*International Journal on Emerging Technologies* **10**(2): 408-414(2019)

**ISSN No. (Print) : 0975-8364**
**ISSN No. (Online) : 2249-3255**

# Revisiting Analytical-Approximate Solution of Time Fractional Rosenau-Hyman Equation via Fractional Reduced Differential Transform Method

***Pankaj Ramani[1], Arif M. Khan[2] and D.L. Suthar[3]***
*[1]Department of Mathematics, Poornima University, Jaipur, India.*
*[2]Department of Mathematics, Jodhpur Institute of Engineering & Technology, Jodhpur India.*
*[3]Department of Mathematics, Wollo University, Dessie, Amhara, Ethiopia.*

*(Corresponding author: Pankaj Ramani)*
*(Received 30 May 2019, Revised 04 August 2019 Accepted 18 August 2019)*
*(Published by Research Trend, Website: www.researchtrend.net)*

**ABSTRACT: In this paper, an approximate analytical solution of time fractional Rosenau-Hyman equation arising in formation of pattern in liquid drops is obtained via fractional reduced differential transform method (FRDTM). The fractional derivatives are taken in Caputo sense. Approximate solution obtained by FRDTM is compared with the exact solutions and it is found that obtained results agreed excellently with exact solution. Then we provide a rigorous convergences analysis and error estimate of results obtained by FRDTM upto merely second approximation. Numerical simulations of the results are depicted through graphical presentations and comparison table of solutions obtained from other numerical solutions showing that present method gives reliable, efficient solution in the form of easily computable and convergent series.**

**Keywords:** Time fractional Rosenau-Hyman equation, Fractional reduced differential transform method, Analytical approximate solution.

**2010 mathematics subject classification**: 34A08, 35A20, 35A22.

## I. INTRODUCTION

The time fractional Rosenau-Hyman equation (FRH) arises in formation of pattern in liquid drops. Fractional Rosenau-Hyman equation studied and solved throughimplementation of VIM and HPM methods by Molliq and Noorani [1] which is given as

$$D_t^{\alpha} u = uD_{xxx}(u) + uD_x(u) + 3D_x(u)D_{xx}(u), \qquad (1)$$

with initial condition

$$u(x,0) = -\frac{8}{3}c\cos^2\left(\frac{x}{4}\right). \qquad (2)$$

Where 0<α ≤ 1, c is an arbitrary constant, t>0 When α = 1, equation (1) reduces to the classical RH equation [2].
Since several physical phenomena arising in engineering and applied sciences can be explained in better way by developing fractional order models. The fractional order equations response and consequent convergence to the integer order classical equations, has gain much attention amongst researcher now days.
The fractional calculus are effectively applied for mathematical modelling of real world problems, e.g. traffic flow models, liquid patterns, earthquake modelling, diffusion models, wave propagation [3-6]. Due to nonlinear nature of the differential equation it is difficult to find exact analytical solutions of these models hence several analytical approximate schemes e.g. ADM [7], MVIM [8, 9], DTM [10], HPM [11] are developed.
The major advantage of these approximate method is huge and complicated computations. To overcome these problems an analytical approximate method that is fractional reduced differential transform method (FRTDM) is developed by Keskin and Oturanc [12,13]. Many researchers explored the subject of fractional differential equations pertaining to different fractional operators like Clarkson and Mansfield [14] studied symmetries of a class of nonlinear third order partial differential equations, Mirza and Roy [15], studied the time fractional three dimensional thermoelastic problem of thin rectangular plate. Further Khan [16] obtained Complex order distribution and caputo fractional derivatives of I- function.
In this paper, we developed the time fractional Rosenau-Hyman equation using FRTDM. The results obtained merely upto second approximation are much better approximation and converge rapidly than VIM and HPM.
The paper is arranged as follows. In section II some basic definitions and preliminaries are given. Section III describes the method (FRTDM). Section IV, exact solutions of FRH equation is obtained via FRTDM and solution behaviour for different fractional order are shown graphically and error analysis and comparison table for solutions obtained from FRTDM, VIM and HPM are given. Section V concludes the present study.

## II. BASIC DEFINITIONS

**Definition 2.1.** The Riemann-Liouville fractional integral operator [3] of order $\alpha > 0$ of function $f : R^+ \to R$ is defined as

$$f\ (\mathrm{x}) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt,\ x > 0 \qquad (3)$$

**Definition 2.2.** The Caputo fractional derivative [3] of order$\alpha > 0,\ n-1 < \alpha \le n,\ x > 0\ n \in \mathbb{N}$ is defined as

$$D^{\alpha} f(x) = I^{n-\alpha} D^{n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t) dt \quad (4)$$

where f (t) has absolute continuous derivatives up to order (n-1).

## III. BASIC IDEA OF FRDTM

In this section, we describe the basic properties of the fractional reduced differential transform method [12].
Let $w(x,t)$ be function of two variables which can be presented as product of two single variable functions as follows

$$w(x,t) = \sum_{i=0}^{\infty} F(i) x^i \sum_{j=0}^{\infty} G(j) t^j = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} W(i,j) x^i t^j \quad (5)$$

Where $W(i,j)$ is spectrum of $w(x,t)$.
The fractional reduced differential transform function of $w(x,t)$, is given as

$$W_k(x) = \frac{1}{\Gamma(k\alpha+1)} \left[ D_t^k w(x,t) \right]_{t=t_0} \quad (6)$$

Where $\alpha$ is order of derivative $(0 < \alpha \le 1)$ and $w(x,t)$ is analytic and continuous differentiable with respect to x and t.
Further the inverse reduced differential transform of $W_k(x)$ is defined as

$$w(x,t) = \sum_{k=0}^{\infty} W_k(x) (t-t_0)^{k\alpha} \quad (7)$$

## IV. FRACTIONAL OPERATIONS OF FRDTM [13]

Let $u(x,t), v(x,t)$and $w(x,t)$are the analytic functions such as $u(x,t) = R_D^{-1}[U_k(x)]$, $v(x,t) = R_D^{-1}[V_k(x)]$ and $w(x,t) = R_D^{-1}[W_k(x)]$ then following properties holds

i. If $u(x,t) = v(x,t) \pm w(x,t)$, then $U_k(x) = V_k(x) \pm W_k(x)$
ii. If$u(x,t) = a v(x,t)$ then $U_k(x) = a V_k(x)$where $a$ is any constant.
iii. If $u(x,t) = x^m t^n v(x,t)$ then $U_k(x) = V_{k-n}(x)$
iv. If $u(x,t) = v(x,t).w(x,t)$ then $U_k(x) = \sum_{r=0}^{k} V_r(x) W_{k-r}(x)$
v. If $u(x,t) = \frac{\partial^r}{\partial x^r} v(x,t)$ then $U_k(x) = \frac{\partial^r}{\partial x^r} V_k(x)$
vi. If $u(x,t) = \frac{\partial^{\gamma\alpha}}{\partial t^{\gamma\alpha}} v(x,t)$ then $U_k(x) = \frac{\Gamma(\alpha k+\alpha\gamma+1)}{\Gamma(\alpha k+1)} V_{k+\gamma}(x)$
vii. If $u(x,t) = v_1(x,t) v_2(x,t) v_3(x,t)$, then $U_k(x) = \sum_{\gamma=0}^{k} \sum_{i=0}^{\gamma} U_i(x) U_{\gamma-i} U_{k-\gamma}(x)$
viii. If $u(x,t) = v_1(x,t) v_2(x,t) v_3(x,t) v_4(x,t)$, then $U_k(x) = \sum_{\gamma=0}^{k} \sum_{i=0}^{\gamma} \sum_{j=0}^{i} U_j(x) U_{i-j}(x) U_{\gamma-i}(x) U_{k-\gamma}(x)$
ix. If $u(x,t) = x^m t^n$, then $U_k(x) = x^m \delta(k-n)$, $\delta(k) = \begin{cases} 1; & k = 0 \\ 0; & k \ne 0 \end{cases}$

## V. FRACTIONAL ROSENAU HYMAN EQUATION VIA FRDTM

Applying FRTDM on equation(1), we find the following recurrence relation

$$U_{k+1}(x) = \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\alpha+1)} \left[ \sum_{r=0}^{k} U_r(x) \frac{\partial^3}{\partial x^3} U_{k-r}(x) + \sum_{r=0}^{k} U_r(x) \frac{\partial}{\partial x} U_{k-r}(x) + 3 \sum_{r=0}^{k} \frac{\partial}{\partial x} U_r(x) \frac{\partial^2}{\partial x^2} U_{k-r}(x) \right] \quad (8)$$

with initial conditions (2)

$$U_0(x) = -\frac{8}{3} c \cos^2 \frac{x}{4} \quad (9)$$

Using recurrence relation (8) and initial condition (9) we get
For k=0

$$U_1(x) = \frac{1}{\Gamma(\alpha+1)} \left( U_0(x) \frac{\partial^3}{\partial x^3} U_0(x) + U_0(x) \frac{\partial}{\partial x} U_0(x) + 3 \frac{\partial}{\partial x} U_0(x) \frac{\partial^2}{\partial x^2} U_0(x) \right)$$

$$U_1(x) = \frac{1}{\Gamma(\alpha+1)} \left[ -\frac{2}{3} c^2 \sin \frac{x}{2} \right]$$

For k=1, we get

$$U_2(x) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \left[ \sum_{r=0}^{1} U_r(x) \frac{\partial^3}{\partial x^3} U_{1-r}(x) + \sum_{r=0}^{1} U_r(x) \frac{\partial}{\partial x} U_{1-r}(x) + 3 \sum_{r=0}^{1} \frac{\partial}{\partial x} U_r(x) \frac{\partial^2}{\partial x^2} U_{1-r}(x) \right]$$

$$U_2(x) = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \left( U_0(x) \frac{\partial^3}{\partial x^3} U_1(x) + U_1(x) \frac{\partial^3}{\partial x^3} U_0(x) + U_0(x) \frac{\partial}{\partial x} U_1(x) + U_1(x) \frac{\partial}{\partial x} U_0(x) + 3 \frac{\partial}{\partial x} U_0(x) \frac{\partial^2}{\partial x^2} U_1(x) + 3 \frac{\partial}{\partial x} U_1(x) \frac{\partial^2}{\partial x^2} U_0(x) \right)$$

$$U_2(x) = \frac{c^3}{3\Gamma(2\alpha+1)} \cos\frac{x}{2}$$

and so on

$$U_3(x) = \frac{c^4}{6\Gamma(3\alpha+1)} \sin\frac{x}{2}$$
$$\vdots$$
$$U_4(x) = -\frac{c^5}{12\Gamma(4\alpha+1)} \cos\frac{x}{2}$$

Therefore the approximate solution of equation (1) is given as

$$u(x,t) = -\frac{8}{3} c \cos^2\frac{x}{4} - \frac{2c^2}{3}\frac{t^\alpha}{\Gamma(\alpha+1)} \sin\frac{x}{2} + \frac{c^3 t^{2\alpha}}{3\Gamma(2\alpha+1)} \cos\frac{x}{2} + \frac{c^4 t^{3\alpha}}{6\Gamma(3\alpha+1)} \sin\frac{x}{2} - \frac{c^5 t^{4\alpha}}{12\Gamma(4\alpha+1)} \cos\frac{x}{2} \dots$$

For $\alpha = 1$, we get

$$u(x,t) = -\frac{4c}{3} - \frac{4c}{3} \cos\frac{x}{2} - \frac{2c^2}{3} t \sin\frac{x}{2} + \frac{c^3}{6} t^2 \cos\frac{x}{2} + \frac{c^4}{36} t^3 \sin\frac{x}{2} - \frac{c^5}{(12)}\frac{t^4}{(24)} \cos\frac{x}{2} + \dots$$

$$u(x,t) = -\frac{4c}{3} - \frac{4c}{3} \cos\frac{x}{2}\left(1 - \frac{\left({ct}/{2}\right)^2}{2!} + \frac{\left({ct}/{2}\right)^4}{4!} \dots\right) - \frac{4c}{3} \sin\frac{x}{2}\left(\left(\frac{ct}{2}\right) - \frac{\left({ct}/{2}\right)^3}{3!} + \cdots\right)$$

$$= -\frac{4c}{3}\left[1 + \cos\frac{x}{2}\cos\frac{ct}{2} + \sin\frac{x}{2}\sin\frac{ct}{2}\right]$$

$$= -\frac{4c}{3}\left[1 + \cos\left(\frac{x - ct}{2}\right)\right]$$

which is the exact solution [14], where $|(x - ct)| \le 2\pi$.

**Table 1: Error analysis of the II approximate solution of Fractional Rosenau-Hyman when** α = 1 **and c=1.**

| x | t | II-Approx. result by FRDTM | Exact | Absolute Error = $\lvert U_{FRDTM} - U_{Exact}\rvert$ |
|---|---|---|---|---|
| $\pi/4$ | 0.2 | -2.6100 | -2.6100 | 0.0000 |
| | 0.4 | -2.6426 | -2.6420 | 0.0006 |
| | 0.6 | -2.6628 | -2.6609 | 0.0019 |
| $\pi/2$ | 0.2 | -2.3657 | -2.3656 | 0.0001 |
| | 0.4 | -2.4458 | -2.4447 | 0.0011 |
| | 0.6 | -2.5166 | -2.5127 | 0.0039 |
| $3\pi/4$ | 0.2 | -1.9642 | -1.9640 | 0.0002 |
| | 0.4 | -2.0797 | -2.0781 | 0.0016 |
| | 0.6 | -2.1902 | -2.1848 | 0.0054 |
| $\pi$ | 0.2 | -1.4667 | -1.4664 | 0.0003 |
| | 0.4 | -1.6000 | -1.5982 | 0.0018 |
| | 0.6 | -1.7333 | -1.7274 | 0.0059 |

**Table 2: Fifth term solution through VIM and HPM [1] when** α = 1.

| x | t | VIM | HPM |
|---|---|---|---|
| $\pi/4$ | 0.2 | -2.6099 | -2.6099 |
| | 0.6 | -2.6609 | -2.6609 |
| | 1.0 | -2.6590 | -2.6589 |
| $\pi/2$ | 0.2 | -2.3655 | -2.3655 |
| | 0.6 | -2. 5126 | -2.5126 |
| | 1.0 | -2.6127 | -2. 6127 |
| $3\pi/4$ | 0.2 | -0.4893 | -0.4893 |
| | 0.6 | -0.71125 | -0.71125 |
| | 1.0 | -0.9579 | -0.9579 |
| $\pi$ | 0.2 | -1.4664 | -1.4664 |
| | 0.6 | -1.7273 | -1.7273 |
| | 1.0 | -1.9725 | -1. 9725 |

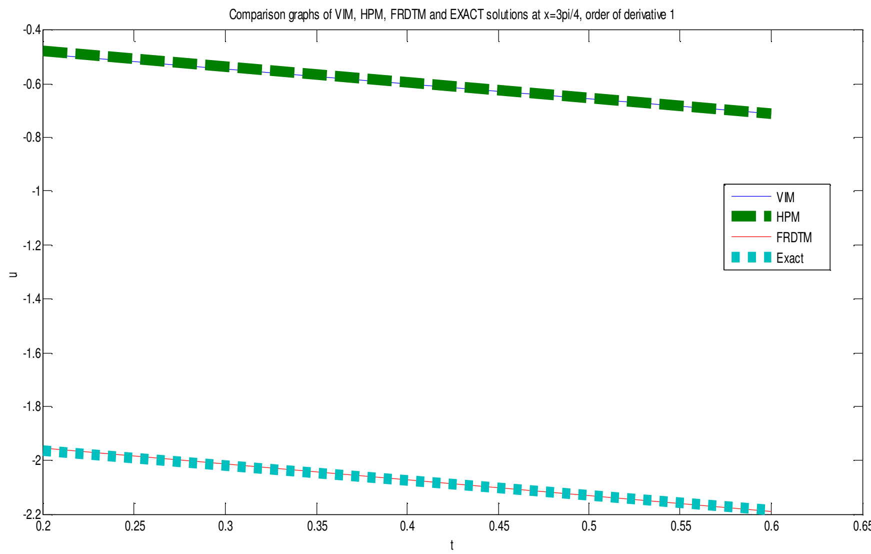


**Fig. 1.** Comparison graph of Exact solution and solutions obtained by FRDTM, VIM, HPM when x=3*pi/4, α = 1.

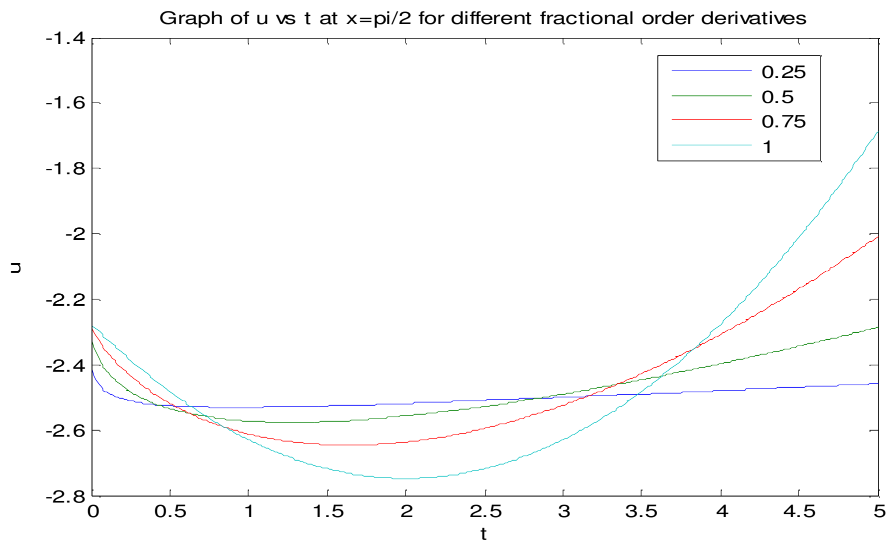


**Fig. 2.** Approximate solution u(x, t) obtained by FRDTM for x = pi/2 when different values of α.

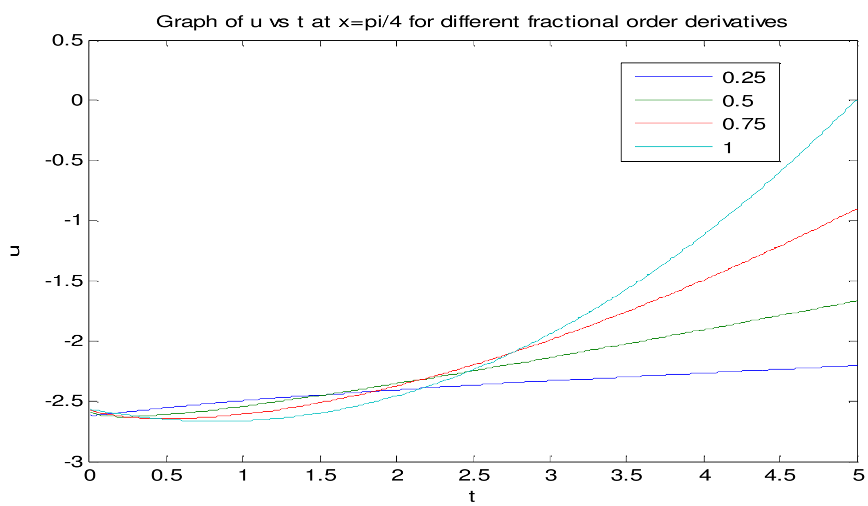


**Fig. 3.** Approximate solution u(x, t) obtained by FRDTM for x = pi/4 when different values of α.

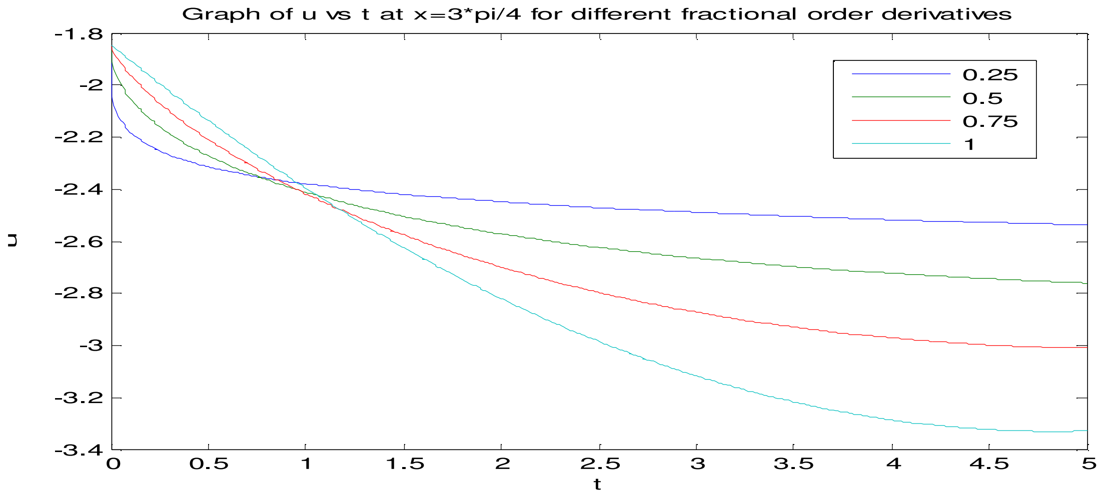


**Fig. 4.** Approximate solution u(x, t) obtained by FRDTM for x = 3* pi/4 when different values of α.

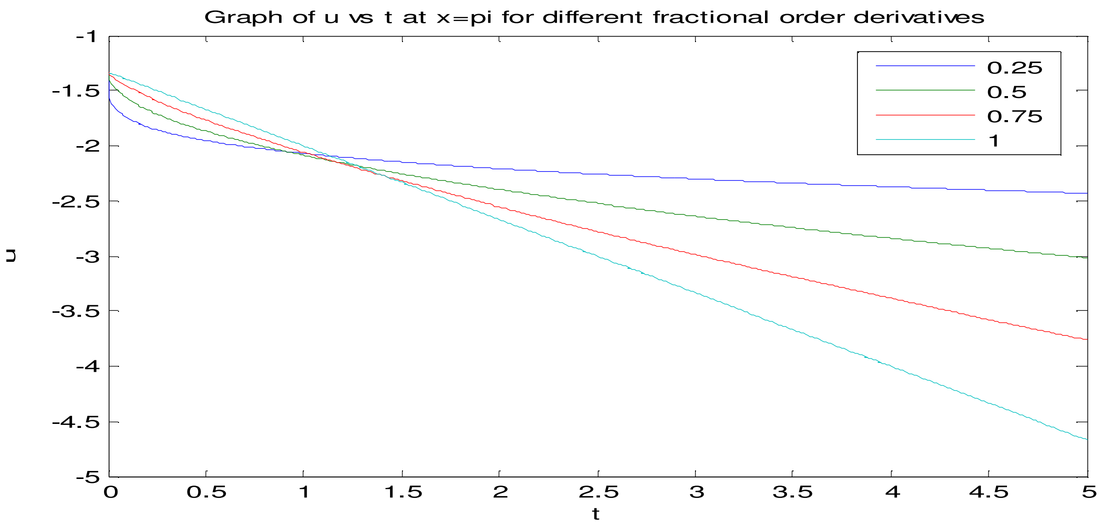


**Fig. 5.** Approximate solution u(x, t) obtained by FRDTM for x = pi when different values of α.

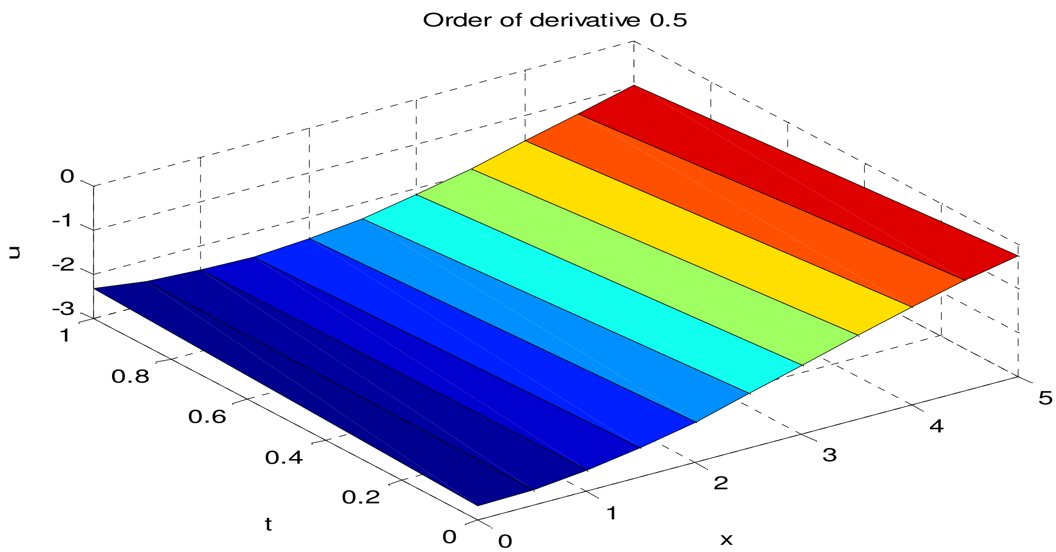


**Fig. 6.** Phase plot of solution behaviour u(x, t) at α = 0.5.

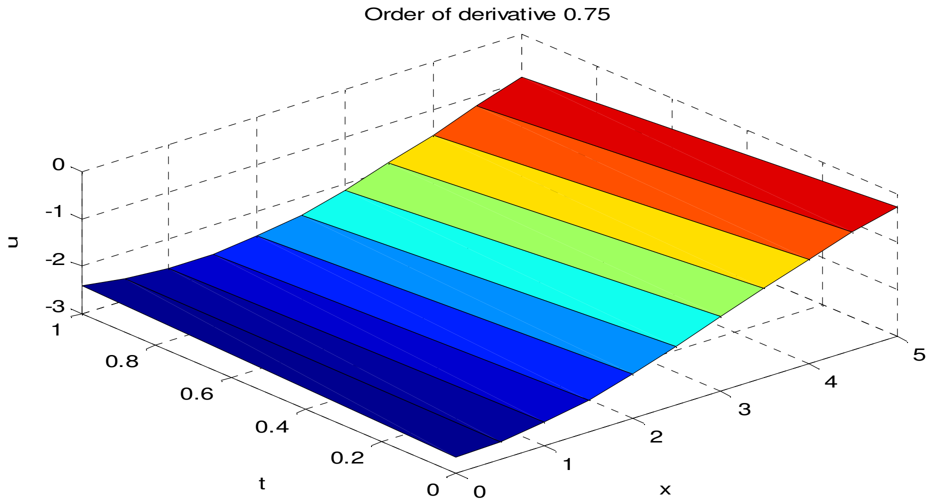


**Fig. 7.** Phase plot of solution behaviour u(x, t) when α = 0.75.

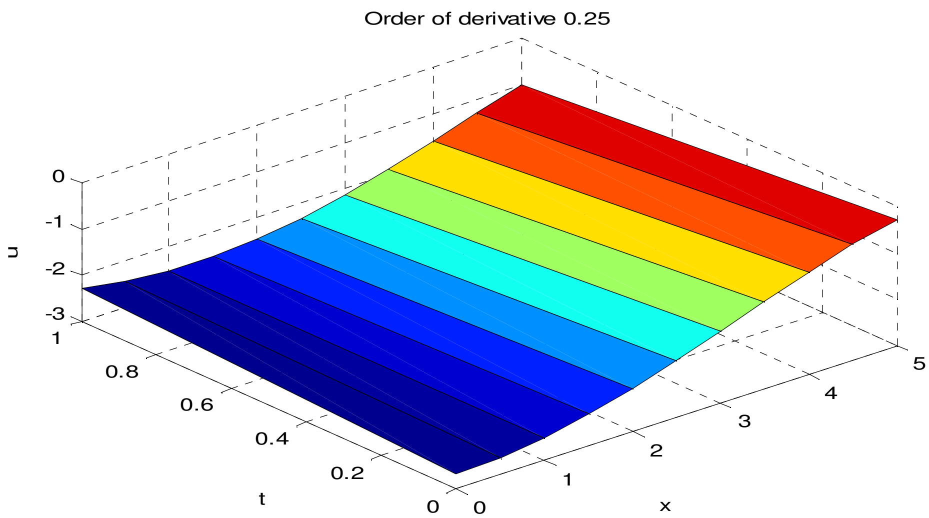


**Fig. 8.** Phase plot of solution behaviour u(x, t) at α = 0.25.

## VI. NUMERICAL RESULTS AND DISCUSSIONS

The comparison of the results obtained of the FRDTM merely at second approximation and the exact solution for α = 1 is given in table 1 for different values of x, t. Table 2 show the approximate solution using fifth iterates of VIM and HPM for α = 1, at different values of x, t. Fig. 1 show that the solution using second iterates of FRDTM converges faster than solution using fifth iterates of VIM and HPM when x=3*pi/4, α = 1. Variation of solution behaviour for different fractional orders are shown by Fig. 2 to 5 when x = pi/2, x=pi/4, x=3*pi/4, x=pi. Finally we present phase plot of u(x, t) for the different values of$\alpha$through Fig. 6-8.

## VII. CONCLUSION

In the present paper the fractional order reduced differential transform method is applied for the Caputo time fractional order Rosenau-Hyman equation arising in the formation of pattern in liquid drops. The proposed solution of FRH equation with an initial condition is obtained in terms of power series, without involving the discretization, perturbation and He's polynomials.Solution obtained by the method agree excellently with fifth terms solutions of VIM and HPM at just second approximation. Performed computation shows that the method is easy to implement with small size of computation and converge faster than approximate solution using the VIMand HPM. Therefore it is very effective and efficient semi-analytical method for recent appearance of non-linear fractional differential equations arises in some fields of applied mathematics.

## ACKNOWLEDGMENT

The authors are grateful to the Dean Research, Poornima University Jaipur and Dr. Dinesh Kumar from Jai Narain Vyas University for providing their valuable suggestions.

**Conflict of Interest:** The authors declares that have no conflicts of interest.

## REFERENCES

[1]. Molliq, R.Y. and Noorani, M.S.M., (2012). Solving the Fractional Rosenau-Hyman Equation via Variational Iteration Method and Homotopy Perturbation Method. Hindawi Publication corporation, IJDE, Article ID:**472030**.

[2]. Rosenau, P. and Hyman, J.M. (1993). Compactons, solitons with finite wavelength. *Physical Review Letters,* Vol. **70**(5): 564-567.
[3]. Hilfer, R., (2000). Application of Fractional Calculus in Physics. World Scientific, Singapore.
[4]. Kilbas, A.A., Srivastava, H.H., Trujillo, J.J., (2006). Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam.
[5]. Cascaval, R.C., Eckstein, E.C., Frota, C.L., Goldstein, J.A., (2002). Fractional telegraph equations. *J. Math. Anal. Appl.,* Vol. **276**(1): 145-159.
[6]. Yang, X.J., (2016). Fractional derivatives of constant and variable orders applied to anomalous relaxation models in heat transfer problems. Arxiv:**1612**.03202.
[7]. Garg, M, Sharma, A., (2011). Solution of space time fractional telegraph equation by Adomian decomposition method. *J. Inequal. Spec. Funct.,* Vol. **2**(1): 1-7.
[8]. Khan A.M., Lalita Mistry,(2016).Stability Analysis and Numerical Solution for the Fractional order Biochemical Reaction Model. *Nonlinear Analysis and Differential Equations HIKARI* Ltd., Vol. **4**(11): 521 – 530.
[9]. Khan A.M., and Amit Chouhan**,** (2016). Approximate Solution of the Fractional Susceptible-Infected-Recovered model by Modified Variational Iteration Method. *Journal of Fractional Calculus and Applications* (Egypt), Vol. **7**(1): 147-153.
[10]. Ravi Kant, A.S.V., Aruna, K., (2008). Differential transform method for solving linear and nonlinear systems of partial differential equations. *Phys. Lett. A ,* Vol. **372**(46): 6896-6898.
[11]. Kumar, S., Singh, O.P. (2010). Numerical Inversion of the abel integral equation using homotopy perturbation method. *Z Naturforsch*, Vol. **65**(1): 677-682.
[12]. Keskin, Y. and Oturanc, G. (2010). Reduced differential transform method, a new approach to fractional partial differential equations. *Nonlinear Sci. Lett.,* Vol. **1**(1): 61-72.
[13]. Keskin, Y., (2010). Ph.D. Thesis. Selcuk University (in Turkish).
[14]. Clarkson, P.A. and Mansfield, E.L. (1997). Priestley, Symmetries of a class of nonlinear third order partial differential equations. *Mathematical and computing modelling,* Vol. **25**(8): 195-212.
[15]. Hamna Mirza and Roy, H.S., (2019). Study of time fractional three dimensional thermoelastic problem of thin rectangular plate. *International Journal of Theoretical & Applied Sciences,* Vol. **11**(1): 52-63.
[16]. Khan, A.M., (2013). Complex order distribution and caputo fractional derivatives of I- function. International *Journal of Theoretical & Applied Sciences*, Vol. **5**(1)**:** 9-13.

**How to cite this article:** Ramani, P., Khan, Arif M. and Suthar, D.L. (2019). Revisiting Analytical-Approximate Solution of Time Fractional Rosenau-Hyman Equation via Fractional Reduced Differential Transform Method. *International Journal of Emerging Technologies*, **10**(2): 408–414.