \delta_i f_1$$

$$\Rightarrow \alpha_i^2 d_i h_i = \alpha_i^2 \delta_i d_1 h_i - 3\alpha_i^2 f_i + 3\alpha_i^2 \delta_i f_1 + \alpha_i A_1 - \alpha_i \beta_i f_i + \alpha_i \beta_i \delta_i f_1$$

$$= \alpha_i^2 h_i d_i - \alpha_i^2 \delta_i h_i d_1 + f_i(\beta_i + 3\alpha_i)\alpha_i - f_1(\beta_i + 3\alpha_i)\alpha_i \delta_i$$

Eliminate $\alpha_i$ from above equation

$$A_1 = \alpha_i h_i d_i - \alpha_i \delta_i h_i d_1 + f_i(\beta_i + 3\alpha_i) - f_1(\beta_i + 3\alpha_i) \quad (11)$$

Put $x = x_n$ in Eqn. (4) this

$$\Rightarrow \theta = \frac{x_n - x_1}{x_n - x_1} \Rightarrow \theta = 1$$

Put the value of $\theta$ in Eqns. (2), (3), (7), (8) and (9), we will get,

$$\psi'(L_i(x_n)) = \delta_i \psi'(x_n) + s'(x_n)$$

By using the second condition

$$d_{i+1} = \delta_i d_n + \frac{1}{h_i} \frac{P'(1)Q(1) - Q'(1)P(1)}{[Q(1)]^2}$$

$$d_{i+1} = \delta_i d_n + \frac{1}{h_i} \frac{(-A_4 + 5A_5)(\gamma_i) - (2\gamma_i - \beta_i)(A_5)}{\gamma_i^2}$$

Multiplying whole equation by $h_i$

$$\Rightarrow d_{i+1} h_i = \delta_i d_n h_i + \frac{-\gamma_i A_4 + 5\gamma_i A_5 + \beta_i A_5 - 2\gamma_i A_5}{\gamma_i^2}$$

Multiplying the above equation with $\gamma_i^2$

$$\Rightarrow \gamma_i^2 d_{i+1} h_i = \gamma_i^2 \delta_i d_n h_i + 3\gamma_i A_5 + \beta_i A_5 - \gamma_i A_4$$

$$\Rightarrow \gamma_i^2 d_{i+1} h_i = \gamma_i^2 \delta_i d_n h_i + 3\gamma_i(\gamma_i f_{i+1} - \gamma_i \delta_i f_n) + \beta_i(\gamma_i f_{i+1} - \gamma_i \delta_i f_n) - \gamma_i A_4$$

$$\Rightarrow \gamma_i^2 d_{i+1} h_i = \gamma_i^2 \delta_i d_n h_i + 3\gamma_i^2 f_{i+1} - 3\gamma_i^2 \delta_i f_n + \beta_i \gamma_i f_{i+1} - \beta_i \gamma_i \delta_i f_n - \gamma_i A_4$$

$$\Rightarrow \gamma_i A_4 = \gamma_i^2 \delta_i h_i d_n - \gamma_i^2 h_i d_{i+1} + f_{i+1}(\beta_i + 3\gamma_i)\gamma_i - f_n(\beta_i + 3\gamma_i)\gamma_i \delta_i$$

Eliminate $\gamma_i$ from above equation

$$A_4 = \gamma_i \delta_i h_i d_n - \gamma_i h_i d_{i+1} + f_{i+1}(\beta_i + 3\gamma_i) - f_n(\beta_i + 3\gamma_i)\delta_i \quad (12)$$

Again differentiate Eqn. (7) with respect to $x$

$$\psi''(l_i(x)) = \delta_i \psi''(x) + s_i''(x) \quad (13)$$

Further after differentiating $s_i'(x)$ we have,

$$s_{i''}(x) = \frac{1}{h_i^2 [Q_i(\theta)]^3} P_{i''}(\theta)[Q_i(\theta)]^2 - Q_i''(\theta) P_i(\theta) Q_i(\theta) + 2P_i(\theta)[Q_i'(\theta)]^2 - 2P_i'(\theta) Q_i'(\theta) Q_i(\theta) \quad (14)$$

Now Differentiate Eqn. (9) with respect to $\theta$, we implies

$$P''(\theta) = 20(1-\theta)^3 A_0 + 12(1-\theta)^2 \theta A_1 - 8(1-\theta)^3 A_1 - 12(1-\theta)^2 \theta A_2 + 6(1-\theta)\theta^2 A_2 + 2(1-\theta)^3 A_2 + 6(1-\theta)^2 \theta A_3 - 12(1-\theta)\theta^2 A_3 + 2\theta^3 A_3 + 12(1-\theta)\theta^2 A_4 - 8\theta^3 A_4 + 20\theta^3 A_5 \quad (15)$$

Differentiate Eqn. (10) with respect to $\theta$ then we get,

$$Q''(\theta) = 2\alpha_i - \beta_i - \beta_i + 2\gamma_i$$

$$\Rightarrow Q''(\theta) = 2\alpha_i - 2\beta_i + 2\gamma_i \quad (16)$$

Put $x = x_1$ in equation (4) this

$$\Rightarrow \theta = \frac{x_1 - x_1}{x_n - x_1} \Rightarrow \theta = 0$$

Putting $x = x_1$, from Eqns. (2), (3), (13) and (14) we will get

$$\psi''(L_i(x_1)) = \delta_i \psi''(x_1) + s''(x_1).$$

According to third condition and put the values of $P''(\theta)$ and $Q''(\theta)$ in above equation:

$$D_i = \delta_i D_1 + \frac{1}{h_i^2 \alpha_i^3}(20A_0 - 8A_1 + 2A_2)\alpha_i^2 - \alpha_i(2\alpha_i - 2\beta_i + 2\gamma_i)A_0 + 2A_0(-2\alpha_i + \beta_i)^2 - 2\alpha_i(-2\alpha_i + \beta_i)(-5A_0 + A_1)$$

Put the values of $A_0, A_1$ and multiply the whole equation by $\alpha_i^3$ and $h_i^2$, we get,

$$\alpha_i^3 h_i^2 D_i = \alpha_i^3 h_i^2 \delta_i D_1 + (20(\alpha_i(f_i - \delta_i f_1)) - 8(\alpha_i h_i d_i - \alpha_i \delta_i h_i d_1 + f_i(\beta_i + 3\alpha_i) - f_1(\beta_i + 3\alpha_i)\delta_i) + 2A_2)\alpha_i^2 - \alpha_i(2\alpha_i - 2\beta_i + 2\gamma_i)$$
$$(\alpha_i(f_i - \delta_i f_1)) + 2(\alpha_i(f_i - \delta_i f_1))(-2\alpha_i + \beta_i)^2 - 2\alpha_i(-2\alpha_i + \beta_i)$$
$$(-5(\alpha_i(f_i - \delta_i f_1)) + \alpha_i h_i d_i - \alpha_i \delta_i h_i d_1 + f_i(\beta_i + 3\alpha_i) - f_1(\beta_i + 3\alpha_i)\delta_i)$$

$$\Rightarrow \alpha_i^3 h_i^2 D_i = \alpha_i^3 h_i^2 \delta_i D_1 + 20\alpha_i^3 f_i - 20\alpha_i^3 \delta_i f_1 - 8\alpha_i^3 h_i d_i + 8\alpha_i^3 \delta_i h_i d_1 - 8\alpha_i^2 \beta_i f_i - 24\alpha_i^3 f_i + 8\alpha_i^2 \beta_i f_1 + 24\alpha_i^3 \delta_i f_1 + 2\alpha_i^2 A_2 - (2\alpha_i - 2\beta_i + 2\gamma_i)(\alpha_i^2 f_i - \alpha_i^2 \delta_i f_1))$$
$$+(2\alpha_i f_i - 2\alpha_i \delta_i f_1))(4\alpha_i^2 + \beta_i^2 - 4\alpha_i \beta_i) - (-4\alpha_i^2 + 2\alpha_i \beta_i)$$
$$(-5\alpha_i f_i + 5\alpha_i \delta_i f_1 + \alpha_i h_i d_i - \alpha_i \delta_i h_i d_1 + \beta_i f_i + 3\alpha_i f_i - \beta_i \delta_i f_1 - 3\alpha_i \delta_i f_1)$$

$$\Rightarrow \alpha_i^3 h_i^2 D_i = \alpha_i^3 h_i^2 \delta_i D_1 + 20\alpha_i^3 f_i - 20\alpha_i^3 \delta_i f_1 - 8\alpha_i^3 h_i d_i + 8\alpha_i^3 \delta_i h_i d_1 - 8\alpha_i^2 \beta_i f_i - 24\alpha_i^3 f_i + 8\alpha_i^2 \beta_i f_1 + 24\alpha_i^3 \delta_i f_1$$
$$+2\alpha_i^2 A_2 + 2\alpha_i^3 \delta_i f_1 - 2\alpha_i^2 \beta_i \delta_i f_1 + 2\alpha_i^2 \gamma_i \delta_i f_1 - 2\alpha_i^3 f_i$$
$$+2\alpha_i^2 \beta_i f_i - 2\alpha_i^2 \gamma_i f_i + 2\alpha_i \beta_i^2 f_i + 8\alpha_i^3 f_i - 8\alpha_i^2 \beta_i f_i - 2\alpha_i \beta_i^2 \delta_i f_1 - 8\alpha_i^3 \delta_i f_1$$

$$\Rightarrow \alpha_i^3 h_i^2 D_i = \alpha_i^3 h_i^2 \delta_i D_1 + 2\alpha_i^2 A_2 + \alpha_i^2 f_i[20\alpha_i - 8\beta_i - 24\alpha_i - 2\alpha_i + 2\beta_i - 2\gamma_i + \frac{2\beta_i^2}{\alpha_i} + 8\alpha_i + 12\alpha_i - \frac{2\beta_i^2}{\alpha_i} - 6\beta_i - 8\beta_i - 20\alpha_i + 10\beta_i + 4\beta_i] + \alpha_i^2 f_1[-20\alpha_i + 8\beta_i + 24\alpha_i + 2\alpha_i - 2\beta_i + 2\gamma_i - \frac{2\beta_i^2}{\alpha_i} - 8\alpha_i - 12\alpha_i + \frac{2\beta_i^2}{\alpha_i} + 6\beta_i + 8\beta_i + 20\alpha_i - 10\beta_i - 4\beta_i]\delta_i + \alpha_i^2 h_i d_i[-8\alpha_i + 4\alpha_i - 2\beta_i] + \alpha_i^2 h_i d_1[8\alpha_i - 4\alpha_i + 2\beta_i]\delta_i$$

$$\Rightarrow \alpha_i^3 h^2 D_i = \alpha_i^3 h_i^2 \delta_i D_1 + 2\alpha_i^2 A_2 + \alpha_i^2 f_i[-6\alpha_i - 6\beta_i - 2\gamma_i] + \alpha_i^2 f_1[6\alpha_i + 6\beta_i + 2\gamma_i]\delta_i + \alpha_i^2 h_i d_i[-4\alpha_i - 2\beta_i] + \alpha_i^2 h_i d_1[4\alpha_i + 2\beta_i]\delta_i$$

$$\Rightarrow 2\alpha_i^2 A_2 = \alpha_i^3 h^2 D_i - \alpha_i^3 h_i^2 \delta_i D_1 + 2\alpha_i^2 f_i[3\alpha_i + 3\beta_i + \gamma_i]$$
$$-2\alpha_i^2 f_1[3\alpha_i + 3\beta_i + \gamma_i]\delta_i$$
$$+2\alpha_i^2 h_i d_i[2\alpha_i + \beta_i] - 2\alpha_i^2 h_i d_1[2\alpha_i + \beta_i]\delta_i$$
$$\Rightarrow A_2 = 0.5\alpha_i h^2 D_i - 0.5\alpha_i h_i^2 \delta_i D_1 + f_i[3\alpha_i + 3\beta_i + \gamma_i]$$
$$-f_1[3\alpha_i + 3\beta_i + \gamma_i]\delta_i - h_i d_i[2\alpha_i + \beta_i] - h_i d_1[2\alpha_i + \beta_i]\delta_i$$

$$A_2 = 0.5\alpha_i h^2 D_i - 0.5\alpha_i h_i^2 \delta_i D_1 + f_i[3\alpha_i + 3\beta_i + \gamma_i]$$
$$-f_1[3\alpha_i + 3\beta_i + \gamma_i]\delta_i + h_i[2\alpha_i + \beta_i][d_i - \delta_i d_1] \quad (17)$$

Put $x = x_n$ in Eqn. (4), we get

$$\Rightarrow \theta = \frac{x_n - x_1}{x_n - x_1} \Rightarrow \theta = 1.$$

Put the value of $\theta$ in Eqns. (2), (3), (13) and (14), we will get

$$\psi''(L_i(x_n)) = \delta_i \psi''(x_n) + s''(x_n).$$

According to third condition and put the values of $P''(\theta)$ and $Q''(\theta)$ in above equation:

$$D_{i+1} = \delta_i D_n + \frac{1}{h_i^2 \gamma_i^3}(2A_3 - 8A_4 + 20A_5)\gamma_i^2 - \gamma_i(2\alpha_i - 2\beta_i$$
$$+2\gamma_i)A_5 + 2A_5(2\gamma_i - \beta_i)^2 - 2\gamma_i(2\gamma_i - \beta_i)(-A_4 + 5A_5)$$

Put the values of $A_4$, $A_5$ and multiply the whole equation by $\gamma_i^3$ and $h_i^2$ we get,

$$\gamma_i^3 h_i^2 D_{i+1} = \gamma_i^3 h_i^2 \delta_i D_n + (2A_3 - 8(\gamma_i \delta_i h_i d_n - \gamma_i h_i d_{i+1}$$
$$+f_{i+1}(\beta_i + 3\gamma_i) - f_1(\beta_i + 3\gamma_i)\delta_i) + 20(\gamma_i(f_{i+1} - \delta_i f_n))\gamma_i^2$$
$$-\gamma_i(2\alpha_i - 2\beta_i + 2\gamma_i)(\gamma_i(f_{i+1} - \delta_i f_n)) + 2(\gamma_i(f_{i+1} - \delta_i f_n))$$
$$(2\gamma_i - \beta_i)^2 - 2\gamma_i$$
$$(2\gamma_i - \beta_i)(-\gamma_i \delta_i h_i d_n) + \gamma_i h_i d_{i+1} - f_{i+1}(\beta_i + 3\gamma_i)$$
$$+f_n(\beta_i + 3\gamma_i)\delta_i + 5(\gamma_i(f_{i+1} - \delta_i f_n))$$

$$\Rightarrow \gamma_i^3 h_i^2 D_{i+1} = \gamma_i^3 h_i^2 \delta_i D_n + 2A_3 - 8\gamma_i^3 \delta_i h_i d_n + 8\gamma_i^3 h_i d_{i+1}$$
$$-8f_{i+1}\beta_i \gamma_i^2 - 24\gamma_i^3 f_{i+1} + 8\beta_i \gamma_i^2 \delta_i f_n + 24\gamma_i^3 f_n \delta_i + 20\gamma_i^3 f_{i+1}$$
$$-20\gamma_i^3 \delta_i f_n - (2\alpha_i - 2\beta_i + 2\gamma_i)(\gamma_i^2 f_{i+1} - \gamma_i^2 \delta_i f_n) + 2(\gamma_i f_{i+1}$$
$$-\gamma_i \delta_i f_n)(4\gamma_i^2 + \beta_i^2 - 4\gamma_i \beta_i) - (4\gamma_i^2 - 2\beta_i \gamma_i)(-\gamma_i \delta_i h_i d_n$$
$$+\gamma_i h_i d_{i+1}) - f_{i+1}(\beta_i + 3\gamma_i) + f_n(\beta_i + 3\gamma_i)\delta_i + 5\gamma_i(f_{i+1} - 5\delta_i f_n)$$
$$\Rightarrow \gamma_i^3 h_i^2 D_{i+1} = \gamma_i^3 h_i^2 \delta_i D_n + 2A_3 - 8\gamma_i^3 \delta_i h_i d_n + 8\gamma_i^3 h_i d_{i+1}$$
$$-8f_{i+1}\beta_i \gamma_i^2 - 24\gamma_i^3 f_{i+1} + 8\beta_i \gamma_i^2 \delta_i f_n + 24\gamma_i^3 f_n \delta_i + 20\gamma_i^3 f_{i+1}$$
$$-20\gamma_i^3 \delta_i f_n - 2\alpha_i \gamma_i^2 f_{i+1} + 2\alpha_i \gamma_i^2 \delta_i f_n + 2\beta_i \gamma_i^2 f_{i+1} - 2\beta_i \gamma_i^2 \delta_i f_n$$
$$-2\gamma_i^3 f_{i+1} + 2\gamma_i^3 \delta_i f_n + 8\gamma_i^3 f_{i+1} - 8\gamma_i^3 \delta_i f_n + 2\beta_i^2 \gamma_i f_{i+1}$$
$$-2\beta_i^2 \gamma_i \delta_i f_n - 8\gamma_i^2 \beta_i f_{i+1} + 8\gamma_i^2 \beta_i \delta_i f_n + 4\gamma_i^3 h_i \delta_i d_n$$
$$-4\gamma_i^3 h_i d_{i+1} + 4\gamma_i^2 \beta_i f_{i+1} + 12\gamma_i^3 f_{i+1} - 4\gamma_i^3 \beta_i \delta_i f_n - 12\gamma_i^3 \delta_i f_n$$
$$-20\gamma_i^3 f_{i+1} + 20\gamma_i^3 \delta_i f_n 2 - \gamma_i^2 \beta_i \delta_i h_i d_n + 2\gamma_i^2 \beta_i h_i d_{i+1} - 2\beta_i^2 \gamma_i f_{i+1}$$
$$-6\gamma_i^2 \beta_i f_{i+1} + 2\beta_i^2 \gamma_i^2 \delta_i f_n + 6\gamma_i^2 \beta_i \delta_i f_n + 10\gamma_i^2 \beta_i f_{i+1} - 10\gamma_i^2 \beta_i \delta_i f_n$$

$$\Rightarrow \gamma_i^3 h_i^2 D_{i+1} = \gamma_i^3 h_i^2 \delta_i D_n + 2A_3 + \gamma_i^2 f_{i+1}[-24\gamma_i - 8\beta_i + 20\gamma_i$$
$$-2\alpha_i - 2\gamma_i + 2\beta_i + 8\gamma_i + \frac{2\beta_i^2}{\gamma_i} - 8\beta_i + 12\gamma_i + 4\beta_i - 20\gamma_i - \frac{2\beta_i^2}{\gamma_i}$$
$$+10\beta_i - 6\beta_i] + \gamma_i^2 f_n[24\gamma_i + 8\beta_i - 20\gamma_i + 2\alpha_i + 2\gamma_i - 2\beta_i - 8\gamma_i$$
$$-\frac{2\beta_i^2}{\gamma_i} + 8\beta_i - 12\gamma_i - 4\beta_i + 20\gamma_i + \frac{2\beta_i^2}{\gamma_i} - 10\beta_i + 6\beta_i]\delta_i$$
$$+\gamma_i^2 h_i d_n[-8\gamma_i + 4\gamma_i - 2\beta_i]\delta_i + \gamma_i^2 h_i d_{i+1}[8\gamma_i - 4\gamma_i + 2\beta_i]$$
$$\Rightarrow \gamma_i^3 h_i^2 D_{i+1} = \gamma_i^3 h_i^2 \delta_i D_n + 2A_3 + \gamma_i^2 f_{i+1}[-2\alpha_i - 6\beta_i - 6\gamma_i]$$
$$+\gamma_i^2 f_n[2\alpha_i + 6\beta_i + 6\gamma_i]\delta_i + \gamma_i^2 h_i d_n[-4\gamma_i - 2\beta_i]\delta_i$$
$$+\gamma_i^2 h_i d_{i+1}[4\gamma_i + 2\beta_i]$$
$$\Rightarrow 2\gamma_i^2 A_3 = \gamma_i^3 h_i^2 D_{i+1} - \gamma_i^3 h_i^2 \delta_i D_n + 2\gamma_i^2 f_{i+1}[\alpha_i + 3\beta_i + 3\gamma_i]$$
$$-2\gamma_i^2 f_n[\alpha_i + 3\beta_i + 3\gamma_i]\delta_i + 2\gamma_i^2 h_i d_n[2\gamma_i + \beta_i]\delta_i$$
$$-2\gamma_i^2 h_i d_{i+1}[2\gamma_i + \beta_i]$$
$$\Rightarrow A_3 = 0.5\gamma_i h_i^2 D_{i+1} - 0.5\gamma_i h_i^2 \delta_i D_n + f_{i+1}[\alpha_i + 3\beta_i + 3\gamma_i]$$
$$-f_n[\alpha_i + 3\beta_i + 3\gamma_i]\delta_i + h_i d_n[2\gamma_i + \beta_i]\delta_i - h_i d_{i+1}[2\gamma_i + \beta_i]$$
$$A_3 = 0.5\gamma_i h_i^2 D_{i+1} - 0.5\gamma_i h_i^2 \delta_i D_n + f_{i+1}[\alpha_i + 3\beta_i + 3\gamma_i]$$
$$- f_n[\alpha_i + 3\beta_i + 3\gamma_i]\delta_i + h_i[2\gamma_i + \beta_i][\delta_i d_n - d_{i+1}] \quad (18)$$

So, these are the values of shape parameters $(A_0, A_1, A_2, A_3, A_4 and A_5)$.

## IV. POSITIVITY CONDITION FOR $C^2$ RQFIF

In this section we have introduced the positivity condition for the $C^2$ rational quintic fractal interpolation function. We will assume that all the $A_i$'s and $\delta_i$ are greater than zero. So let's start with $A_0$.

For $A_0 > 0$, put the value of $A_0$ from Eqn. (5) we get,

$$\Rightarrow \alpha_i(f_i - \delta_i f_1) > 0$$
$$\Rightarrow \alpha_i f_i - \alpha_i \delta_i f_1 > 0$$
$$\Rightarrow \alpha_i f_i > \alpha_i \delta_i f_1$$

Eliminate $\alpha_i$ from both side if the above equation then,

$$\delta_i < \frac{f_i}{f_1} \quad (19)$$

Above equation is satisfied iff $f_1 \neq 0$

For $A_1 > 0$, put the value of $A_1$ from Eqn. (11), we get

$$\alpha_i h_i d_i - \alpha_i \delta_i h_i d_1 + f_i(3\alpha_i + \beta_i) - f_1(3\alpha_i + \beta_i)\delta_i > 0$$
$$\Rightarrow \alpha_i h_i d_i - \alpha_i \delta_i h_i d_1 + 3\alpha_i f_i + f_i \beta_i - 3\alpha_i \delta_i f_1 - f_1 \beta_i \delta_i > 0$$
$$\Rightarrow \alpha_i h_i(d_i - \delta_i d_1) + 3\alpha_i(f_i - \delta_i f_1) + \beta_i(f_i - \delta_i f_1) > 0$$
$$\Rightarrow \beta_i(f_i - \delta_i f_1) > \alpha_i h_i(\delta_i d_1 - d_i) + 3\alpha_i(\delta_i f_1 - f_i)$$
$$\beta_i > \frac{\alpha_i h_i(\delta_i d_1 - d_i) + 3\alpha_i(\delta_i f_1 - f_i)}{f_i - \delta_i f_1} \quad (20)$$

Above equation is satisfied iff $f_i - \delta_i f_1 > 0$

For $A_2 > 0$, put the value of $A_2$ from Eqn. (17) we get

$$\Rightarrow 0.5\alpha_i h^2 D_i - 0.5\alpha_i h_i^2 \delta_i D_1 + f_i[3\alpha_i + 3\beta_i + \gamma_i]$$
$$-f_1[3\alpha_i + 3\beta_i + \gamma_i]\delta_i + h_i[2\alpha_i + \beta_i][d_i - \delta_i d_1] > 0$$

$$\Rightarrow 0.5\alpha_i h^2 D_i - 0.5\alpha_i h_i^2 \delta_i D_1 + 3\alpha_i f_i + 3\beta_i f_i + \gamma_i f_i - 3\alpha_i \delta_i f_1$$
$$-3\beta_i \delta_i f_1 - \gamma_i \delta_i f_1 + 2\alpha_i h_i d_i - 2\alpha_i h_i \delta_i d_1 - \beta_i h_i \delta_i d_1 + h_i \beta_i d_i > 0$$

$$\Rightarrow 0.5\alpha_i h^2 D_i - 0.5\alpha_i h_i^2 \delta_i D_1 + (3\alpha_i + \gamma_i)(f_i - \delta_i f_1)$$
$$+2\alpha_i h_i (d_i - \delta_i d_1) + \beta_i (3(f_i - \delta_i f_1) + h_i (d_i - \delta_i d_1)) > 0$$

$$\Rightarrow \beta_i (3(f_i - \delta_i f_1) + h_i (d_i - \delta_i d_1)) > 0.5\alpha_i h_i^2 \delta_i D_1$$
$$-0.5\alpha_i h^2 D_i + (3\alpha_i + \gamma_i)(\delta_i f_1 - f_i) + 2\alpha_i h_i (\delta_i d_1 - d_i)$$

$$\beta_i > \frac{0.5\alpha_i h_i^2 \delta_i D_1 - 0.5\alpha_i h^2 D_i + (3\alpha_i + \gamma_i)(\delta_i f_1 - f_i) + 2\alpha_i h_i (\delta_i d_1 - d_i)}{3(f_i - \delta_i f_1) + h_i (d_i - \delta_i d_1)} \quad (21)$$

This inequality satisfies iff

$3(f_i - \delta_i f_1) + h_i (d_i - \delta_i d_1) > 0$

For $A_3 > 0$, put the value of $A_3$ from Eqn. (18), we get

$$\Rightarrow 0.5\gamma_i h_i^2 D_{i+1} - 0.5\gamma_i h_i^2 \delta_i D_n + f_{i+1}[\alpha_i + 3\beta_i + 3\gamma_i]$$
$$-f_n[\alpha_i + 3\beta_i + 3\gamma_i]\delta_i + h_i[2\gamma_i + \beta_i][\delta_i d_n - d_{i+1}] > 0$$

$$\Rightarrow 0.5\gamma_i h_i^2 D_{i+1} - 0.5\gamma_i h_i^2 \delta_i D_n + \alpha_i f_{i+1} + 3\beta_i f_{i+1} + 3\gamma_i f_{i+1}$$
$$-\alpha_i \delta_i f_n - 3\beta_i \delta_i f_n - 3\gamma_i \delta_i f_n + 2\gamma_i \delta_i h_i d_n - 2\gamma_i h_i d_{i+1}$$
$$-\beta_i h_i d_{i+1} + \beta_i \delta_i h_i d_n > 0$$

$$\Rightarrow 0.5\gamma_i h_i^2 D_{i+1} - 0.5\gamma_i h_i^2 \delta_i D_n + (\alpha_i + 3\gamma_i)(f_{i+1} - \delta_i f_n)$$
$$+2\gamma_i h_i (\delta_i d_n - d_{i+1}) + \beta_i (3(f_{i+1} - \delta_i f_n) + h_i (\delta_i d_n - d_{i+1})) > 0$$

$$\Rightarrow \beta_i (3(f_{i+1} - \delta_i f_n) + h_i (\delta_i d_n - d_{i+1})) > 0.5\gamma_i h_i^2 \delta_i D_n$$
$$-0.5\gamma_i h_i^2 D_{i+1} + (\alpha_i + 3\gamma_i)(\delta_i f_n - f_{i+1}) + 2\gamma_i h_i (d_{i+1} - \delta_i d_n)$$

$$\beta_i > \frac{1}{3(f_{i+1} - \delta_i f_n) + h_i (\delta_i d_n - d_{i+1})}(0.5\gamma_i h_i^2 \delta_i D_n - 0.5\gamma_i h_i^2 D_{i+1} + (\alpha_i + 3\gamma_i)(\delta_i f_n - f_{i+1}) + 2\gamma_i h_i (d_{i+1} - \delta_i d_n)) \quad (22)$$

Above inequality satisfies iff,

$3(f_{i+1} - \delta_i f_n) + h_i (\delta_i d_n - d_{i+1}) > 0$

For $A_4 > 0$, put the value of $A_4$ from Eqn. (12) we get

$$\gamma_i \delta_i h_i d_n - \gamma_i h_i d_{i+1} + f_{i+1}(\beta_i + 3\gamma_i) - f_n(\beta_i + 3\gamma_i)\delta_i > 0$$

$$\gamma_i \delta_i h_i d_n - \gamma_i h_i d_{i+1} + \beta_i f_{i+1} + 3\gamma_i f_{i+1} - \beta_i \delta_i f_n - 3\gamma_i \delta_i f_n > 0$$

$$\gamma_i \delta_i h_i d_n - \gamma_i h_i d_{i+1} + 3\gamma_i (f_{i+1} - \delta_i f_n) + \beta_i (f_{i+1} - \delta_i f_n) > 0$$

$$\Rightarrow \beta_i (f_{i+1} - \delta_i f_n) > \gamma_i h_i d_{i+1} - \gamma_i \delta_i h_i d_n + 3\gamma_i (\delta_i f_n - f_{i+1})$$

$$\beta_i > \frac{\gamma_i h_i d_{i+1} - \gamma_i \delta_i h_i d_n + 3\gamma_i (\delta_i f_n - f_{i+1})}{f_{i+1} - \delta_i f_n} \quad (23)$$

Above inequality satisfies if and only if,

$f_{i+1} - \delta_i f_n > 0$

For $A_5 > 0$, put the value of $A_5$ from Eqn. (6) we get

$$\Rightarrow \gamma_i (f_{i+1} - \delta_i f_n) > 0$$

$$\Rightarrow \gamma_i f_{i+1} - \gamma_i \delta_i f_n > 0$$

Eliminate $\gamma_i$ from both sides of inequality, we'll get,

$$\Rightarrow f_{i+1} > \delta_i f_n$$

$$\Rightarrow \delta_i f_n < f_{i+1}$$

$$\Rightarrow \gamma_i f_{i+1} > \gamma_i \delta_i f_n$$

$$\delta_i < \frac{f_{i+1}}{f_n}; f_n \neq 0 \quad (24)$$

So the final positivity conditions are:

$$\beta_i = max\left\{\frac{\alpha_i h_i (\delta_i d_1 - d_i) + 3\alpha_i (\delta_i f_1 - f_i)}{f_i - \delta_i f_1}, \frac{0.5\alpha_i h_i^2 \delta_i D_1 - 0.5\alpha_i h^2 D_i + (3\alpha_i + \gamma_i)(\delta_i f_1 - f_i) + 2\alpha_i h_i (\delta_i d_1 - d_i)}{3(f_i - \delta_i f_1) + h_i (d_i - \delta_i d_1)}, \frac{(0.5\gamma_i h_i^2 \delta_i D_n - 0.5\gamma_i h_i^2 D_{i+1} + (\alpha_i + 3\gamma_i)(\delta_i f_n - f_{i+1}) + 2\gamma_i h_i (d_{i+1} - \delta_i d_n))}{3(f_{i+1} - \delta_i f_n) + h_i (\delta_i d_n - d_{i+1})}, \frac{\gamma_i h_i d_{i+1} - \gamma_i \delta_i h_i d_n + 3\gamma_i (\delta_i f_n - f_{i+1})}{f_{i+1} - \delta_i f_n}\right\} \quad (25)$$

and, $\delta_i = min\left\{\frac{f_i}{f_1}, \frac{f_{i+1}}{f_n}\right\}$ (26)

So these are the positivity conditions for rational quintic fractal interpolation function.

**Example:** Consider the data set {(0, 0), (3, 10), (5, 14), (8, 35), (9, 50)}. Then, FIF for various selection of vertical scaling factors are shown in Fig. 3. The graph of interpolated function is given in

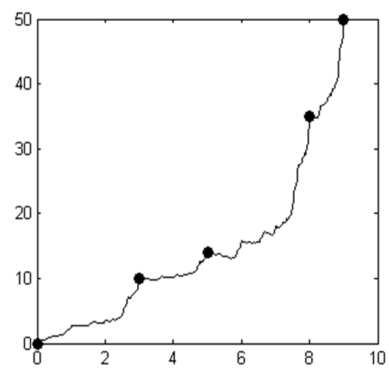


**Fig. 3.** Rational quintic fractal interpolation function.

## V. CONCLUSION

In this paper, we have introduced "$C^2$ Rational Quintic Fractal Interpolation Function" with six shape-parameters and three free-parameters. We further find the condition over vertical scaling factors and shape parameters for preserving the positivity condition.

## CONFLICT OF INTEREST

Authors have no any conflict of interest.

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