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# Some Coupled Coincidence Point Result Using Altering Distances with Ordered Metric Spaces

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**ABSTRACT: In this manuscript, we use mixed g -monotone property to find results on coupled coincidence point for nonlinear contractive maps under ordered metric spaces by using altering distances function. So many methods are in existing literature to prove the result on coupled coincidence point. In this manuscript we are using g- monotone property which provides the effective result. We provide examples to support the result. Also, an application for integral equations given to help of these outcomes.**

**Keywords:** Coupled coincidence fixed point, partial ordered, complete metric.

2010 AMS Subject Classification: 47H10, 54H25.

## I. INTRODUCTION

Ran and Reurings (2004) firstly obtained the existence for contractive type mappings which have fixed points [25]. The authors demonstrated a few applications of their outcomes to linear and non-linear matrix equations. Numerous researchers develop and considered the conclusions under different contractive conditions in spaces which is order metric e.g., in [3, 4, 8, 17, 23, 24, 30]. Khan *et al.,* (1984) [19] presented altering distance functions. They have applied this function and its augmentations in numerous papers, some of them [7, 10, 16, 18, 22, 28].

The study to existence and oneness of coupled fixed points under maps fulfilling certain contractive conditions has been an interesting field of mathematics. For coupled fixed points, Bhaskar *et al*., [9] presented ideas of mixed monotone mappings and showed the certain theorems. Later on, Ciric and Lakshmikantham (2009) [14] presented a new notion as coupled coincidence point with commutative maps of the mixed g- monotone. Results established by them in a partially ordered space for mappings which is mixed g-monotone. In recent years, several authors have got results of coupled fixed and coincidence point under different type of maps on abstract metric spaces like partially ordered, complete, cone and G- metric [6, 11, 13, 15, 20, 21, 27, 29]. After using the concept of $g$ - monotone property in our result we get another way to find the result on coupled coincidence point.

Abbas *et al.*, (2010) [1] presented a new perception of w and w* - compatible maps. Abbas *et al*., (2011) [2] used the idea in G- metric spaces to represent a oneness theorem for coupled fixed point using nonlinear contractive maps [5, 12, 26].

Definitions given below by Bhaskar and Lakshmikantham (2006) [9].

**Definition 1.1. [9].** A point $(z,s) \in W \times W$ of $R: W \times W \rightarrow W$ is coupled fixed point if it possesses $R(z, s) = z$ and $R(s,z) = z$.

**Definition 1.2. [9].** Assume $(W, \leq)$ partially ordered set, $R: W \times W \rightarrow W$. R has property of mixed monotone when $R(z,v)$ is monotonically non decreasing and non-increasing in z and v respectively, therefore for any $z, v \in W$,

$z_1, z_2 \in W, z_1 \leq z_2 \Longrightarrow R(z_1, v) \leq R(z_2, v)$

and

$v_1, v_2 \in W, v_1 \leq v_2 \Longrightarrow R(z, v_1) \leq R(z, v_2)$

The subsequent definition introduced by Ciric and Lakshmikantham in [14].

**Definition 1.3. [14].** For mappings $R: W \times W$ to W and $h: W \rightarrow W$, point $(z,v) \in W \times W$ is coupled coincidence point if it possesses $R(z,v) = hz, R(v,z) = hv$.

**Definition 1.4. [14]** Let W be a set of non-empty type. Two Mappings R and s are commutative where $R: W \times W \rightarrow W$ and $s: W \rightarrow W$ if $sR(z,v) = R(sz, sv)$, for all $z, v \in W$.

Information given by Khan *et al*., (1984) [19] about the function of altering distance type.

**Definition 1.5. [19].** $\Gamma$ is a function of altering distance type if $\Gamma: [0,\infty) \rightarrow [0,\infty)$ such that:

(i) $\Gamma$ is increasing and smooth.

(ii) $\Gamma(z) = 0$ iff $z = 0$.

Abbas *et al.*, (2010) [1] gave the idea of w and w*-compatible maps and applied the idea to derive useful theorem in cone metric for coupled fixed point.

**Definition 1.6. [1].** Maps $R: W \times W \rightarrow W$ and $h: W \rightarrow W$ are called

(i) w-compatible if $s(R(z,v)) = R(hz, hv)$ whenever $hz = R(z,v)$ and $hv = R(v,z)$;

(ii) $w*$ –compatible if $h(R(z,z)) = R(hz, hz)$ whenever $hz = R(z,z)$.

The objective of this manuscript is to substantiate certain results on coupled coincidence point for maps utilizing the property mixed g- monotone, including altering distance functions under ordered metric spaces. Lastly, for integral equations we existent application.

## II. MAIN THEOREM

**Theorem 2.1.** Presume $(W, \leq, d)$ be a space which is complete ordered metric. Assume that $h: W \to W$ and $R: W \times W \to W$ show continuous mappings and R hold the property of mixed g- monotone, $h$ commutes with $R$ satisfy

$$\phi\left(d\left(R(z,v), R(t,u)\right)\right) \leq \phi\left(M\left((z,v),(t,u)\right)\right) - \varphi\left(M\left((z,v),(t,u)\right)\right) + \theta\left(N\left((z,v),(t,u)\right)\right) \quad (1)$$

where

$M((z,v),(t,u)) = \max\{d(hz,ht), d(hv,hu),$
$d(R(z,v),hz), d(R(t,u),ht),$
$d(R(v,z),hv), \ d(R(u,t),hu)\}$

and

$N((z,v),(t,u)) = \min\{d(R(z,v),ht), d(R(t,u),hz),$
$d(R(z,v),hz), d(R(t,u),ht)\}$

for each $z, v, t, u \in W$ with $hz \geq ht$ and $hv \leq hu$, here $\varphi$ and $\phi$ represent functions of altering distance type, $\theta: [0,\infty) \to [0,\infty)$ is continuous function defined as $\theta(x) = 0$ iff $x = 0$. Presuppose that $R(W \times W) \subseteq h(W)$ and furthermore for each $z_0, v_0 \in W$ with $hz_0 \leq R(z_0, v_0)$, $hv_0 \geq R(v_0, z_0)$, then h and R possesses coupled coincidence point in W.

**Proof.** Suppose that $z_0, v_0 \in W$ with $h z_0 \leq R(z_0, v_0)$ and $hv_0 \geq R(v_0, z_0)$. As $R(W \times W) \subseteq h(W)$, take $z_1, v_1 \in W$ then $hz_1 = R(z_0, v_0)$and $hv_1 = R(v_0, z0)$.

We can take $z_2, v_2 \in W$ then $hz_2 = R(z_1, v_1)$ and $hv_2 = R(v_1, z_1)$. As R possesses the property of mixed g-monotone, we have $hz_0 \leq hz_1 \leq hz_2$ and $hv_2 \leq hv_1 \leq hv_0$. Persistent this process for two sequences $\{z_n\}$ and $\{v_n\}$ in W, we can create

$hz_n = R(z_{n-1}, v_{n-1}) \leq hz_{n+1} = R(z_n, v_n)$

and $hv_{n+1} = R(v_n, z_n) \leq hv_n = R(v_{n-1}, z_{n-1})$

If, for any integer n, we have $(hz_{n+1}, hv_{n+1}) = (hz_n, hv_n)$, then $R(z_n, v_n) = hz_n$ and $R(v_n, z_n) = hv_n$ , that is $(z_n, v_n)$ is coincidence point of R and h.

So, we presume that $(hz_{n+1}, hv_{n+1}) \neq (hz_n, hv_n)$, for each $n \in N$, that is, we accept that one of $hz_{n+1} \neq hz_n$ or $hv_{n+1} \neq hv_n$.

For each $n \in N$, we have after employing the inequality (1),

$\phi(d(hz_{n+1}, hz_n)) = \phi(d(R(z_n, v_n), R(z_{n-1}, v_{n-1})))$
$\leq \phi(M((z_n, v_n),(z_{n-1}, v_{n-1})))$
$- \varphi(M((z_n, v_n),(z_{n-1}, v_{n-1})))$
$+ \theta(N((z_n, v_n),(z_{n-1}, v_{n-1})))$,

and

$\phi(d(hv_{n+1}, hv_n)) = \phi(d(R(v_n, z_n), R(v_{n-1}, z_{n-1})))$
$\leq \phi(M((v_n, z_n),(v_{n-1}, z_{n-1})))$
$- \varphi(M((v_n, z_n),(v_{n-1}, z_{n-1})))$
$+ \theta(N((v_n, z_n),(v_{n-1}, z_{n-1})))$,

where,

$M((z_n, v_n),(z_{n-1}, v_{n-1})) = M((v_n, z_n),(v_{n-1}, z_{n-1}))$
$= \max\{d(hz_n, hz_{n-1}), d(hv_n, hv_{n-1}),$
$d(R(z_n, v_n), hz_n), d(R(z_{n-1}, v_{n-1}), hz_{n-1}),$
$d(R(v_n, z_n), hv_n), d(R(v_{n-1}, z_{n-1}), hv_{n-1})\}$
$= \max\{d(hz_n, hz_{n-1}), d(hv_n, hv_{n-1}),$
$d(hz_{n+1}, hz_n), d(hv_{n+1}, hv_n)\}$

and

$N(hz_{n+1}) =$
$\min\{d(R(z_n, v_n), hz_{n-1}), d(R(z_{n-1}, v_{n-1}), hz_n),$
$d(R(z_n, v_n), hz_n), d(R(z_{n-1}, v_{n-1}), hz_{n-1})\}$
$= \min\{d(hz_{n+1}, hz_{n-1}), d(hz_n, hz_n),$
$d\{(hz_{n+1}, hz_n), d(hz_n, hz_{n-1})\} = 0$

Similarly,

$N((v_n, z_n),(v_{n-1}, z_{n-1})) = 0$

Let us consider three cases.

**Case I:** $M((z_n, v_n),(z_{n-1}, v_{n-1})) = d(hz_{n+1}, hz_n)$.

We claim that

$M((z_n, v_n),(z_{n-1}, v_{n-1})) = d(hz_{n+1}, hz_n) = 0$.

If $d(hz_{n+1}, hz_n) \neq 0$,

then

$\phi(d(hz_{n+1}, hz_n)) \leq \phi(d(hz_{n+1}, hz_n))$
$- \varphi(d(hz_{n+1}, hz_n))$
$< \phi(d(hz_{n+1}, hz_n))$ as $\varphi \geq 0$.

which is a contradiction.

Since $M((z_n, v_n),(z_{n-1}, v_{n-1})) = 0$.

**Case II:** $M((z_n, v_n),(z_{n-1}, v_{n-1})) = d(hv_{n+1}, hv_n)$.

Analogous to the proof as Case I, we can prove this.

**Case III:** $M\left((z_n, v_n),(z_{n-1}, v_{n-1})\right) =$
$\max\{d(hz_n, hz_{n-1}), d(hv_n, hv_{n-1})\}$

so

$$\phi(d(hz_{n+1}, hz_n)) \leq \phi(\max\{d(hz_n, hz_{n-1}), d(hv_n, hv_{n-1})\} - \varphi(\max\{d(hz_n, hz_{n-1}), d(hv_n, hv_{n-1})\}) \quad (2)$$

and

$$\phi(d(hv_{n+1}, hv_n) \leq \phi(\max\{d(hv_n, hv_{n-1}), d(hz_n, hz_{n-1})\} - \varphi(\max\{d(hv_n, hv_{n-1}), d(hz_n, hz_{n-1})\}) \quad (3)$$

Now, by (2) and (3), we have for all $n \in N$,

$\phi(d(hz_{n+1}, hz_n)) \leq \phi(\max\{d(hz_n, hz_{n-1}),$
$d(hv_n, hv_{n-1}) - \varphi(\max\{d(hz_n, hz_{n-1}), d(hv_n, hv_{n-1})\})$

As $\varphi \geq 0$.

$\phi(d(hz_{n+1}, hz_n))$
$\leq \phi(\max\{d(hz_n, hz_{n-1}), d(hv_n, hv_{n-1})\})$,

we have, after employing the concept is non-decreasing,

$$d(hz_{n+1}, hz_n) \leq \max\{d(hz_n, hz_{n-1}), d(hv_n, hv_{n-1}) \quad (4)$$

Similarly, we get

$\phi(d(hv_{n+1}, hv_n))$
$\leq \phi(\max\{d(hv_n, hv_{n-1}), d(hz_n, hz_{n-1})\})$
$- \varphi(\max\{d(hv_n, hv_{n-1}), d(hz_n, hz_{n-1})\})$
$\leq \phi(\max\{d(hv_n, hv_{n-1}), d(hz_n, hz_{n-1})\})$,

and consequently

$$d(hv_{n+1}, hv_n) \leq \max\{d(hv_n, hv_{n-1}), d(hz_n, hz_{n-1}))\} \quad (5)$$

by (4) and (5), we have

$\max\{d(hz_{n+1}, hz_n), d(hv_{n+1}, hv_n)\} \leq$
$\max\{d(hz_n, hz_{n-1}), d(hv_n, hv_{n-1})\}$,

and thus, the sequence

$\max\{d(hz_{n+1}, hz_n), d(hv_{n+1}, hv_n)$

is decreasing and non-negative. Which infers that $\exists\, a \geq 0$ such that

$$\lim_{n\to\infty} \max\{d(hz_{n+1}, hz_n), d(hv_{n+1}, hv_n)\} = a \quad (6)$$

It is certainly observed if $\phi: [0,\infty) \to [0,\infty)$ non-decreasing, $\phi(\max(l,m)) = \max(\phi(l), \phi(h))$ for $l, m \in [0,\infty)$.

Applying this, (2) and (3), we get

$$\max\{\phi(d(hz_{n+1}, hz_n)), \phi(d(hv_{n+1}, hv_n))\} = \phi(\max\{d(hz_{n+1}, hz_n), d(hv_{n+1}, hv_n)\}) \leq \phi(\max\{d(hz_n, hz_{n-1}), d(hv_n, hv_{n-1})\}) - \varphi(\max\{dd(hz_n, hz_{n-1}), d(hv_n, hv_{n-1})\}) \quad (7)$$

Letting $n \to \infty$ in (7) and consider (6), we have

$\phi(a) \leq \phi(a) - \varphi(a) \leq \phi(a) \Rightarrow \varphi(a) = 0$.

As $\varphi$ is a function of altering distance, so $a = 0$ implies

$$\max\{d(hz_{n+1}, hz_n), d(hv_{n+1}, hv_n) = 0 \text{ as} \quad (8)$$

Thus

$\lim_{n\to\infty} \max d(hz_{n+1}, hz_n) = \lim_{n\to\infty} \max d(hv_{n+1}, hv_n) = 0$

Next, we claim that $\{hz_n\}, \{hv_n\}$ two Cauchy sequences. We will prove that for each $0 < \varepsilon$, there exists $p \in N$ such type if $n, m \geq p$,

$\max\{d(hz_{m(s)}, hz_{n(s)}), d(hv_{m(s)}, hv_{n(s)})\} < \varepsilon$

Presuppose the exceeding statement is wrong. So, we can get sequence $\{hz_{m(s)}\}, \{hz_{n(s)}\}$ with $n(s) > m(s) > s$ such that

$\max\{d(hz_{m(s)},, hz_{n(s)}), d(hv_{m(s)}, hv_{n(s)})\} \geq \varepsilon$ (9)

Furthermore, we can take n(s) according to m(s) such that it is smallest index with $n(s) > m(s)$ and satisfy (9). Then

$\max\{d(hz_{m(s)}, hz_{n(s)-1}), d(hv_{m(s)}, hv_{n(s)-1})\} < \varepsilon$ (10)

From triangle inequality

$d(hz_{n(s)}, hz_{m(s)}) \leq d(hz_{n(s)}, hz_{n(s)-1}) + d(hz_{n(s)-1}, hz_{n(s)})$ (11)

Similarly

$d(hv_{n(s)},hv_{m(s)}) \leq d(hv_{n(s)},hv_{n(s)-1})+d(hv_{n(s)-1},hv_{m(s)})$ (12)

From (11) and (12), we have

$\max\{d(hz_{n(s)},hz_{m(s)}),d(hv_{n(s)},hv_{m(s)})\} \leq \max\{d(hz_{n(s)},hz_{n(s)-1}),d(hv_{n(s)},hv_{n(s)-1})\} + \max\{d(hz_{n(s)-1},hz_{m(s)}),d(hv_{n(s)-1},hv_{m(s)})\}$ (13)

From (9), (10) and (13), we get

$\varepsilon \leq \max\{d(hz_{n(s)},hz_{m(s)}),d(hv_{n(s)},hv_{m(s)})\} \leq \max\{d(hz_{n(s)},hz_{n(s)-1}),d(hv_{n(s)},hv_{n(s)-1})\}+\varepsilon$ (14)

Letting $n \to \infty$ in (14) and consider (8) we have

$\max\{d(hz_{n(s)},hz_{m(s)}),d(hv_{n(s)},hv_{m(s)})\}=\varepsilon$ (15)

Again, the triangle inequality, we have

$d(hz_{n(s)-1},hz_{m(s)-1}) \leq d(hz_{n(s)-1},hz_{m(s)})+d(hz_{m(s)},hz_{m(s)-1})$ (16)

and

$d(hv_{n(s)-1},hv_{m(s)-1}) \leq d(hv_{n(s)-1},hv_{m(s)})+d(hv_{m(s)}, hv_{m(s)-1})$ (17)

From (16) and (17), we have

$\max\{d(hz_{n(s)-1},hz_{m(s)-1}),d(hv_{n(s)-1},hv_{m(s)-1})\} \leq \max\{d(hz_{n(s)-1},hz_{m(s)}),d(hv_{n(s)-1},hv_{m(s)})\}+ \max\{d(hz_{m(s)},hz_{m(s)-1}),d(hv_{m(s)},hv_{m(s)-1})\}$ (18)

From (10), we have

$\max\{d(hz_{n(s)-1},hz_{m(s)-1}),d(hv_{n(s)-1},hv_{m(s)-1})\} \leq \max\{d(hz_{m(s)},hz_{m(s)-1}),d(hv_{m(s)},hv_{m(s)-1})\}+\varepsilon$ (19)

Apply the triangle inequality

$d(hz_{n(s)},hz_{m(s)}) \leq d(hz_{n(s)},hz_{n(s)-1})+ d(hz_{n(s)-1},hz_{m(s)-1})+d(hz_{m(s)-1},hz_{m(s)})$ (20)

and

$d(hv_{n(s)},hv_{m(s)}) \leq d(hv_{n(s)},hv_{n(s)-1}) +d(hv_{n(s)-1},hv_{m(s)-1})+d(hv_{m(s)-1}),hv_{m(s)})$. (21)

From (20), (21) and (9), we get

$\varepsilon \leq \max\{d(hz_{n(s)},hz_{m(s)}),d(hv_{n(s)},hv_{m(s)})\} \leq \max\{d(hz_{n(s)},hz_{n(s)-1}),d(hv_{n(s)},hv_{n(s)-1})\} + \max\{d(hz_{n(s)-1},hz_{m(s)-1}), d(hv_{n(s)-1},hv_{m(s)-1})\} + \max\{d(hz_{m(s)-1},hz_{m(s)}),d(hv_{m(s)-1}),hv_{m(s)})\}$ (22)

From (22) and (19), we have

$\varepsilon - \max\{d(hz_{n(s)},hz_{n(s)-1}),d(hv_{n(s)},hv_{n(s)-1})\} - \max\{d(hz_{m(s)-1},hz_{m(s)}),d(hv_{m(s)-1}),hv_{m(s)})\} \leq \max\{d(hz_{n(s)-1},hz_{m(s)-1}), d(hv_{n(s)-1},hv_{m(s)-1})\} < \max\{d(hz_{m(s)-1},hz_{m(s)}),d(hv_{m(s)-1},hv_{m(s)})\}+\varepsilon$ (23)

Letting $s \to \infty$ in (23) and taking (8), we have

$\lim_{s\to\infty} \max\{d(hz_{n(s)-1},hz_{m(s)-1}), d(hv_{n(s)-1},hv_{m(s)-1})\}=\varepsilon$ (24)

Now, utilizing the inequality (1), we have

$\phi(d(hz_{n(s)},hz_{m(s)})) = \phi(d(R(z_{n(s)-1}, v_{n(s)-1}), d(R(z_{m(s)-1}, v_{m(s)-1}))) \leq \phi(M((z_{n(s)-1},v_{n(s)-1}),(z_{m(s)-1},v_{m(s)-1}))) - \varphi(M((z_{n(s)-1},v_{n(s)-1}),(z_{m(s)-1},v_{m(s)-1}))) + \theta(N((z_{n(s)-1},v_{n(s)-1}),(z_{m(s-1},v_{m(s)-1})))$ (25)

Where

$M((z_{n(s)-1},v_{n(s)-1}),(z_{m(s)-1},v_{m(s)-1})) = \max\{d(hz_{n(s)-1},hz_{m(s)-1}), d(hv_{n(s)-1},hv_{m(s)-1}), d(hz_{n(s)},hz_{n(s)-1}),d(hz_{m(s)},hz_{m(s)-1}), d(hv_{n(s)},hv_{(s)-1}),d(hv_{m(s)},hv_{m(s)-1})\}$

and

$N((z_{n(s)-1},v_{n(s)-1}),(z_{m(s)-1},v_{m(s)-1}))= \min\{d(hz_{n(s)},hz_{m(s)-1}),d(hz_{m(s)},hz_{n(s)-1}), d(hz_{n(s)},hz_{n(s)-1}),d(hz_{m(s)},hz_{m(s)-1})\}$.

Similarly

$\phi(d(hv_{n(s)},hv_{m(s)}))=\phi(d(R(v_{n(s)-1},z_{n(s)-1}), d(R(v_{m(s)-1},z_{m(s)-1}))) \leq \phi(M((v_{n(s)-1},z_{n(s)-1}),(v_{m(s)-1},z_{m(s)-1}))) - \varphi(M((v_{n(s)-1},z_{n(s)-1}),(v_{m(s)-1},z_{m(s)-1}))) + \theta(N((v_{n(s)-1},z_{n(s)-1}),(v_{m(s)-1},z_{m(s)-1})))$ (26)

Where

$M((v_{n(s)-1},z_{n(s)-1}),(v_{m(s)-1},z_{m(s)-1})= \max\{d(hz_{n (s)-1},hz_{m(s)-1}),d(hv_{n(s)-1},hv_{m(s)-1}), d(hv_{n(s)},hv_{n(s)-1}),d(hv_{m(s)},hv_{m(s)-1}), d(hz_{n(s)},hz_{n(s)-1}),d(hz_{m(s)},hz_{m(s)-1})\}$

and

$N((v_{n(s)-1},z_{n(s)-1}),(v_{m(s)-1},z_{m(s)-1}))= \min\{d(hv_{n (s)},hv_{m(s)-1}),d(hv_{m(s)},hv_{n(s)-1}), d(hv_{n(s)},hv_{n(s)-1}),d(hv_{m(s)},hv_{m(s)-1})\}$

From (25) and (26), we have

$\max\{\phi(d(hz_{n(s)},hz_{m(s)}),d(hv_{n(s)},hv_{m(s)}))\} \leq \phi(a_n)-\varphi(a_n)+\theta(b_n)$,

where

$a_n = \max\{d(hz_{n(s)-1},hz_{m(s)-1}), d(hv_{n(s)-1},hv_{m(s)-1})\, d(hz_{n(s)},hz_{n(s)-1}), d(hv_{n(s)},hv_{n(s)-1}), d(hz_{m(s)},hz_{m(s)-1}), d(hv_{m(s)},hv_{m(s)-1})\}$,

and

$b_n=\min\{d(hz_{n(s)},hz_{m(s)-1}),d(hv_{n(s)},hv_{m(s)-1}), d(hz_{m(s)},hz_{n(s)-1}),d(hv_{m(s)},hv_{n(s)-1}), d(hz_{n(s)},hz_{n(s)-1}),d(hv_{n(s)},hv_{n(s)-1}), d(hz_{m(s)},hz_{m(s)-1}),d(hv_{m(s)},hv_{m(s)-1})\} \leq \min\{d(hz_{n(s)},hz_{n(s)-1}),d(hv_{n(s)},hv_{n(s)-1}), d(hz_{m(s)},hz_{m(s)-1}),d(hv_{m(s)},hv_{m(s)-1})\}$.

Finally, lettings $\to \infty$ in last two in equalities and using (24), (15) and (8), the continuity of $\phi,\varphi$ and $\theta$, we have

$\phi(\varepsilon) \leq \phi(\max(\varepsilon,0,0)) - \varphi(\max(\varepsilon,0,0)) + \theta(\min(0,0)) \leq \phi(\varepsilon)$

and as a result, $\varphi(\varepsilon) = 0$. As $\varphi$ is function of an altering distance type, $\varepsilon = 0$ and which is contradiction. This shows our requirement.

As (W, d) is a complete metric, $\exists$ z,v $\in$ W, then

$z = \lim hz_n= \lim R(z_n,v_n) = R(\lim z_n, \lim v_n)$, as $n \to \infty$

$v = \lim hv_n= \lim R(v_n,z_n) = R(\lim v_n, \lim z_n)$. as $n \to \infty$ (27)

Since h is continuous, therefore by (27), we get

$\lim h(hz_n) = hz, \lim h(hv_n) = hv$ as $n \to \infty$ (28)

Commutativity of h and R generates

$h(hz_{n+1}) = h(R(z_n,v_n)) = R(hz_n,hv_n)$

$h(hv_{n+1}) = h(R(v_n,z_n)) = R(hv_n,hz_n)$ (29)

Now $\{h(hz_{n+1})\}$ convergent to $R(z,v)$ and $\{h(hv_{n+1})\}$ convergent to $R(v,z)$ by the continuity of R. Apply (28) and using uniqueness of the limit, $R(z,v) = hz$ and $R(v,z) = hv$, that is, h and R represent coupled coincidence point. Hence proof is complete.

In the subsequent result we drop the continuity of R .

**Theorem 2.2.** Presuppose every condition of result 2.1 are satisfied. Furthermore, presume that under the partial order ≤, h is monotone, and W has the subsequent condition

(i) $v_n \geq v$, ∀ n, if non-increasing sequence $\{v_n\}$ in W converges to some point v ∈W,

(ii) $z_n \leq z$, ∀ n, if non-decreasing sequence$\{z_n\}$ in W converges to some point z ∈ W,

after that conclusion of result (1) also hold.

**Proof.** Succeeding the proof of result (1). Then $\lim_{n\to\infty} hz_n = z$ and $\lim_{n\to\infty} hv_n = v$

To prove $hz = R(z,v), hv = R(v,z)$.

$\{hz_n\}$ non-decreasing and $hz_n \to z$ and $\{hv_n\}$ non-increasing and $hv_n \to v$, from the assumptions (1) and (2) that $hz_n \leq z$ and $hv_n \geq v$ ∀n ∈N, no loss of generality in addition,, that h is nondecreasing one can assume, regarding partial order ≤, $h^2z_n \leq hz$ and $h^2v_n \geq hv$, $\forall n \in N$, where $h^2x = h(hx)$ for all x ∈ W.

Utilizing the condition (1) we obtain

$$\phi(d(R(z,v),g^2z_{n+1}) = \phi(d(R(z,v),R(hz_n,hv_n)))$$
$$\leq \phi(M((z,v),(hz_n,hv_n))) - \varphi(M((z,v),(hz_n,hv_n)))$$
$$+\theta(N((z,v),(hz_n,hv_n))) \quad (30)$$

and

$$\phi(d(R(v,z),h^2v_{n+1}) = \phi(d(R(v,z),R(hv_n,hz_n)))$$
$$\leq \phi(M((v,z),(hv_n,hz_n))) - \varphi(M((v,z),(hv_n,hz_n)))$$
$$+\theta(N((v,z),(hv_n,hz_n))) \quad (31)$$

where

$$M((z,v),(hz_n,hv_n)) = M((v,z),(hv_n,hz_n)) =$$
$$\max\{d(hz,hz_n),d(hv,hv_n),d(R(z,v),hz),$$
$$d(R(hz_n,hv_n),hhz_n),d(R(v,z),hv), d(R(hv_n,hz_n),hhv_n)\},$$

and

$$N((z,v),(hz_n,hv_n)) = \min\{d(R(z,v),h^2z_n), d(h^2z_{n+1},hz),$$
$$d(R(z,v),hz),d(h^2z_{n+1},h^2z_n)\} \quad (32)$$

Similarly

$$N((v,z),(hv_n,hz_n)) = \min\{d(R(v,z),h^2v_n),d(h^2v_{n+1},hv),$$
$$d(R(v,z),hv),d(h^2v_{n+1},h^2v_n)\} \quad (33)$$

Now we claim that

$$\max\{d(R(z,v),hz), d(R(v,z),hv)\} = 0 \quad (34)$$

If this is wrong, then $\max\{d(R(z,v),hz), d(R(v,z),hv)\} > 0$.

Since $\lim_{n\to\infty} hz_n = z$, $\lim_{n\to\infty} hv_n = v$,

there exists N in set of natural number for each n >N,

$$M((z,v),(hz_n,hv_n)) = M((v,z),(hv_n,hz_n)) =$$
$$\max\{d(R(z,v),hz),d(R(v,z),hv)\}$$

Connecting this with (30), (31) and (32), we have for all n >N,

$$\phi(\max\{d(R(z,v),h^2z_{n+1}),d(R(v,z),h^2v_{n+1})\}) =$$
$$\max\{\phi(d(R(z,v),h^2z_{n+1}),\phi(d(R(v,z),h^2v_{n+1}))\}$$
$$\leq \phi(\max\{d(R(z,v),hz),d(R(v,z),hv)\} - \varphi(\max\{d(R(z,v),hz),$$
$$d(R(v,z),hv)\} + \theta(\min\{d(R(z,v),h^2z_n),d(h^2z_{n+1},hz),$$
$$d(R(z,v),hz),d(h^2z_{n+1},h^2z_n), d(R(v,z),h^2v_n), d(h^2v_{n+1},hv),$$
$$d(R(v,z),hv), d(h^2v_{n+1},h^2v_n)\})$$

Letting $n \to \infty$, it follows that

$$\phi(\max\{d(R(z,v),hz),d(R(v,z),hv)\})$$
$$\leq \phi(\max\{d(R(z,v),hz),d(R(v,z),hv)\})$$
$$-\varphi(\max\{d(R(z,v),hz),d(R(v,z),hv)\}$$
$$+\theta(\min\{d(R(z,v),hz),d(hz,hz),d(R(z,v),hz),$$
$$d(hz,hz)\ d(R(v,z),hv),d(hv,hv),d(R(v,z),hv),d(hv,hv)\})$$

Apply property of θ, we get

$$\phi(\max\{d(R(z,v),hz),d(R(v,z),hv)\})$$
$$\leq \phi(\max\{d(R(z,v),hz),d(R(v,z),hv)\})$$
$$-\varphi(\max\{d(R(z,v),hz),d(R(v,z),hv)\}\}$$
$$\leq \phi(\max\{d(R(z,v),hz),d(R(v,z),hv)\})$$

and consequently,

$$\varphi(\max\{d(R(z,v),hz),d(R(v,z),hv)\} = 0.$$

A contradiction since φ is a function of an altering distance type. So, (34) holds, then, it proceeds that $hz = R(z,v)$ and $hv = R(v,z)$.

**Theorem 2.3.** Under the hypothesis of Theorem 2.2, presuppose that $hv_0 \leq hz_0$. Then, it proceeds $hz = R(z,v) = R(v,z) = hv$. Furthermore, if R and h are w-compatible, then R and h possess coupled coincidence point of the type $(u,u)$.

**Proof.** If $hv_0 \leq hz_0$, then $hv \leq hv_n \leq hv_0 \leq hz_0 \leq hz_n \leq hz$ for each n ∈ N. Thus, if $hz \neq hv$ (and then $d(hz,hv) \neq 0$ and $d(hv,hz) \neq 0$), using inequality (1), we obtain

$$\phi(d(hv,hz)) = \phi\Big(d\big(R(v,z),R(z,v)\big)\Big)$$
$$\leq \phi\Big(M\big((v,z),(z,v)\big)\Big) - \varphi\Big(M\big((v,z),(z,v)\big)\Big)$$
$$+\theta\Big(N\big((v,z),(z,v)\big)\Big),$$

Where $M\big((v,z),(z,v)\big) = \max\{d(hv,hz),d(hz,hv)\}$,

and $N\big((v,z),(z,v)\big) = 0$.

Hence

$$\phi(d(hv,hz)) \leq \phi(\max\{d(hv,hz),d(hz,hv)\}) -$$
$$\varphi(\max\{d(hv,hz),d(hz,hv)\}) \quad (35)$$

Since $hv \leq hz$, hence using the same idea we have

$$\varphi(d(hz,hv)) \leq \varphi(\max\{d(hv,hz),d(hz,hv)\}) -$$
$$\phi(\max\{d(hv,hz),d(hz,hv)\}) \quad (36)$$

From (35) and (36), we have

$$\phi(\max\{d(hv,hz),d(hz,hv)\})$$
$$= \max\{\phi(d(hv,hz)),\phi(d(hz,hv))\}$$
$$\leq \phi(\max\{d(hv,hz),d(hz,hv)\})$$
$$-\varphi(\max\{d(hv,hz),d(hz,hv)\})$$
$$\leq \phi(\max\{d(hv,hz),d(hz,hv)\}).$$

and consequently,

$$\varphi(\max\{d(hv,hz),d(hz,hv)\}) = 0.$$

As φ is a function of altering distance type, we obtain $d(hv,hz) = 0, d(hz,hv) = 0$, a contradiction. Hence $hz = hv$, that is, $hz = R(z,v) = R(v,z) = hv$. Now, let $u = hz = hv$. Since R and h are w- compatible, so

$$hu = h(hz) = h\big(R(z,v)\big) = R(hz,hv) = R(u,u)$$

Therefore, h and R possess $(u,u)$ as a coupled coincidence point.

To guarantee uniqueness of common coupled fixed point. We need the subsequent idea for the partial order relation, if $(W,\leq)$ partially ordered set, for (z, v), $(z',v') \in W \times W$,

$$(z,v) \leq (z',v') \Longleftrightarrow z \leq z' \text{ and } v' \leq v \quad (37)$$

**Theorem 2.4.** Including the equation (37) to the assumption of result (1) (respectively result (2)), assume that, to each $(z,v),(z',v') \in W \times W$, there exists $(d,s) \in W \times W$ i.e. comparable to $(z,v)$ and $(z',v')$. Then R and h have preciously one common coupled fixed point.

**Proof.** From result (1), R and h is nonempty set of coupled coincidence points. Now we will prove if $(z,v)$and$(z',v')$ two coupled coincidence points, then

$$h(z) = R(z,v), h(v) = R(v,z)$$

and

$$h(z') = R(z',v'), h(v') = R(v',z'),$$

then

$$hz = hz' \text{ and } hv = hv' \quad (38)$$

Take an element (d, s) ∈ W ×W comparable with both of them.

Let $d_0 = d$, $s_0 = s$ and choose $d_1, s_1 \in W$ so that $hd_1 = R(d_0,s_0)$ and $hs_1 = R(s_0,d_0)$.

Then, in a similar way we can define sequences $\{hd_n\}$ and $\{hs_n\}$ as used in Theorem (1), as follows $hd_{n+1}= R(d_n,s_n)$ and $hs_{n+1}= R(s_n,d_n)$.

Since $(hz, hv) = \left(R(z,v), R(v,z)\right)$ and $\left(R(d,s), R(s,d)\right) = (hd_1, hs_1)$ are comparable, then $hz \le hd_1$ and $hv \ge hs_1$. With the help of mathematical induction, it is simple to prove

$hz \le hd_n, hv \ge hs_n, \ \forall\, n \in N.$

Using contractive condition (1)

$$\phi(d(hz,hd_{n+1})) = \phi(d(R(z,v),R(d_n,s_n))) \le \phi(M((z,v),(d_n,s_n))) - \varphi(M((z,v),(d_n,s_n))) + \theta(N((z,v),(d_n,s_n))), \tag{39}$$

where

$M((z,v),(d_n,s_n)) = \max\{d(hz,hd_n),d(hv,hs_n),d(R(z,v),hz), d(R(d_n,s_n),hd_n),d(R(v,z),hv),d(R(s_n,d_n),hs_n)\}$

$= \max\{d(hz,hd_n),d(hv,hs_n)\}$.

and

$N((z,v),(d_n,s_n)) = \min\{d(R(z,v),hd_n),d(R(d_n,s_n),hz), d(R(z,v),hz), d(R(d_n,s_n),hd_n)\}$

$= 0$.

Therefore

$$\phi(d(hz,hd_{n+1})) \le \phi(\max\{d(hz,hd_n),d(hv,hs_n)\}) - \varphi(\max\{d(hz,hd_n),d(hv,hs_n)\}) \le \phi(\max\{d(hz,hd_n),d(hv,hs_n)\}) \tag{40}$$

and analogously

$$\phi(d(hv,hs_{n+1})) \le \phi(\max\{d(hv,hs_n),d(hz,hd_n)\}) - \varphi(\max\{d(hv,hs_n),d(hz,hd_n)\}) \le \phi(\max\{d(hv,hs_n),d(hz,hd_n)\}) \tag{41}$$

From (40) and (41) and utilizing concept of ϕ is non-decreasing, we find

$$\phi(\max\{d(hz,hd_{n+1}),d(hv,hs_{n+1})\}) = \max\{\phi(d(hz,hd_{n+1})),\phi(d(hv,hs_{n+1}))\} \le \phi(\max\{d(hz,hd_n),d(hv,hs_n)\}) - \varphi(\max\{d(hz,hd_n),d(hv,hs_n)\}) \le \phi(\max\{d(hz,hd_n),d(hv,hs_n)\}) \tag{42}$$

This implies that

$\max\{d(hz,hd_{n+1}),d(hv,hs_{n+1})\} \le \max\{d(hz,hd_n),d(hv,hs_n)\}$, and consequently the sequence $\max\{d(hz,hd_{n+1}),d(hv,hs_{n+1})\}$ is decreasing and non-negative so,

$$\lim_{n\to\infty} \max\{d(hz,hd_{n+1}),d(hv,hs_{n+1})\} = a, \tag{43}$$

for certain $a \ge 0$. Using (43) and taking $n \to \infty$ in (42), we have

$\phi(a) \le \phi(a) - \varphi(a) \le \phi(a), \Rightarrow \varphi(a) = 0$

and thus $a = 0$.

Lastly,

$$\lim_{n\to\infty} \max\{d(hz,hd_{n+1}),d(hv,hs_{n+1})\} = 0 \tag{44}$$

This implies, when $n \to \infty$

$$\lim d(hz,hd_{n+1}) = \lim d(hv,hs_{n+1}) = 0, \tag{45}$$

Similarly, as $n \to \infty$

$$\lim d(hz',hd_{n+1}) = \lim d(hv',hs_{n+1}) = 0 \tag{46}$$

From (45) and (46), we have

$hz = hz', hv = hv'$.

Since $hz = R(z,v)$ and $hv= R(v,z)$,

by commutativity of h and R, we have

$h(hz) = h\left(R(z,v)\right) = R(hz,hv)$ and

$$h(hv) = h\left(R(v,z)\right) = R(hv,hz) \tag{47}$$

Denote $hz = a$ and $hv = b$. Then, from (47), it follows that

$$ha = R(a,b) \text{ and } hb = R(b,a) \tag{48}$$

Thus R and h have $(a,b)$ as other coupled coincidence point, and $a = hz = ha$ and $b = hv = hb$.

Hence, R and h have $(a,b)$ as a coupled common fixed point.

To show the only one coupled fixed point, presume (m, p) is other coupled common fixed point for R and h. Then $m = hm = R(m,p)$ and $p = hp = R(p,h)$. Since the pair $(m,p)$ is coupled coincidence point of R and h, since $hm = ha$ and $hp = hb$. So, $m = hm = ha = a$ and $p = hp = hb = b$. Therefore, we get only one coupled fixed point.

**Theorem 2.5.** Taking the hypothesis of result (2), presuppose that in addition to each $(z,v),(z',v') \in W \times W, (d,s) \in W \times W$ that is comparable to $\left(R(z,v),R(v,z)\right)$ and $\left(R(z',v'),R(v',z')\right)$. If R and h are w-compatible, we can say R and h have only one common coupled fixed point of the type $(s,s)$.

**Proof.** By result (2), R and h is nonempty which is set of coupled fixed points. Take $(z,v)$ and $(z',v')$ coupled coincidence points of R and h. Apply the technique of the result (4), we can show

$$hz = hz' \text{ and } hv = hv \tag{49}$$

if $(z,v)$ is a coupled coincidence point of R and h, then $(v,z)$ is also a coupled coincidence point of R and h. Therefore by (49) we have $hz = hv$. Put $t = hz = hv$. Since $hz = R(z,v)$, $hv = R(v,z)$ and R and h are w-compatible, we have $hs = h(hz) = h\left(R(z,v)\right) = R(hz,hv) = R(s,s)$. Thus, $(s,s)$ is a common coupled fixed point of R and h. So, $hs = hz = hv = s$ and hence we have $s = hs = R(s,s)$. Hence, $(s,s)$ is a common coupled fixed point of R and h.

To show common coupled fixed point of R and h are unique, we take another coupled fixed point $(p,q)$ be of R and h, that is, $p = hp = R(p,q)$ and $q = hq = R(q,p)$. Clearly, we have $hs = hp$ and $hs = hq$. Therefore $s = p = q$. Thus, R and h have a unique common coupled fixed point of the type $(s,s)$.

**Example 2.6.** Suppose $W = \{0,1,2\}$ and define $d : W \times W \to R^+$ as $d(z,v) = \max\{z,v\}$. Let $R : W \times W \to W$ as $R(z,v) = z$ for all $z,v \in W$ and $h : W \to W$ with $h(0) = 1, h(1) = 2, h(2) = 2$ for all $z \in W$.

Let $\phi,\varphi : [0,\infty) \to [0,\infty)$ and $\theta : [0,\infty) \to [0,\infty)$ be defined by $\phi(u) = u$ and $\varphi(u) = \frac{1}{2}(u), \theta(u) = u$. Then, $\phi,\varphi,\theta$ hold the properties discussed in result (1).

First, we verify that h commutes with R, that is, $h\left(R(z,v)\right) = R(hz,hv)$.

**Case-1:** If $z = 0, v = 0$ then, $hR(0,0) = g(0) = 1$ and $R\left(h(0),g(0)\right) = 1$.

**Case-2:** If $z = 0, v = 1$ then, $hR(0,1) = g(0) = 1$ and $R\left(h(0),g(1)\right) = 1$.

**Case-3:** If $z = 1, v = 0$ then, $hR(1,0) = g(1) = 2$ and $R\left(h(1),g(0)\right) = 2$.

**Case-4:** If $z = 1, v = 1$ then, $hR(1,1) = g(1) = 2$ and $R\left(h(1),g(1)\right) = 2$.

**Case-5:** If $z = 0, v = 2$ then, $hR(0,2) = g(0) = 1$ and $R\left(h(0),g(2)\right) = 1$.

**Case-6:** If $z = 2, v = 0$ then, $hR(2,0) = g(2) = 2$ and $R\left(h(2),g(0)\right) = 2$.

**Case-7:** If $z = 2, v = 1$ then, $hR(2,1) = g(2) = 2$ and $R\left(h(2),g(1)\right) = 2$.

**Case-8:** If $z = 1, v = 2$ then, $hR(1,2) = g(1) = 2$ and $R\left(h(1),g(2)\right) = 2$.

**Case-9:** If $z = 2, v = 2$ then, $hR(2,2) = g(2) = 2$ and $R(h(2), g(2)) = 2$.

In all above cases it satisfies the condition.

Next, we verify that R and h satisfies inequality (1) and let $hz \geq hu$ and $hz \leq hv$. Then, the following cases arises.

**Case-1:** If $z = v = t = u = 0$, then $d(R(0,0), R(0,0)) = d(0,0) = 0$ and

$M((z,v),(t,u)) = M((0,0),(0,0))$
$= \max\{d(h(0),h(0)), d(h(0),g(0)), d(R(0,0),g(0)), d(R(0,0),h(0)), d(R(0,0),h(0)), d(R(0,0),h(0))\}$
$= \max\{d(1,1), d(1,1), d(0,1), d(0,1), d(0,1), d(0,1)\}$
$= \max\{1,1,1,1,1,1\} = 1.$

also

$N((z,v),(t,u)) = N((0,0),(0,0))$
$= \min\{d(R(0,0),h(0)), d(R(0,0),h(0)), d(R(0,0),h(0)), d(R(0,0),h(0))\}$
$= \min\{d(0,1), d(0,1), d(0,1), d(0,1)\}$
$= \min\{1,1,1,1\} = 1$

As, $\phi(0) = 0 < \phi(1) - \varphi(1) + \theta(1) = \frac{3}{2}$

**Case-2:** If $z = 1, v = 0, t = 1, u = 0$, then $d(R(1,0), R(1,0)) = d(1,1) = 1$ and

$M((z,v),(t,u)) = M((1,0),(1,0))$
$= \max\{d(h(1),h(1)), d(h(0),h(0)), d(R(1,0),h(1)), d(R(1,0),h(1)), d(R(0,1),h(0)), d(R(0,1),h(0))\}$
$= \max\{d(2,2), d(1,1), d(1,2), d(1,2), d(0,1), d(0,1)\}$
$= \max\{2,1,2,2,1,1\} = 2.$

also

$N((z,v),(t,u)) = N((1,0),(1,0)) = \min\{d(R(1,0),h(1)), d(R(1,0),h(1)), d(R(1,0),h(1)), d(R(1,0),h(1))\}$
$= \min\{d(1,2), d(1,2), d(1,2), d(1,2)\}$
$= \min\{2,2,2,2\} = 2.$

As, $\phi(1) = 1 < \phi(2) - \varphi(2) + \theta(2) = 3$,

**Case-3:** If $z = 1, v = 0, t = 1, u = 1$, then as $d(R(1,0), R(1,1)) = 1, M((1,0),(1,1)) = 2$ and $N((1,0),(1,1)) = 2$,

**Case 4:**

If $z = 1, v = 1, t = 1, u = 1$, then as $d(R(1,0), R(1,1)) = 1$, $M((1,0),(1,1)) = 2$ and $N((1,0),(1,1)) = 2$,

**Case-5:** If $z = v = t = u = 2$, then as $d(R(1,0), R(1,1)) = 1, M((1,0),(1,1)) = 2$ and $N((1,0),(1,1)) = 2$,

**Case-6:** If $z = 2, v = 0, t = 2, u = 0$, then as $d(R(1,0), R(1,1)) = 1, M((1,0),(1,1)) = 2$ and $N((1,0),(1,1)) = 2$,

**Case-7:** If $z = 2, v = 1, t = 2, u = 1$, then as $d(R(1,0), R(1,1)) = 1, M((1,0),(1,1))2$ and $N((1,0),(1,1)) = 2$,

**Case-8:** If $z = t = u = 2, v = 0$, then as $d(R(1,0), R(1,1)) = 1, M((1,0),(1,1)) = 2$ and $N((1,0),(1,1)) = 2$,

**Case-9:** If $z = 2, v = t = u = 1$, then as $d(R(1,0), R(1,1)) = 1, M((1,0),(1,1)) = 2$ and $N((1,0),(1,1)) = 2$,

In all the cases inequality (1) is verified.

So, $\phi, \varphi$ and $\theta$ satisfy all the hypothesis of result (1). Further (2,2) is coupled coincidence point of R and h.

**Example 2.7.** Suppose $W = \mathbb{R}$ with usual metric. We Define $R: W \times W \to W$ as $R(z,v) = \frac{1}{5}(z^2 + v^2 + zv)$ for all $z, v \in W$ and $h: W \to W$ with $h(z) = z$ for all $z \in W$.

Suppose $\phi, \varphi: [0,\infty) \to [0,\infty)$ and $\theta: [0,\infty) \to [0,\infty)$ defined as $\phi(u) = u$ and $\varphi(u) = \frac{1}{4}(u), \theta(u) = u$. Then, $\phi$, $\varphi$, $\theta$ have the properties present in Theorem (1).

Now, let $hz \geq ht$ and $hv \leq gu$. So, we obtain

$$\phi\left(d(R(z,v), R(t,u))\right) = d(R(z,v), R(t,u))$$
$$= \left|\frac{1}{5}\left(\frac{1}{z^2 + v^2 + zv}\right) - \frac{1}{5}(t^2 + u^2 + tu)\right|$$
$$= \frac{1}{5}|(z^2 - t^2) + (v^2 - u^2) + (zv - tu)|$$
$$\leq \frac{1}{5}\left(\frac{1}{|z - t| + |v - u| + |zv - tu|}\right)$$
$$\leq \frac{1}{5}(|(z - t)| + |(v - u)| + |v||z - t| + |t||v - u|)$$
$$\leq \frac{1}{5}(|z - t| + |v - u| + |z - t| + |v - u|)$$
$$= \frac{1}{5}(2d(hz, ht) + 2d(hv, hu))$$
$$\leq \frac{3}{4}M((z,v),(t,u))$$
$$= M((z,v),(t,u)) - \frac{1}{4}M((z,v),(t,u))$$
$$= \phi(M((z,v),(t,u)) - \varphi(M((z,v),(t,u)))$$
$$\leq \phi(M((z,v),(t,u)) - \varphi(M((z,v),(t,u))) + \theta(N((z,v),(t,u)))$$

where

$$M((z,v),(t,u)) = \max\{d(hz,ht), d(hv,hu), d(R(z,v),hz), d(R(t,u),ht), d(R(z,v),ht), d(R(t,u),hz)\}$$

and

$$N((z,v),(t,u)) = \min\{d(R(z,v),hz), d(R(t,u),ht), d(R(z,v),ht), d(R(t,u),hz)\}$$

Thus, all of the assumptions of result (1) are verified. Furthermore, R and h, have (0,0)as the coupled coincidence point.

**Theorem 2.8.** Suppose $(W, \leq, d)$ be complete metric space and $h: W \to W$ and $R: W \times W \to W$ be two continuous maps and F possess the property of mixed g- monotone and h commutes with R, such that

$$\phi\left(d(R(z,v), R(t,u))\right) \leq \phi\left(M((z,v),(t,u))\right) - \varphi\left(M((z,v),(t,u))\right) \quad (50)$$

where

$$M((z,v),(t,u)) = \max\{d(hz,ht), d(hv,gu), (R(z,v),hz), d(R(t,u),ht), d(R(v,z),hv), d(R(u,t),hu)\}$$

$\forall z,v,t,u \in W$ with $hz \geq ht$ and $hv \leq gu$, here $\varphi$ and $\phi$ are functions of altering distance type. Presuppose that $R(W \times W) \subseteq h(W)$. Furthermore for each $z_0, v_0 \in W$ with $hz_0 \leq R(z_0, v_0)$ and $hv_0 \geq R(v_0, z_0)$. Presume that W has the subsequent properties of result 2.2 Then R and h have a coupled coincidence point in W.

**Corollary 2.9.** Assume that $(W, \leq, d)$ be ordered complete metric space. Let $R: W \times W \to W$ be a mapping satisfying (50) (with $g = IX$) for all $z, v, t, u \in W$

with $z \geq t$ and $v \leq u$. W possess the property of mixed monotone and assume either
(i) R is continuous or
(ii) W has the subsequent properties of result 2.2
If there exist $z_0$, $v_0 \in W$ such that $z_0 \leq R(z_0, v_0)$ and $v_0 \geq R(v_0,z_0)$, then R has coupled fixed point.

**Corollary 2.10.** Assume $(W, \leq, d)$ be a metric space which is ordered complete. Let $R: W \times W \to W$ is a continuous mapping on W possessing the property mixed monotone, then there exists $p \in [0,1)$ satisfy
$d(R(z,v), R(t,u))$
$\leq p \max\{d(hz,ht), d(hv,gu), d(R(z,v),hz),$
$d(R(t,u),ht), d(R(v,z),hv), d(R(u,t),hu)\}$
$\forall z,v,t,u \in W$ with $z \geq t$ and $v \leq u$. Pre suppose either R is continuous or W has the subsequent condition (1) and (2) of result (2) furthermore for each $z_0, v_0 \in W$ with $z_0 \leq R(z_0,v_0)$ and $v_0 \geq R(v_0,z_0)$, then R has a coupled fixed point.

**Proof.** Using result (8) and choose as $\phi$ = identity, $\varphi = (1-k)\phi$, we get the result.

**Corollary 2.11.** Suppose R satisfy all conditions of Theorems (8), left (50) which is replaced by the subsequent condition. Then Lebesgue- integrable function μ on R+ which is positive, occurred such that $\int_0^{\varepsilon} \mu(t)dt > 0$, for each $\varepsilon > 0$, then, R has coupled fixed point.

$$\int_0^{\phi\left(d(R(z,v),R(t,u))\right)} \mu(u)du \leq \int_0^{\phi\left(d(R(z,v),R(t,u))\right)} \mu(u)du - \int_0^{\varphi\left(d(R(z,v),R(t,u))\right)} \mu(u)du \quad (51)$$

where

$$M\big((z,v),(t,u)\big) = \max\left\{\begin{matrix} d(hz,ht), d(hv,hu), d(R(z,v),hz), d(R(t,u),ht), \\ d(R(v,z),hv), d(R(u,t),hu) \end{matrix}\right\}.$$

**Proof.** Consider $\phi : [0,\infty) \to [0,\infty)$ defined by

$$\Gamma = \int_0^x \mu(t)dt$$

show altering distance function.
Then (51) becomes

$$\Gamma\left(\phi\left(d\big(R(x,y),R(u,v)\big)\right)\right) \leq \Gamma\left(\phi\left(M\big((x,y),(u,v)\big)\right)\right) - \Gamma\left(\varphi\left(M\big((x,y),(u,v)\big)\right)\right),$$

where

$$M\big((z,v),(t,u)\big) = \max\{d(hz,ht), d(hv,hu), d(R(z,v),hz), d(R(t,u),ht), d(R(v,z),hv), d(R(u,t),hu)\}.$$

Taking $\phi_1 = \Gamma o \phi, \varphi_1 = \Gamma o \varphi$ and applying Theorems (2.8), we get the result.

## III. EXPERIMENT TO INTEGRAL EQUATIONS

In this segment we discuss the existence and oneness of solutions for nonlinear integral equation by utilizing the outcome showed under Section 2.
Take an integral equation of the following type:

$$z(h) = \int_0^1 \big(k_1(h,a) + k_2(h,a)\big)\Big(R\big(a,z(a)\big) + g\big(a,z(a)\big)\Big)da + T(h),\ h \in [0,1] \quad (52)$$

We will analyze (52) under the accompanying presumptions:
(a) $k_i: [0,1] \times [0,1] \to R (i = 1,2)$ are smooth and $k_1(h,s) \geq 0$ and $k_2(h,s) \leq 0$.
(b) $T \in C[0,1]$.
(c) $g, R: [0,1] \times R \to R$ be two smooth functions.
(d) Two constants $\nu, \mu > 0$ exist such that for all $z,v \in R, z \geq v$
$0 \leq R(h, z) - R(h, v) \leq \nu[\ln[(z - v)+ 1]$
$-\mu [\ln[(z - v)+ 1] \leq R(h, z) - R(h, v) \leq 0$
(e) There exist $\gamma, \delta \in C[0,1]$ such that

$$\gamma(h) \leq \int_0^1 \left(k_1(h,a)\Big(R\big(a,\gamma(a)\big) + g\big(a,\delta(a)\big)\Big)da + \int_0^1 k_2(h,a)\right)\Big(R\big(a,\delta(a)\big) + g\big(a,\gamma(a)\big)\Big)da + T(h),$$

$$\delta(h) \geq \int_0^1 \left(k_1(h,a)\Big(R\big(a,\delta(a)\big) + g\big(a,\gamma(a)\big)\Big)da + \int_0^1 k_2(h,a)\right)\Big(R\big(a,\gamma(a)\big) + g\big(a,\delta(a)\big)\Big)da + T(h),$$

(f) $2\max(\nu,\mu)\|k_1 - k_2\|_\infty \leq 1$, where $\|k_1 - k_2\|_\infty = \sup\{(k_1(h,a) - k_2(h,a)) : h,a \in [0,1]\}$.
Suppose $W = C[0,1]$ be the space with the standard metric of smooth functions defined on [0, 1] defined by $d(z,v) = \sup_{h\in[0,1]} |z(h) - v(h)|$, for $z,v \in C[0,1]$,
With a partial order this space equipped given by
$z,v \in C[0,1], z \leq v \iff z(h) \leq v(h)$,
for some $h \in [0,1]$.
If in $W \times W$ we take the order given by
$(z,v),(t,u) \in W \times W$,
$(z,v) \leq (t,u) \Leftrightarrow z \leq t$ and $v \geq u$,
and for some $z,v \in W$ we have $\max(z,v), \min(z,v) \in W$, so condition (37) is satisfied.
Furthermore, in [23] it is showed that $(C[0,1],\leq)$ satisfies presumption (1). our result now formulated as .

**Theorem 3.1.** Under presumptions (a)-(f), Eqn. (52) hold solution in C[0,1] which is unique.

**Proof.** We take the mapping $R: W \times W \to W$ defined as

$$R(z,v)(h) = \int_0^1 \Big(k_1(h,a)\Big(R\big(a,z(a)\big) + g\big(a,v(a)\big)\Big)da + \int_0^1 k_2(h,a)\Big)\Big(R\big(a,v(a)\big) + g\big(a,z(a)\big)\Big)da + T(h),$$

for $h \in [0,1]$.
By virtuousness of our presumptions, R is well defined for $z,v \in W$ then $R(z,v) \in W$.
On priority, we show that R has the property of mixed monotone.
for $z_1 \leq z_2$ and $h \in [0,1]$, we get

$$R(z_1,v)(h) - R(z_2,v)(h)$$
$$= \int_0^1 \left(k_1(h,a)\Big(R\big(a,z_1(a)\big) + g\big(a,v(a)\big)\Big)da + \int_0^1 k_2(h,a)\right)\Big(R\big(a,v(a)\big) + g\big(a,z_1(a)\big)\Big)da$$
$$+ T(h) - \int_0^1 \left(k_1(h,a)\Big(R\big(a,z_2(a)\big) + g\big(a,v(a)\big)\Big)da - \int_0^1 k_2(h,a)\right)\Big(R\big(a,v(a)\big) + g\big(a,z_2(a)\big)\Big)da - T(h),$$

$$= \int_0^1 (k_1(h,a)(R(a,z_1(a)) - R(a,z_2(a)))da + \int_0^1 k_2(h,a))g(a,z_1(a)) - g(a,z_2(a)))da \quad (53)$$

Taking into account that $z_1 \leq z_2$ and our presumptions,
$R(a,z_1(a)) - R(a,z_2(a)) \leq 0$,
$g(a,z_1(a)) - g(a, z_2(a)) \geq 0$,

and from (53) we get

$R(z_1,v)(h) - R(z_2,v)(h) \leq 0$

and this proves that $R(z_1,v) \leq R(z_2,v)$.

Similarly, if $v_1 \geq v_2$ and $h \in [0,1]$,

$R(z,v_1) \leq R(z,v_2)$.

Thus, R has the property of mixed monotone.

We estimate $d(R(z,v),R(t,u))$ for $z \geq t$ and $v \leq u$.

As R has the property of mixed monotone,

$R(z,v) \geq R(t,u)$ and we have

$$d(R(z,v),R(t,u)) = \sup_{h\in[0,1]} |R(z,v)(h) - R(t,u)(h)| = \sup_{h\in[0,1]} (R(z,v)(h) - R(t,u)(h))$$

$$= \sup_{h\in[0,1]} \Big[\int_0^1 (k_1(h,a)(R(a,z(a)) + g(a,v(a)))da + \int_0^1 k_2(h,a))(R(a,v(a)) + g(a,z(a)))da + T(h) - \int_0^1 (k_1(h,a)(R(a,t(a)) + g(a,u(a)))da - \int_0^1 k_2(h,a))(R(a,u(a)) + g(a,t(a)))da - T(h)\Big] \quad (54)$$

$$= \sup_{h\in[0,1]} \Big[\Big[\int_0^1 k_1(h,a)\big[(R(a,z(a) - R(a,t(a))) - (g(a,u(a)) - g(a,v(a)))\big]da - \int_0^1 k_2(h,a)\big[(R(a,u(a) - R(a,v(a))) - (g(a,z(a)) - g(a,t(a)))\big]da\Big]$$

By our presumptions (that $z \geq t$ and $v \leq u$)

$R(a,z(a))-R(a,t(a)) \leq \nu \ln[(z(a)-t(a))+1]$

$g(a,u(a))-g(a,v(a)) \geq -\mu \ln[(u(a)-v(a))+1]$

$R(a,u(a))-R(a,v(a)) \leq \nu \ln[(u(a)-v(a))+1]$

$g(a,z(a))-g(a,t(a)) \geq -\mu \ln[(z(a)-t(a))+1]$.

Consider these last in equalities, $k_2 \leq 0$ and (54), we have

$$d(R(z,v),R(t,u)) \leq \sup_{h\in[0,1]} \Big[\int_0^1 k_1(h,a)\nu\big[\ln[(z(a) - t(a)) + 1]\big] + \mu\big[\ln[(u(a) - v(a)) + 1]\big]da + \int_0^1 (-k_2(h,a))\nu\big[\ln[(u(a) - v(a)) + 1]\big]\mu[\ln[(z(a) - t(a)) + 1]]da\Big] \quad (55)$$

$$= \max(\nu,\mu) \sup_{h\in[0,1]} \Big[\int_0^1 (k_1(h,a) - k_2(h,a))\ln[(z(a) - t(a)) + 1]da + \int_0^1 (k_1(h,a) - k_2(h,a)) \ln[(u(a) - v(a)) + 1]da\Big]$$

Defining

$I = \int_0^1 (k_1(h,a) - k_2(h,a)) \ln[(z(a) - t(a)) + 1]da$

$II = \int_0^1 (k_1(h,a) - k_2(h,a)) \ln[(u(a) - v(a)) + 1]da$

and utilizing the Cauchy - Schwartz inequality in (a) we have

$$I \leq \left(\int_0^1 (k_1(h,a) - k_2(h,a))^2 da\right)^{\frac{1}{2}} \left(\int_0^1 (\ln[(z(a) - t(a)) + 1])^2 da\right)^{1/2}$$

$$\leq \|k_1 - k_2\|_\infty . (\ln\|z - t\| + 1)$$

$$= \|k_1 - k_2\|_\infty . (\ln (d(z,t) + 1) \quad (56)$$

In similar way, we can find the following estimate for (II):

$$II \leq \|k_1 - k_2\|_\infty . (\ln (d(v,u) + 1) \quad (57)$$

from (55)- (57), we get

$$d(R(z,v),R(t,u)) \leq \max(\nu,\mu)\|k_1 - k_2\|_\infty [(\ln(d(z,t) + 1)) + (\ln(d(v,u) + 1))]$$

$$\leq \max(\nu,\mu) \|k_1 - k_2\|_\infty [(\ln (\max (d(z,t), d(v,u), d(R(z,v),z), d(R(t,u),t), d(R(v,z),v), d(R(u,t),u)) + 1)) + (\ln (d(z,t), d(v,u), d(R(z,v),z), d(R(t,u),t), d(R(v,z),v), d(R(u,t),u)) + 1))]$$

$$= 2 \max(\nu,\mu) \|k_1 - k_2\|_\infty \ln (\max(d(z,t), d(v,u), d(R(z,v),z), d(R(t,u),t), d(R(v,z),v), d(R(u,t),u) + 1))$$

From last inequality and presumption (f) give us

$$d(R(z,v),R(t,u)) \leq (\ln(\max(d(z,t), d(v,u), d(R(z,v),z), d(R(t,u),t), d(R(v,z),v), d(R(u,t),u)) + 1))$$

or, equivalently,

$$d(R(z,v),R(t,u)) \leq (\max(d(z,t), d(v,u), d(R(z,v),z), d(R(t,u),t), d(R(v,z),v), d(R(u,t),u)) - [(\max(d(z,t), d(v,u), d(R(z,v),z), d(R(t,u),t), d(R(v,z),v), d(R(u,t),u)) - \ln(\max(d(z,t), d(v,u), d(R(z,v),z), d(R(t,u),t), d(R(v,z),v), d(R(u,t),u)) + 1))] \quad (58)$$

Put $\phi(z) = z$ and $\varphi(z) = z - \ln(z + 1)$. Clearly, $\varphi$ and $\phi$ are altering distance functions therefore from (58) we have

$$\phi(d(R(z,v),R(t,u)) \leq \phi(d(\max(d(z,t), d(v,u), d(R(z,v),z), d(R(t,u),t), d(R(v,z),v), d(R(u,t),u)) - \varphi(d(\max(d(z,t), d(v,u), d(R(z,v),z), d(R(t,u),t), d(R(v,z),v), d(R(u,t),u))))$$

This shows that the mapping R satisfies the condition occurring in Corollary (9).

Finally, let $\gamma, \delta$ be the functions occurring in presumption (e); then, by (e), we get

$\gamma \leq R(\gamma,\delta), \delta \geq R(\delta,\gamma)$.

Using Corollary (9), the existence of $(z,v) \in W \times W$ we deduce such that $z = R(z,v)$ and $v = R(v,z)$, that is, $(z,v)$ is a solution of equation (52).

This finishes the proof.

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