International Journal on Emerging Technologies **10**(2b): 105-108(2019)

ISSN No. (Print) : 0975-8364
ISSN No. (Online) : 2249-3255

Self-Comparison of Convergence Speed in Agarwal, O'Regan and Sahu's S-Iteration

Naveen Kumar[1] and Surjeet Singh Chauhan Gonder[2]
[1,2]Department of Mathematics, Chandigarh University, Gharuan, Mohali-140413, Punjab, India.

(Corresponding author: Naveen Kumar)
(Received 02 May 2019, Revised 10 July 2019, Accepted 08 August 2019)
(Published by Research Trend, Website: www.researchtrend.net)

ABSTRACT: We examine the self-comparison of the speed of convergence of the iterative method Agarwal, O'Regan & Sahu's S-iteration [4] (see also [5]) with rate of convergence of the iterations obtained after interchanging the coefficients introduced in the original one. This paper shows that the interchange of the coefficients introduced in such type of schemes play a vital role in self-comparing. We self-compare the convergence rate for this particular iteration method and provide a numerical to illustrate the result which explains the justification of the existence of the previous method.

Keywords: Contractive mapping; Agarwal, O'Regan & Sahu's S-iteration; speed of convergence.

I. INTRODUCTION

Although several authors have compared the fastness of convergence of many iteration plans like Berinde [1], in 2004, compared Picard iteration with Mann iteration, in 2006, B.V. Prasadnc [2] compared Mann iteration with Ishikawa iteration, in 2007, Popescu [3] compared Picard iteration with Mann iteration for their rate of convergence and S. Fathollahi [11] has compared the speed of convergence of several iterative plans and found that the coefficients included in these types of iterative schemes have a vital part in finding their convergence speed, in this review paper we analyse the convergence rate of Agarwal, O'Regan & Sahu's S-iteration procedure.

II. PREMILINARIES

Definition 2.1 [5] Assume (M*, d) as a complete metric space and Q: M*→ M* as a self-map of M*. We define a set containing all fixed points of Q as

$$F_Q = \{m \in M^* \mid Qm = m\}$$

Then

$$d(Qm, Qn) \le a\, d(m, n) \quad \text{for all } m, n \in M^* \text{ and } a \in [0, 1)$$

is known as the Banach contraction's condition.

Definition 2.2 [17] Let $\{u_m\}$ and $\{v_m\}$ are two real sequences of converging to 'u' and 'v' separately. Consider

$$\lim_{n\to\infty} \left| \frac{u_m - u}{v_m - v} \right| = l'$$

(1) If $l' = 0$, then $\{u_m\}_{m=0,1,2,...}$ is said to fast converge to 'u' than $\{v_m\}_{m=0,1,2,...}$ to 'v'.

(2) If $0 < l' < 1$, then $\{u_m\}$ and $\{v_m\}$ have the same convergence rate.

Definition 2.3 [4] ([5] and [16]) The Agarwal, O'Regan and Sahu's S-iteration is given as:

$$u_{n+1} = (1 - a_n)\, Tu_n + a_n Tv_n$$
$$v_n = (1 - b_n)\, u_n + b_n Tu_n$$

where $\{a_n\}$ and $\{b_n\}$ are the sequences of positive numbers in [0, 1].

III. RESULT AND DISCUSSION

Consider all the possible cases of Agarwal, O'Regan and Sahu's S-iteration emphasis after interchanging the associated coefficients:

$$u_{n+1} = (1 - a_n)\, Tu_n + a_n Tv_n$$
$$v_n = (1 - b_n)\, u_n + b_n Tu_n\, n \ge 1 \qquad (1)$$

and

$$u_{n+1} = a_n Tu_n + (1 - a_n)\, Tv_n$$
$$v_n = b_n u_n + (1 - b_n)\, Tu_n\, n \ge 1 \qquad (2)$$

and

$$u_{n+1} = (1 - a_n)\, Tu_n + a_n Tv_n$$
$$v_n = b_n u_n + (1 - b_n) Tu_n\, n \ge 1 \qquad (3)$$

and

$$u_{n+1} = a_n Tu_n + (1 - a_n) Tv_n$$
$$v_n = (1 - b_n)\, u_n + b_n Tu_n\, n \ge 1 \qquad (4)$$

Let $\{u_n\}$ be the sequence in the iteration plan (1), Then:

$$\begin{aligned}
\|v_n - p\| &= \|(1 - b_n)\, u_n + b_n Tu_n - p\| \\
&= \|(1 - b_n)\, u_n + b_n Tu_n - b_n p + b_n p - p\| \\
&\le b_n \|Tu_n - p\| + (1 - b_n)\, \|u_n - p\| \\
&\le \lambda b_n \|u_n - p\| + (1 - b_n)\, \|u_n - p\| \\
&\le [\lambda\, b_n + (1 - b_n)]\, \|u_n - p\|
\end{aligned}$$

Also,

$$\begin{aligned}
\|u_{n+1} - p\| &= \|(1 - a_n)\, Tu_n + a_n Tv_n - p\| \\
&\le (1 - a_n)\, \|Tu_n - p\| + a_n \|Tv_n - p\| \\
&\le \lambda\, (1 - a_n)\, \|u_n - p\| + a_n \lambda\, \|v_n - p\|
\end{aligned}$$

$$\le \lambda\, (1 - a_n)\, \|u_n - p\| + a_n \lambda\, [\lambda\, b_n + (1 - b_n)]\, \|u_n - p\|$$
$$= [\lambda\, (1 - a_n) + \lambda^2 a_n b_n + \lambda\, a_n (1 - b_n)]\, \|u_n - p\|$$

Since $a_n, b_n \in (\frac{1}{2}, 1)$ for all $n \ge 1$,

$1 - a_n < \frac{1}{2}$ and $a_n b_n < 1$ and $a_n (1 - b_n) < \frac{1}{2}$

Thus

$$[\lambda\, (1 - a_n) + \lambda^2 a_n b_n + \lambda\, a_n (1 - b_n)] < \tfrac{1}{2}\lambda + \lambda^2 + \tfrac{1}{2}\lambda$$
$$= \lambda + \lambda^2$$

Take $p_n = [\lambda + \lambda^2]^n \, \| u_1 - p \|$

Let $\{u_n\}$ be the sequence in the iteration plan (3.12), we obtain:

$$\begin{aligned}
\|v_n - p\| &= \|b_n u_n + (1 - b_n)\, Tu_n - p\| \\
&\le b_n \|u_n - p\| + (1 - b_n)\, \|Tu_n - p\| \\
&\le b_n \|u_n - p\| + (1 - b_n)\, \lambda\, \|u_n - p\| \\
&= [b_n + \lambda\, (1 - b_n)\, \|u_n - p\| \quad \text{for all } n \ge 1.
\end{aligned}$$

and

$$\begin{aligned}
\|\underline{u}_{n+1} - p\| &= \|a_n Tu_n + (1 - a_n) Tv_n - p\| \\
&\le a_n \|Tu_n - p\| + (1 - a_n)\, \|Tv_n - p\| \\
&< \lambda\, a_n\, \|u_n - p\| + (1 - a_n)\, \lambda\, \|v_n - p\| \\
&\le \lambda\, a_n\, \|u_n - p\| + (1 - a_n)\, \lambda\, [b_n + \lambda\, (1 - b_n)]
\end{aligned}$$

$\|u_n - p\|$

$$\le [\lambda\, a_n + \lambda\, (1 - a_n)\, b_n + \lambda^2 (1 - a_n)\, (1 - b_n)]$$

$\|u_n - p\|$

Since $a_n, b_n \in (\frac{1}{2}, 1)$ for all $n \geq 1$

$a_n < 1$ and $(1 - a_n) b_n < \frac{1}{2}$ and $(1 - a_n)(1 - b_n) < \frac{1}{4}$

Thus

$[\lambda a_n + \lambda (1 - a_n) b_n + \lambda^2 (1 - a_n)(1 - b_n)] < \lambda + \frac{1}{2}\lambda + \frac{1}{4}\lambda^2$

$= \frac{6\lambda + \lambda^2}{4}$

Take $q_n = \left[\frac{6\lambda + \lambda^2}{4}\right]^n \| u_1 - p \|$

Let $\{u_n\}$ is the sequence in the iteration plan (3.13), we obtain:

$\|v_n - p\| = \|b_n u_n + (1 - b_n) T u_n - p\|$
$\leq b_n \|u_n - p\| + (1 - b_n) \|T u_n - p\|$
$\leq b_n \|u_n - p\| + (1 - b_n) \lambda \|u_n - p\|$
$= [b_n + \lambda (1 - b_n)] \|u_n - p\|$ for all $n \geq 1$.

Again

$\|u_{n+1} - p\| = \|(1 - a_n) T u_n + a_n T v_n - p\|$
$\leq (1 - a_n) \|T u_n - p\| + a_n \|T v_n - p\|$
$\leq \lambda (1 - a_n) \|u_n - p\| + a_n \lambda \|v_n - p\|$
$\leq \lambda (1 - a_n) \|u_n - p\| + a_n \lambda [b_n + k(1 - b_n)] \|u_n - p\|$
$\leq [\lambda (1 - a_n) + \lambda a_n b_n + \lambda^2 a_n (1 - b_n)] \|u_n - p\|$

Since $a_n, b_n \in (\frac{1}{2}, 1)$ for all $n \geq 1$,

$(1 - a_n) < \frac{1}{2}$ and $a_n b_n < 1$ and $a_n (1 - b_n) < \frac{1}{2}$

Thus

$[\lambda (1 - a_n) + \lambda a_n b_n + \lambda^2 a_n (1 - b_n)] < \frac{1}{2}\lambda + \lambda + \frac{1}{2}\lambda^2$

$= \frac{3\lambda + \lambda^2}{2}$

Take $r_n = \left[\frac{3\lambda + \lambda^2}{2}\right]^n \| u_1 - p \|$

Let $\{u_n\}$ is the sequence in the iteration (3.14), we get:

$\|v_n - p\| = \|(1 - b_n) u_n + b_n T u_n - p\|$
$= \|(1 - b_n) u_n + b_n T u_n - b_n p + b_n p - p\|$
$\leq b_n \|T u_n - p\| + (1 - b_n) \|u_n - p\|$
$\leq k b_n \|u_n - p\| + (1 - b_n) \|u_n - p\|$
$= [k b_n + (1 - b_n)] \|u_n - p\|$

and

$\|u_{n+1} - p\| = \|a_n T u_n + (1 - a_n) T v_n - p\|$
$\leq a_n \|T u_n - p\| + (1 - a_n) \|T v_n - p\|$
$< \lambda a_n \|u_n - p\| + (1 - a_n) \lambda \|v_n - p\|$
$\leq \lambda a_n \|u_n - p\| + (1 - a_n) \lambda [\lambda b_n + (1 - b_n)] \|u_n - p\|$
$\leq [\lambda a_n + \lambda^2 (1 - a_n) b_n + \lambda (1 - a_n)(1 - b_n)] \|u_n - p\|$

Since $a_n, b_n \in (\frac{1}{2}, 1)$ for all $n \geq 1$,

$a_n < 1$ and $(1 - a_n) b_n < \frac{1}{2}$ and $(1 - a_n)(1 - b_n)] < \frac{1}{4}$

Thus

$[\lambda a_n + \lambda^2 (1 - a_n) b_n + \lambda (1 - a_n)(1 - b_n)] < \lambda + \frac{\lambda^2}{2} + \frac{\lambda}{4}$

$= \frac{2\lambda^2 + 5\lambda}{4}$

Take $s_n = \left[\frac{2\lambda^2 + 5\lambda}{4}\right]^n \| u_1 - p \|$

Major comparison of all iterations:

$$\lim_{n \to \infty} \left| \frac{p_n}{q_n} \right| = \frac{[\lambda + \lambda^2]^n \|u_1 - p\|}{\left[\frac{6\lambda + \lambda^2}{4}\right]^n \|u_1 - p\|} = 0.$$

$$\lim_{n \to \infty} \left| \frac{q_n}{r_n} \right| = \frac{\left[\frac{6\lambda + \lambda^2}{4}\right]^n \|u_1 - p\|}{\left[\frac{3\lambda + \lambda^2}{2}\right]^n \|u_1 - p\|} = 0.$$

$$\lim_{n \to \infty} \left| \frac{r_n}{s_n} \right| = \frac{\left[\frac{3\lambda + \lambda^2}{2}\right]^n \|u_1 - p\|}{\left[\frac{2\lambda^2 + 5\lambda}{4}\right]^n \|u_1 - p\|} = 0.$$

$$\lim_{n \to \infty} \left| \frac{p_n}{r_n} \right| = \frac{[\lambda + \lambda^2]^n \|u_1 - p\|}{\left[\frac{3\lambda + \lambda^2}{2}\right]^n \|u_1 - p\|} = 0.$$

$$\lim_{n \to \infty} \left| \frac{p_n}{s_n} \right| = \frac{[\lambda + \lambda^2]^n \|u_1 - p\|}{\left[\frac{2\lambda^2 + 5\lambda}{4}\right]^n \|u_1 - p\|} = 0.$$

$$\lim_{n \to \infty} \left| \frac{q_n}{s_n} \right| = \frac{\left[\frac{6\lambda + \lambda^2}{4}\right]^n \|u_1 - p\|}{\left[\frac{2\lambda^2 + 5\lambda}{4}\right]^n \|u_1 - p\|} = 0.$$

Based on the above comparisons, we analyse that the iterative scheme (1) converges speedier than (2), (2) converges speedier as that of (3), (2) converges speedier as that of (4), (3) converges speedier as compared with (4) and (1) converges speedier than (3). So all in all, we examine that the original iterative scheme (1) gives fastest convergence rather that the cases (2), (3) and (4).

IV. MATHEMATICAL ILLUSTRATION

Let us take the interval [1, 60], $x_0 = 20$, $a_n = 0.75$, $b_n = 0.85$ for all n = 0, 1, 2, . . .

Define T: C → C by $T(x) = \sqrt{x}$ with $x \in C$, where T is contraction map.

In following two Tables, we compare the four cases of the Agarwal, O'Regan and Sahu's S-iteration:

$u_{n+1} = (1 - a_n) T u_n + a_n T v_n$
$v_n = (1 - b_n) u_n + b_n T u_n \quad n \geq 1$. . (5)

and

$u_{n+1} = a_n T u_n + (1 - a_n) T v_n$
$v_n = b_n u_n + (1 - b_n) T u_n \; n \geq 1$. . (6)

and

$u_{n+1} = (1 - a_n) T u_n + a_n T v_n$
$v_n = b_n u_n + (1 - b_n) T u_n \; n \geq 1$ (7)

and

$u_{n+1} = a_n T u_n + (1 - a_n) T v_n$
$v_n = (1 - b_n) u_n + b_n T u_n \; n \geq 1$. . (8)

V. CONCLUSION

Based on the entire study conducted on the speed of convergence of Agarwal, O'Regan and Sahu's S-iteration scheme, we examine that the original two step iteration (5) in the Agarwal, O'Regan and Sahu's S-iteration plan converges speedier as compared with the other iterations (6), (7) and (8) obtained after interchange of the coefficients introduced in the (6), (7) and (8) Agarwal, O'Regan and Sahu's S-iteration. So we conclude that the interchange of coefficients may effect the result in self-comparing of iterations.

ACKNOWLEDGEMENT

The authors would like to express their gratitude to the referee for his valuable comments and suggestions that led to a significant improvement of the manuscript.

CONFLICT OF INTEREST

Author has no any conflict of interest.

Table 1: Cases of the Agarwal, O'Regan and Sahu's S-iteration.

Iteration	S-iteration u_{n+1} (4.1)	S-iteration u_{n+1} (4.2)	S-iteration u_{n+1} (4.3)	S-iteration u_{n+1} (4.4)
1	3.07398388	4.40501883	4.27078460	4.00608526
2	1.48600908	2.07778971	2.00538370	1.88046605
3	1.14631646	1.43307945	1.39244543	1.32927412
4	1.04781320	1.19339398	1.16983456	1.1362056
5	1.01605890	1.09068721	1.07698647	1.05882987
6	1.00544396	1.04352663	1.03565157	1.02589524
7	1.00185133	1.02112747	1.01667510	1.01149426
8	1.00063025	1.01031140	1.00783588	1.00512108
9	1.00021463	1.00504603	1.00369030	1.00228541
10	1.00007310	1.00247258	1.00173974	1.00102068
11	1.00002489	1.00121235	1.00082058	1.00045599
12	1.00000847	1.00059462	1.00038713	1.00020374
13	1.00000288	1.00029169	1.00018265	1.00009104
14	1.00000097	1.0001431	1.00008618	1.00004068

Table 2: Cases of the Agarwal, O'Regan and Sahu's S-iteration.

Iteration	S-iteration v_n(4.1)	S-iteration v_n (4.2)	S-iteration v_n (4.3)	S-iteration v_n (4.4)
1	6.80131556	17.67082039	17.67082039	6.80131556
2	1.95138391	4.05908805	3.94015506	2.30220541
3	1.25906812	1.98233935	1.91699350	1.44767563
4	1.08200990	1.39768447	1.36058150	1.17939139
5	1.02725533	1.17824874	1.15659771	1.07647087
6	1.00920668	1.08373808	1.07110543	1.03346988
7	1.00312713	1.04022736	1.03295428	1.01481942
8	1.00106415	1.01953462	1.01541929	1.00659524
9	1.00036235	1.00953606	1.00724704	1.00294184
10	1.00012341	1.00466710	1.00341327	1.00131355
11	1.00004203	1.00228702	1.00160920	1.00058678
12	1.00001431	1.00112139	1.00075902	1.00026217
13	1.00000487	1.00055001	1.00035809	1.00011714
14	1.00000097	1.00026981	1.00016895	1.00005234

From Table 1 and Table 2, we observe that the iterative scheme (5) converges quicker than (6), (7) converges speedier than (7), (6) converges faster than (8), (7) converges quicker than (7) and (6) converges speedier than (4.3). Hence the original iterative scheme (4.1) has the fastest convergence than the other iterations (6), (7) & (8).

REFERENCES

[1] Berinde, V. (2004). Picard iteration converges faster than the Mann iteration for a class of quasi-contractive operators. *Fixed Point Theory Appl.*, (2), 97-105.

[2] Babu, G. V. R. & Vara Prasad, K. N. V. V. (2006). Mann iteration converges faster than Ishikawa iteration for the class of Zamfirescu operators. *Fixed Point Theory Appl.*, Article ID 49615.

[3] Popescu, O. (2007). Picard iteration converges faster than Mann iteration of quasi-contractive operators. *Math.Commun.***12**(2): 195-202.

[4] Mogbademu, A. A. (2012). Modified multi-step iterative methods for a family of p-strongly pseudo contractive maps.*Bulletin of Mathematical Analysisand Applications,* 83-90.

[5] Babu, G. V. R. and Vara Prasad, K. N. V. V. (2006) Comparison of fastness of the convergence among Krasnoselskij, Mann and Ishikawa iterations in Banach spaces. *Fixed Point Theory Appl.* Article ID 35704.

[6] Noor, M. A..(2000). New approximation schemes for general variational inequalities.*Journal of Mathematical Analysis and Applications*, Vol. **251**, No.1,pp 217–229.

[7] Mogbadem, A. A. (2006). Modified Noor iterative Procedure for three asymptotically Pseudo contractive maps, *International Journal of Open Problems, Compt., Mathematics*. (6), 95-102.

[8] Zhiqun Xue & Guiwen, Lv. (2013).The convergence of the modified Mann and Ishikawa iterations in Banach spaces.*Journal of Inequalities and Applications*, 188.

[9] Chang, S. S. (2001). Iterative approximation problem of fixed point for asymptotically non expansive mappings in Banach spaces.*Acta Math. Appl. Sin.* **24:** 236-241.

[10] Olisama, V. O., Mogbademu, A. A. & Olaleru, J.O. (2014). Modified Multi-Step Iterative Processes for a Finite Family of Asymptotically Pseudocontrative Maps. *Applied Mathematical Sciences*. Vol. **8**, no. 19: 901 – 910.

[11] Fathollahi, S., Ghiura, A., Postolache, M., and Rezapour, S. (2015). A comparative study on the convergence rate of some iteration methods involving contractive mappings. *Fixed Point Theory Appl.*

[12] Akewe, H., Olaoluwa, H. (2012). On the Convergence of Modified three-step Iteration process for generalized Contractive-like operators. *Bulletin of Mathematical Analysis and Applications*, Volume 4 Issue**3:** 78-86.

[13] Kima, T. H., Xu, & Stron, H. K. (2005). Convergence of modified Mann iterations.*Non-linear Analysis*, **61**: 51– 60.

[14] Saluja, G. S. (2015). Convergence of Modified S-iteration Process for Two Generalized Asymptotically

Quasi-nonexpansive Mappings in CAT (0) Spaces. *Mathematica Moravica*, Vol. **19**, 1: 19–31.
[15] Bosede, A. O. (2010). Strong Convergence results for the Jungck–Mann iteration processes. *Bulletin of Math. Anal. Appl.* **2**(3): 65-73.
[16] Chauhan, S.S., Utreja, K., Imdad, M., Ahmadullah, Md. (2017).Strong Convergence Theorems for a Quasi-contractive type Mapping Employing a new Iterative Scheme with an Application. *Honam Mathematical J.* **39**, No. 1, pp. 1-25.
[17] Kumar, N., Chauhan & Gonder, S. S. (2018). Analysis of Jungck-Mann and Jungck-Ishikawa iteration schemes for their speed of convergence.*AIP Publishing*.
[18] Kumar, N., Chauhan & Gonder, S. S. (2018). A Review on the convergence speed in the Agarwal etal.andModified-Agarwal Iterative Schemes.*Universal Review*, Vol. **7**, Issue X,pp. 163-167.
[19] Kumar N., Chauhan & Gonder, S. S. (2018). An Illustrative Analysis of Modified-Agarwal and Jungck-Mann Iterative Procedures for their Speed of Convergence, Universal *Review*, pp. 168-173.
[20] Kumar, N., Chauhan & Gonder, S. S. (2018). Examination of the Speed of Convergence of the Modified-Agarwal Iterative Scheme. *Universal Review*, pp. 174-179.