Statistical \( \Lambda \) – Convergence of Order \( \alpha \) in Probabilistic Normed Spaces

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ABSTRACT: In this paper, we discuss the statistical \( \Lambda \)–convergence and statistical \( \Lambda \)–Cauchy in probabilistic normed space and extend this concept further for order \( \alpha \) and present some basic results of statistical \( \Lambda \)–convergence for order \( \alpha \) in probabilistic normed space by giving some illustrative examples.

Keywords: Statistical convergence, Statistical \( \Lambda \)-convergence, \( \ell \)-norm, Probabilistic Normed Space (\( \ell^P \)-space)

I. INTRODUCTION
The term statistical convergence in general directly connected with an extension of the usual convergence concept for the sequences. This concept has been studied by various authors parallel to the theory of ordinary convergence. This important theory of convergence for the sequences was initiated by Fast [7] and Steinhaus [20] simultaneously. The advancement in this convergence began after the work of Schoenberg [17] and Salát [16]. Later on numerous researchers worked on statistical convergence for different types of sequences and in different spaces like intuitionistic fuzzy normed space [11], random normed space [13], probabilistic normed space [9] etc.. The concept of statistical convergence was further explored using the summability theory by Fridy [8], Conner [6], Meenakshi et al., [2], Salát [16] etc. Çolak [3-5] extended the work on statistical convergence as well as \( \lambda \) – statistical convergence was presented by Mursaleen [12]. The basic structure of a statistical metric space was proposed by Menger [10], that is also known as probabilistic metric space which was later on explored by Schweizer and Sklar [18,19]. The term probabilistic metric space is applicable in those settings where exact distance between two points is unpredictable. Probabilistic normed space is considered as a very important class in the probabilistic metric space. Karakus [9] also demonstrated statistical convergence basic concepts in PN–spaces. Recently, Srivastava and Mursaleen [15] also associated their work on statistical \( \Lambda \)–convergence in PN–spaces. Later, Meenakshi et al., [1] also gave overview of \( \Lambda \)– statistical convergence for order \( \alpha \) in the setup of a random 2-normed space. Our present work generalize the statistical \( \Lambda \)-convergence in probabilistic normed space for order \( \alpha \).

II. PRELIMINARIES
First we review some basic terms required for our work which is based on natural density.

Definition 2.1 The term natural density of any set \( \mathcal{M} \subseteq \mathbb{N} \) (set of natural numbers) is given as
\[
\delta(\mathcal{M}) = \lim_{n \to \infty} \frac{1}{n} \sum_{a \in \mathcal{M}} 1
\]
where \( [\cdot, \cdot) \) represents the order of the enclosed set.

Definition 2.2 A sequence \( x = (x_m) \) is statistically convergent to \( \xi \) if for each \( \varepsilon > 0 \),
\[
\mathcal{M}(\varepsilon) = \{ m \leq n : |x_m - \xi| < \varepsilon \}
\]
has natural density zero. We can write it as \( \text{St lim} x_m = \xi \).

Definition 2.3 A function \( \Phi : \mathbb{R} \to \mathbb{R}_+^0 \) (set of positive real numbers) is known as a distribution function if it is a non-decreasing and left continuous when,
\[
\inf \Phi(c) = 0 \quad ; \quad c \in \mathbb{R}_-
\]
\[
\sup \Phi(c) = 1 \quad ; \quad c \in \mathbb{R}_+
\]
and \( \mathcal{F}_c = \{ \Phi : \Phi \text{ is a distribution function with } \Phi(0) = 0 \} \).

If \( k \in \mathbb{R}_+^0 \), then \( \Phi_k \in \mathcal{F}_c \), where \( \Phi_k(c) = \begin{cases} 0 & (c \leq k) \\ 1 & (c > k) \end{cases} \) for all \( \Phi \in \mathcal{F}_c \).

Definition 2.4 A \( \ell \)-norm (triangular form) is the map \( * : [0,1] \times [0,1] \to [0,1] \) which is continuous, commutative, associative and non-decreasing.

Definition 2.5 [19] Let \( \mathcal{S} : \mathbb{R} \to \mathcal{F} \) be a map such that
\( \mathcal{S}(X) = \mathcal{S}(x) \) and \( \mathcal{S}(p) \) is value of \( \mathcal{S} \) at \( p \in \mathbb{R}_+^0 \) (where \( \mathcal{S} \) is a linear space and \( \mathcal{F} \) is a set of distribution functions). Then \( \mathcal{S} \) is known as probabilistic norm and triplet \( (\mathbb{R}, \mathcal{S}, *) \) with \( * \) as a \( \ell \)-norm is known as probabilistic normed space or PN–space if it satisfies below four properties:

(i) \( \mathcal{S}(0) = 0 \)

(ii) \( \mathcal{S}_x(p) = 1 \), \( \forall p > 0 \) if \( x = 0 \).

(iii) \( \mathcal{S}_{xy}(p) = \mathcal{S}_x(p/|y|) \) where \( y \neq 0 \in \mathbb{R}_+^0 \).

(iv) \( \mathcal{S}_{x+y}(p+q) = \mathcal{S}_x(p) \ast \mathcal{S}_y(q) \), \( \forall x, y \in \mathcal{S} \) and \( p, q \in \mathbb{R}_+^0 \).

Example 2.1 If \( (\mathbb{R}, ||\cdot||) \) is a normed space, \( \beta \in \mathcal{F} \) such that \( \beta(0) = 0 \) and \( \beta \neq \Phi \) where
\[
\Phi(p) = \begin{cases} 1 & (p > 0) \\ 0 & (p \leq 0), x \in \mathbb{R} \quad p \in \mathbb{R} \quad \text{with} \\
\end{cases}
\]
\[
\mathcal{S}_x(p) = \begin{cases} \beta(p) & (p \leq 0), x \in \mathbb{R} \quad \text{and} \quad p \in \mathbb{R} \quad \text{with} \\
\end{cases}
\]
Then triplet \( (\mathbb{R}, \mathcal{S}, *) \) is known as a PN–space.

First we mention \( \Delta \) – convergent sequence as given by Mursaleen [14]. Suppose \( \lambda = (\Lambda_{m,n})_{m,n}^\infty \) is a real sequence with \( \lim_{m \to \infty} \lambda_{m,n} = \infty \) and \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \). In such case, a sequence \( x = (x_m)_{m=0}^\infty \) is \( \Delta \)-convergent to some number \( \xi \) when \( \Lambda_{m,n}(x) \to \xi \) as \( m \to \infty \), and
\[
\Lambda_{m,n}(x) = \frac{1}{m} \sum_{l=0}^n \lambda_{m,l} (x_l - \xi) = x_m \quad (m \in \mathbb{N})
\]
Next, we discuss the idea of \( \Delta \)-convergence along with statistical \( \Lambda \)-convergence in PN–spaces.
Definition 2.6 [15] Let \((X;\Lambda^+\cdot\alpha)\) be a PN-space. A sequence \(x = (x_n)_{n=0}^\infty\) is called convergent to some \(x^*\) with respect to \(\Lambda^+\cdot\alpha\) if for each \(\varepsilon > 0\) and \(\phi \in (0,1)\) there exists a number \(m_0 \in N\) with \(\Lambda^+\cdot\alpha(x_n - x^*) < \varepsilon\) for all \(n \geq m_0\) and it can be written as, 

\[
\lim_{n \to \infty} x_n = x^*.
\]

Remark 2.1 Consider \((X;\|\|))\) is a real normed space with \(\|x(p) = \frac{p}{p+\|x\|}, p \geq 0\) and \(x \in X\).

Definition 2.7 [9] Let \((X;\Lambda^+\cdot\alpha)\) be a PN-space. A sequence \(x = (x_n)_{n=0}^\infty\) is called statistically convergent to some \(x^*\) with respect to \(\Lambda\) if for each \(\varepsilon > 0\) and \(\phi \in (0,1)\), we have

\[
\delta((m \in N : \Lambda_\alpha(x_m - x^*) < \varepsilon) \leq 1 - \phi) = 0.
\]

Definition 2.8 [15] Let \((X;\Lambda^+\cdot\alpha)\) be a PN-space. A sequence \(x = (x_n)_{n=0}^\infty\) is called statistically \(\Delta\)-convergent with respect to \(\Lambda\) if for each \(\varepsilon > 0\) and \(\phi \in (0,1)\), we have

\[
\delta((m \in N : \Lambda_\alpha(x_m - x^*) < \varepsilon) \leq 1 - \phi) = 0.
\]

Symbolically, St. \(\Lambda^+\cdot\alpha\)-lim \(x_n = x^*\).

Definition 2.10 [9] Let \((X;\Lambda^+\cdot\alpha)\) be a PN-space. A sequence \(x = (x_n)_{n=0}^\infty\) is called statistically Cauchy with respect to \(\Lambda\) if for each \(\varepsilon > 0\) and \(\phi \in (0,1)\), there exists a number \(m_0 \in N\) with \(\Lambda_\alpha(x_m - x_n) < \varepsilon\) whenever \(m, n \geq m_0\) and it can be written as, 

\[
\lim_{n \to \infty} x_n = x^*.
\]

Definition 2.11 [15] Let \((X;\Lambda^+\cdot\alpha)\) be a PN-space. A sequence \(x = (x_n)_{n=0}^\infty\) is called statistically \(\Delta\)-Cauchy with respect to \(\Lambda\) if for each \(\varepsilon > 0\) and \(\phi \in (0,1)\), there exists a number \(m_0 \in N\) with \(\Lambda_\alpha(x_m - x_n) < \varepsilon\) whenever \(m, n \geq m_0\) and it can be written as, 

\[
\lim_{n \to \infty} x_n = x^*.
\]

Definition 2.12 [3] Consider \(\lambda = (\lambda_n)_{n=0}^\infty\) is a non-decreasing real sequence of positive numbers and \(\alpha \in (0,1]\). A sequence \(x = (x_n)_{n=0}^\infty\) is called statistically \(\Delta\)-convergent of order \(\alpha\) to some \(x^*\) whenever

\[
\lim_{n \to \infty} \frac{1}{n(1 - \lambda_n + n)} \sum_{k=n}^{\infty} |x_k - x_n| < \varepsilon = 0.
\]

III. MAIN RESULTS

With the help of the definitions given in the previous segment, we systematically analyse the idea of statistical \(\Lambda\)-convergence as well as statistical \(\Lambda\)-Cauchy of order \(\alpha\) in PN-space as follows:

Definition 3.1 Let \((X;\Lambda^+\cdot\alpha)\) be a PN-space and \(\alpha \in (0,1]\). A sequence \(x = (x_n)_{n=0}^\infty\) is called statistically \(\Delta\)-convergent of order \(\alpha\) or \(\Delta^+\cdot\alpha\)-convergent to some \(x^*\) with respect to \(\Lambda\) if for each \(\varepsilon > 0\) and \(\phi \in (0,1)\), we have

\[
\delta((m \in N : \Lambda_\alpha(x_m - x_n) < \varepsilon) \leq 1 - \phi) = 0.
\]

i.e. for \(\lambda_n \to \infty\),

\[
\lim_{n \to \infty} \frac{1}{n(1 - \lambda_n + n)} \sum_{k=n}^{\infty} |x_k - x_n| < \varepsilon = 0.
\]

or

\[
\delta((m \in N : \Lambda_\alpha(x_m - x_n) < \varepsilon) > (1 - \Phi) = 1 - \varepsilon = 0.
\]

where \(\varepsilon\) is called the statistical \(\Delta\)-limit.
Theorem 3.3 Let \((X; \mathcal{A}, \alpha)\) be a PN–space and \(\alpha \in (0,1)\). A sequence \(x = (x_m)_{m=0}^{\infty}\) is statistically \(\Lambda^\alpha\)-convergent if and only if it is statistically \(\Lambda^\alpha\)-Cauchy.

**Proof:** Let \(S\mathfrak{A}\frac{\alpha}{2} - \lim x_m = \xi\).

Now, for all \(\Phi \in (0,1)\) and any \(\varepsilon > 0\), we have

\[\delta_{\mathcal{A}}(x_m < \xi) = 0\]

Then, for \(\varepsilon > 0\) and \(\Phi \in (0,1)\), we have

\[\delta_{\mathcal{A}}(x_m < \xi) = 0\]

This shows that \(\delta_{\mathcal{A}}(x_m < \xi) = 0\). Therefore, \(S\mathfrak{A}\frac{\alpha}{2} - \lim x_m = \xi\).

Conversely, suppose the sequence \(x = (x_m)_{m=0}^{\infty}\) is statistically \(\Lambda^\alpha\)-convergent, then there exists \(\xi \in X\) and \(x_m \to \xi\) as \(m \to \infty\).

Thus, sequence \(x = (x_m)_{m=0}^{\infty}\) is statistical \(\Lambda^\alpha\)-convergent, then

\[\delta_{\mathcal{A}}(x_m < \xi) = 0\]

This shows that \(\delta_{\mathcal{A}}(x_m < \xi) = 0\). Therefore, \(S\mathfrak{A}\frac{\alpha}{2} - \lim x_m = \xi\).

**Theorem 3.4** Let \((X; \mathcal{A}, \alpha)\) be a PN–space. If \(S\mathfrak{A}\frac{\alpha}{2} - \lim x_m = \xi_1\) and \(S\mathfrak{A}\frac{\alpha}{2} - \lim y_m = \xi_2\), then

(i) \(S\mathfrak{A}\frac{\alpha}{2} - \lim x_m = \xi_1\) and \(S\mathfrak{A}\frac{\alpha}{2} - \lim y_m = \xi_2\)

(ii) \(S\mathfrak{A}\frac{\alpha}{2} - \lim (y_m - x_m) = \xi_1 + \xi_2\)

**Proof:**

(i) As \(S\mathfrak{A}\frac{\alpha}{2} - \lim x_m = \xi_1\) and \(S\mathfrak{A}\frac{\alpha}{2} - \lim y_m = \xi_2\), let \(\varepsilon > 0\) and \(\Phi \in (0,1)\), choose \(\Phi \in (0,1)\) such that \((1 - \Phi) > 1 - \theta\).

Take \(M_1(\mathcal{A}, \alpha, \Phi) = \{m \in N: 3(\mathcal{A}x_m - \xi_1) < 1 - \Phi\}\) and \(M_2(\mathcal{A}, \alpha, \Phi) = \{m \in N: 3(\mathcal{A}y_m - \xi_2) < 1 - \Phi\}\).

Since, \(S\mathfrak{A}\frac{\alpha}{2} - \lim x_m = \xi_1\), \(S\mathfrak{A}\frac{\alpha}{2} - \lim y_m = \xi_2\), we get \(\delta_{\mathcal{A}}(M_2(\mathcal{A}, \alpha, \Phi)) = 0\), \(\forall \varepsilon > 0\).

As \(M_1(\mathcal{A}, \alpha, \Phi, \varepsilon), M_2(\mathcal{A}, \alpha, \Phi, \varepsilon) \in \mathcal{A}\), then \(\delta_{\mathcal{A}}(M(\mathcal{A}, \alpha, \Phi)) = 0\), \(\forall \varepsilon > 0\).

Then,

\[\delta_{\mathcal{A}}(\mathcal{A}x_m + \mathcal{A}y_m - \xi_1 - \xi_2) = \delta_{\mathcal{A}}(\mathcal{A}x_m - \xi_1) + \delta_{\mathcal{A}}(\mathcal{A}y_m - \xi_2)
\]

\[\geq 3(\mathcal{A}x_m - \xi_1) + 3(\mathcal{A}y_m - \xi_2) > 1 - \Phi + 1 - \Phi = 2 - 2\Phi > 1 - \theta\]

Therefore, \(S\mathfrak{A}\frac{\alpha}{2} - \lim x_m = \xi_1\) and \(S\mathfrak{A}\frac{\alpha}{2} - \lim y_m = \xi_2\).

(ii) Let \(S\mathfrak{A}\frac{\alpha}{2} - \lim x_m = \xi_1\). Then for \(\Phi \in (0,1)\) and \(\varepsilon > 0\),

\[\delta_{\mathcal{A}}(x_m < \xi_1) = 0\]

Therefore, \(S\mathfrak{A}\frac{\alpha}{2} - \lim x_m = \xi_1\).

Therefore, \(S\mathfrak{A}\frac{\alpha}{2} - \lim x_m = \xi_1\).

**IV. CONCLUSIONS**

This particular paper has taken the concept of statistically \(\Lambda\)-convergent and studied its generalization with order \(\alpha\) in the environment of PN–spaces. Also we have given some illustrative examples to reveal the concepts.

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**CONFLICT OF INTEREST**

Authors have no any conflict of interest.
REFERENCES


