*International Journal of Theoretical & Applied Sciences,* **2**(2)**:** 49-53(2010) **ISSN : 0975-1718**

# Integrals involving Appell's function of matrix argument in complex case

***V.K. Gaur and Satyaveer Singh***

*Deptt. of Mathematics, Dungar College, Bikaner (RJ)*

**Deptt. of Mathematics JJT University, Chudela, Jhunjhunu (RJ)*

**ABSTRACT : In this paper we investigated certain integral representation of Appell function $F_1$, $F_2$, and $F_3$ of real hermition positive definite matrices. The results for scalar argument readily follow by taking the matrices of order unity. Some of the results obtained are believed to be new.**

**Keywords :** Appell function, geometric probability, dircihlet densities, canonical correlatio, lauricella function, hypergeometric function

## I. INTRODUCTION

Following the technique developed by Herz [2], Hua [3], and Mathai [7, 8] the authors define the Appell function $F_1, F_2$ and $F_3$ of two Matrix arguments in terms of certain double integrals of matrix arguments. The definition of Appell function $F_4$ in terms of integrals is not given because a convenient integral is not available. These definition are useful in deriving several uninteresting, useful and new properties of Appell functions of matrix arguments. It is not out of place to mention that there are many practical problems in which Appell functions occur such as geometric probability time series model, queing models, engineering problems, etc. Exton [1] has given a detailed account of statistical distribution where various special function of one and more variable are involved. Gamma, Beta and Dircihlet densities of matrix variates occur in the distributions of various test statistics in multivariable analysis and in the distribution connected with the concepts of generalized variance and canonical correlation matrices. These results can be seen in the monograph by Mathai and Saxena [5] Mathai and Saxena [6] also discussed various problems of statistical distribution where Lauricella function of several complex variables occur naturally. Other useful applications of special function can be seen from the work of Saxena and Sethi [10]. Recently Mathai [8] defined and established several integral representations for Lauricella function of real symmetric positive definite matrices.

In multivariate statistical theory series expansions of Herz hypergeometric function were developed by James, Constantine, Mathai, Sethi and others statistical distributions. We refer to James [4], Muirhead [9], and Takemura [11], Mathai [8], Sethi and Vyas [12] for extensive surveys of the statistical literature. Although series expansions for hypergeometric function appear in the literature for both the real and complex fields due to James [4].

The object of this paper is to derive certain integral representations for Appell function $F_1, F_2$ and $F_3$ of two matrix arguments.

## II. APPELL FUNCTION $F_1$ OF MATRIX ARGUMENT

The Appell function of matrix argument will be denoted by $F_1(\alpha, \beta, \beta' ; \gamma ; \tilde{X}, \tilde{Y})$ analogous to the corresponding scalar case and is defined by the integral.

$$F_1(\alpha, \beta, \beta' ; \gamma ; \tilde{X}, \tilde{Y}) = \frac{\tilde{\Gamma}_m(\gamma)}{\tilde{\Gamma}_m(\alpha)\tilde{\Gamma}_m(\gamma-\alpha)} \int_0^I \left|\det \tilde{U}\right|^{\alpha-m} \left|\det(I-\tilde{U})\right|^{\gamma-\alpha-m} \left|\det(I-\tilde{U}\tilde{X})\right|^{-\beta} \left|\det(I-\tilde{U}\tilde{Y})\right|^{-\beta} d\tilde{U} \quad ...(2.1)$$

For

$$\mathrm{Re}(\gamma) > m-1,\ \mathrm{Re}(\alpha) > m-1,\ \mathrm{Re}(\gamma-\alpha) > m-1$$

$$\tilde{X} = \tilde{X}' > 0,\ \tilde{Y} = \tilde{Y}' > 0\ \left\|\det \tilde{X}\right\| < 1\ \left\|\det \tilde{Y}\right\| < 1$$

Certain integral representation of Appell function $F_1$ of matrix arguments are investigated in the form of following two theorem.

**Theorem 1 :** For

$$\mathrm{Re}(\delta) > m-1\ \mathrm{Re}(\beta) > m-1\ \mathrm{Re}(\beta') > m-1$$

$$\left\|\det \tilde{X}\right\| < 1, \left\|\det \tilde{Y}\right\| < 1$$

We have

$$F_1(\alpha, \beta; \beta^1; \gamma^1; \tilde{X}, \tilde{Y})$$

$$= \frac{\tilde{\Gamma}_m(\delta)\tilde{\Gamma}_m(\delta')}{\tilde{\Gamma}_m(\beta)\tilde{\Gamma}_m(\beta')\tilde{\Gamma}_m(\delta-\beta)\tilde{\Gamma}_m(\delta'-\beta')}$$

$$\int_0^1\int_0^1 \left|\det \tilde{U}\right|^{\beta-m} \left|\det \tilde{V}\right|^{\beta'-m} \left|\det 1-\tilde{U}\right|^{\delta-\beta-m}$$

$$\left|\det(1-\tilde{V})\right|^{\delta'-\beta'-m} F_1(\alpha, \delta, \delta'; \gamma; \tilde{X}^{1/2}\tilde{U}\tilde{X}^{1/2}, \tilde{Y}^{1/2}\tilde{V}\tilde{Y}^{1/2})\, d\tilde{U}\, d\tilde{V} \quad ...(2.2)$$

**Proof :** If we employ (2.1) on the R.H.S of (2.2) invert the order of integration, which is premissible due to the absolute convergence of the integrals involved in the process and integrate with the help of the integral.

$$ {}_2F_1(\alpha, \beta, \gamma'; \tilde{Z}^{1/2}\tilde{X}\tilde{Z}^{1/2}) = \frac{\tilde{\Gamma}_m(\gamma)}{\tilde{\Gamma}_m(\alpha)\,\tilde{\Gamma}_m(\gamma-\alpha)} $$

$$ \int_0^1 \left|\det \tilde{U}\right|^{\alpha-m} \left|\det(I-\tilde{U})\right|^{\gamma-\alpha-m} \left|\det(I-\tilde{U}^{1/2}\tilde{Z}^{1/2}\tilde{X}\tilde{Z}^{1/2}\tilde{U}^{1/2})\right|^{-\beta} d\tilde{U} \quad ...(2.3) $$

Where

$Re(\alpha) > (m-1)$; $Re(\gamma-\alpha) > (m-1)$

$Re(\gamma) > (m-1)$; and $\left|\det \tilde{X}\tilde{Z}\right\| < 1$

We also have

$$ {}_2F_1(\alpha, \beta; \beta;\ \tilde{Z}) = {}_1F_0(a;\ \tilde{Z}) = \left|\det(I-\tilde{Z})\right|^{-\alpha} \quad ...(2.4) $$

For $\left|\det \tilde{Z}\right\| < 1$

The results (2.2) follow subject to the following condition.

$$ \left\|\det \tilde{X}^{1/2}\tilde{U}\tilde{X}^{1/2}\right\| \le \left\|\det \tilde{X}\right\|.\left\|\det\tilde{U}\right\| \text{ since } \left|\det \tilde{U}\right\| < 1 $$

and

$$ \left\|\det \tilde{Y}^{1/2}\tilde{V}\tilde{Y}^{1/2}\right\| \le \left\|\det \tilde{V}\right|.\left|\det \tilde{Y}\right\| \le \left|\det \tilde{Y}\right\| $$

since $\left|\det \tilde{V}\right\| < 1$

The next theorem can be proved in the same way.

**Theorem 2 :** If

$Re(\beta) > (m-1)$; $Re(\beta^1) > (m-1)$

$\left|\det \tilde{X}\right\| < 1$ $\left\|\det \tilde{Y}\right\| < 1$

Then the following results hold

$$ \tilde{\Gamma}_m(b)\tilde{\Gamma}_m(b')F_1(a, b, b', g, \tilde{X}, \tilde{Y}) = \int_{\tilde{R}>0}\int_{\tilde{S}>0} \exp(-\mathrm{tr}(\tilde{R}+\tilde{S}). $$

$$ \left|\det \tilde{R}\right|^{b-m} \left|\det(\tilde{S})\right|^{b'-m} $$

$$ {}_1F_1(a; g; \tilde{X}^{1/2}\tilde{R}\tilde{X}^{1/2} + \tilde{Y}^{1/2}\tilde{S}\tilde{Y}^{1/2})d\tilde{R}d\tilde{S} \quad ...(2.5) $$

## III. APPELL FUNCTION $F_2$ OF MATRIX ARGUMENT

The Appell function $F_2$ of matrix argument will be denoted by $F_2(a, b, b'; g; g', \tilde{X}, \tilde{Y})$ analogous to the corrosponding scalar case and is defined dy the integral

$$ F_2(a, b, b'; g; g', \tilde{X}, \tilde{Y}) = \frac{\tilde{\Gamma}_m(g)\tilde{\Gamma}_m(g')}{\tilde{\Gamma}_m(b)\tilde{\Gamma}_m(b')\tilde{\Gamma}_m(g-b)\tilde{\Gamma}_m(g'-b')} $$

$$ \int_0^1\int_0^1 \left|\det \tilde{U}\right|^{b-m} \left|\det \tilde{V}\right|^{b'-m} \left|\det(I-\tilde{U})\right|^{g-b-m} $$

$$ .\left|\det(I-\tilde{U})\right|^{g'-b'-m} \left|\det(I-\tilde{X}^{1/2}\tilde{U}\tilde{X}^{1/2} - \tilde{Y}^{1/2}\tilde{V}\tilde{Y}^{1/2}\right|^{-a} d\tilde{U}d\tilde{V} \cdots (3.1) $$

where $Re(b, b', g-b, g'-b') > m-1$ $\tilde{X} = \tilde{X} > 0$ $\tilde{Y} = \tilde{Y} > 0$

$\tilde{U} = \tilde{U}'$, $\tilde{V} = \tilde{V}'$ $\left\|\tilde{X}\right\| + \left\|\tilde{Y}\right\| < 1$

It may be pointed out to the reader that in definition (3.1) one may use

$$ \det(I - L'\tilde{U}L - M\tilde{V}M)^{-\alpha} \quad ...(3.2) $$

where L,M are any matrices such that

- $M'M = \tilde{Y}$

This does not mean that the determinant in (3.2) is equal to the one in (3.2). But if we write $H = X^{1/2}L^{-1}$; $K = \tilde{Y}^{1/2}M^{-1}$ then $H'H = I, K'K = I$ *i.e.,* H and K are orthogonal.

Substituting $\tilde{X}^{1/2}\tilde{U}\tilde{X}^{1/2} = L'H'\tilde{U}HL$,

$\tilde{Y}^{1/2}\tilde{V}\tilde{Y}^{1/2} = M'K'\tilde{V}KM$ in

(3.1) and transforming to $\tilde{R} = H'UHS = K'\tilde{V}K$ we obtain the definition in terms of (3.2).

**Theorem 3 :** If

$Re(\alpha) > (m-1)$, $\tilde{X} = \tilde{X}' > 0$ $\tilde{Y} = \tilde{Y}' > 0$ $\left|\tilde{X}\right\| + \left\|\tilde{Y}\right\| < 1$

The following results holds :

$$ F_2(a, b, b'; g; g', \tilde{X}, \tilde{Y}) = \frac{1}{\tilde{\Gamma}_m(a)} \int_{\tilde{R}>0} \left|\det \tilde{R}\right|^{a-m} \exp(-\mathrm{tr}\tilde{R}) $$

$$ {}_1F_1(b; g; \tilde{X}^{1/2}\tilde{R}\tilde{X}^{1/2})\,{}_1F_1(b'; g', \tilde{Y}^{1/2}\tilde{R}\tilde{Y}^{1/2})d\tilde{R} \quad ...(3.3) $$

**Proof :** If we substitute for

$\left|\det(I - \tilde{X}^{1/2}\tilde{U}\tilde{X}^{1/2} - \tilde{Y}^{1/2}\tilde{V}\tilde{Y}^{1/2})\right|^{-2}$ on the R.H.S of (3.1) interchange the order of integration which is premissible under the condition stated with the theorem and then integrate outand with the help of the result.

$$ {}_1F_1(a; b; \tilde{X}^{1/2}\tilde{Z}\tilde{X}^{1/2}) = \frac{\tilde{\Gamma}_m(b)}{\tilde{\Gamma}_m(a)\tilde{\Gamma}_m(b-a)} \int_0^1 \left|\det \tilde{U}\right|^{a-m} $$

$$ .\left|\det(I-\tilde{U})\right|^{b-a-m}.\exp(\mathrm{tr}\tilde{X}^{1/2}\tilde{Z}\tilde{X}^{1/2}\tilde{U})d\tilde{U} \quad ...(3.4) $$

Vaild for $Re(\beta) > (m-1)$ $Re(\beta) > (m-1)$

$Re(\beta-\alpha) > (m-1)$ and $\left\|\tilde{X}\tilde{Z}\right| > 1$

since $\mathrm{tr}\tilde{X}^{1/2}\tilde{Z}\tilde{X}^{1/2}\tilde{U} = \mathrm{tr}\tilde{U}^{1/2}\tilde{X}^{1/2}\tilde{Z}\tilde{X}^{1/2}\tilde{U}^{1/2}$ we obtain (3.3)

**Theorem 4 :** If

Re(α), Re(δ – α) > m – 1 and $\|\tilde{X}\| + \|\tilde{Y}\| < 1$

the following results holds :

$$F_2(a, b, b^{'}; g; g^{'}, \tilde{X}, \tilde{Y}) = \frac{\tilde{\Gamma}_m(d)}{\tilde{\Gamma}_m(a)\tilde{\Gamma}_m(d-a)}$$

$$\int_0^I |\det \tilde{Z}|^{a-m} |\det(I - \tilde{Z})|^{d-a-m}$$

$$F_2(d, b, b^{'}; g, g^{'}, \tilde{Z}^{1/2}\tilde{X}\tilde{Z}^{1/2}, \tilde{Z}^{1/2}\tilde{Y}\tilde{Z}^{1/2}) d\tilde{Z} \quad ...(3.5)$$

**Proof :** Writing $F_2$ as a double integral and using (3.1), we have

$$F_2(d, b, b^{'}; g; g^{'}; \tilde{Z}^{1/2}\tilde{X}\tilde{Z}^{1/2}, \tilde{Z}^{1/2}\tilde{Y}\tilde{Z}^{1/2})$$

$$= \frac{\tilde{\Gamma}_m(g)\tilde{\Gamma}_m(g^{'})}{\tilde{\Gamma}_m(b)\tilde{\Gamma}_m(b^{'})\tilde{\Gamma}_m(g-b)\tilde{\Gamma}_m(g^{'}-b^{'})}$$

$$.\int_0^I\int_0^I |\det \tilde{U}|^{b-m} |\det(\tilde{V})|^{b^{'}-m} |\det(I-\tilde{U})|^{g-b-m}$$

$$.|\det(I-\tilde{V})|^{g^{'}-m} |\det(I - \tilde{Z}^{1/2}\tilde{X}^{1/2}\tilde{U}\tilde{X}^{1/2}\tilde{Z}^{1/2} - \tilde{Z}^{1/2}\tilde{Y}^{1/2}\tilde{V}\tilde{Y}^{1/2}\tilde{Z}^{1/2})|$$

$$.|\det(I - \tilde{Z}^{1/2}\tilde{X}^{1/2}\tilde{U}\tilde{X}^{1/2}\tilde{Z}^{1/2} - \tilde{Z}^{1/2}\tilde{Y}^{1/2}\tilde{V}\tilde{Y}^{1/2}\tilde{Z}^{1/2})|^{-d} d\tilde{U}d\tilde{V} \cdots (3.6)$$

Substuting (3.6) on the R.H.S of (3.5) and integrating our with the help of the result (2.1) the result follows :

**Theorem 5 :** If

$\mathrm{Re}(d-b), \mathrm{Re}(d^{'}-b^{'}) > m-1$

$\mathrm{Re}(g-d), \mathrm{Re}(g^{'}-d^{'}) > m-1$

$\mathrm{Re}(d), \mathrm{Re}(d^{'}) > m-1$

$\mathrm{Re}(b), \mathrm{Re}(b^{'}) > m-1$

$\mathrm{Re}(g), \mathrm{Re}(g^{'}) > m-1 \quad \|\tilde{X}\| < 1 \quad$ and $\quad \|\tilde{Y}\| < 1$

The following results hold.

$$F_2(a, b, b^{'}; g; g^{'}, \tilde{X}, \tilde{Y}) = \frac{\tilde{\Gamma}_m(d)\tilde{\Gamma}_m(d^{'})}{\tilde{\Gamma}_m(b)\tilde{\Gamma}_m(b^{'})\tilde{\Gamma}_m(d-b)\tilde{\Gamma}_m(d^{'}-b^{'})}$$

$$.\int_0^I\int_0^I |\det \tilde{U}|^{d-m} |\det(\tilde{V})|^{d-m} |\det(I-\tilde{U})|^{d-b-m}$$

$$F_2(a, d, d^{'}; g; g^{'}; X^{1/2}\tilde{U}\tilde{X}^{1/2}; \tilde{Y}^{1/2}\tilde{V}\tilde{Y}^{1/2}) d\tilde{U}d\tilde{V} \quad ...(3.7)$$

as pointed out by the reference, a proof of theorem (5) will follow directly from the result

$$\int_0^I \tilde{U}^{b-M} |\det(I-\tilde{U})|^{d-b-m} {}_1F_1(d; g; \tilde{R}^{1/2}\tilde{X}^{1/2}\tilde{U}\tilde{X}^{1/2}\tilde{R}^{1/2}) d\tilde{U}$$

$$= \frac{\tilde{\Gamma}_m(b)\tilde{\Gamma}_m(d-b)}{\tilde{\Gamma}_m(d)} {}_1F_1(b; d; \tilde{R}^{1/2}\tilde{X}\tilde{R}^{1/2}) \quad ...(3.8)$$

**Theorem 6 :**

$$F_2(a, b, b^{'}; g; g^{'}, \tilde{X}, \tilde{Y}) = \frac{\tilde{\Gamma}_m(g)\tilde{\Gamma}_m(g^{'})}{\tilde{\Gamma}_m(d)\tilde{\Gamma}_m(d^{'})\tilde{\Gamma}_m(g-d)\tilde{\Gamma}_m(g^{'}-d^{'})}$$

$$.\int_0^I\int_0^I |\det \tilde{U}|^{d-m} |\det(\tilde{V})|^{d^{'}-m} |\det(I-\tilde{U})|^{g-d-m}$$

$$|\det(I-\tilde{U})|^{g^{'}-d^{'}-m} F_2(a, b, b^{'} d; d^{'}, \tilde{X}^{1/2}\tilde{U}\tilde{X}^{1/2}, \tilde{Y}^{1/2}\tilde{V}\tilde{Y}^{1/2}) d\tilde{U}d\tilde{V} \quad ...(3.9)$$

**Proof :** IF we substitute for $F_2$ on the R.H.S of (3.9), it gives 4-fold integral as

$$\sum_1 = X_1 \int_0^I\int_0^I\int_0^I\int_0^I |\det \tilde{U}|^{b-m} |\det \tilde{V}|^{b-m} |\det(I-\tilde{U})|^{d-b-m}$$

$$|\det(I-\tilde{V})|^{d^{'}-b^{'}-m} |\det \tilde{R}|^{d-m} |\det \tilde{S}|^{d^{'}-m} ]$$

$$|\det(I-\tilde{V})|^{d^{'}-b^{'}-m} |\det \tilde{R}|^{d-m} |\det \tilde{S}|^{d^{'}-m}$$

$$|\det(I-\tilde{R})|^{g-d-m} |\det(I-\tilde{S})|^{g^{'}-d^{'}-m}$$

$$|\det(I - \tilde{X}^{1/2}\tilde{R}^{1/2}\tilde{U}\tilde{R}^{1/2}\tilde{X}^{1/2} - \tilde{Y}^{1/2}\tilde{S}^{1/2}\tilde{V}\tilde{S}^{1/2}\tilde{Y}^{1/2}|^{a} d\tilde{U}d\tilde{V}d\tilde{R}d\tilde{S} \cdots (3.10)$$

where

$$X_1 = \frac{\tilde{\Gamma}_m(g)\tilde{\Gamma}_m(g^{'})}{\tilde{\Gamma}_m(b)\tilde{\Gamma}_m(b^{'})\tilde{\Gamma}_m(d^{'}-b^{'})\tilde{\Gamma}_m(d-b)\tilde{\Gamma}_m(g-d)\tilde{\Gamma}_m(g^{'}-d^{'})}$$

Consider the transformation of to and to through

$\tilde{R}^{1/2}\tilde{U}\tilde{R}^{1/2} = \tilde{\wedge}_1 \Rightarrow |\det \tilde{R}|^m d\tilde{U} = d\tilde{\wedge}_1$

$\tilde{S}^{1/2}\tilde{V}\tilde{S}^{1/2} = \tilde{\wedge}_2 \Rightarrow |\det \tilde{S}|^m d\tilde{V} = d\tilde{\wedge}_2$

$0 < \tilde{\wedge}_1 < R, \; 0 < \tilde{\wedge}_2 < S,$ and

$\tilde{U} = R^{-1/2}\tilde{\wedge}_1 R^{-1/2}, \tilde{V} = \tilde{S}^{-1/2}\tilde{\wedge}_2\tilde{S}^{-1/2}$

Substuting the above transformation in (3.10) we observe that

$$\sum d_1 = \tilde{X}_1 \int_{\tilde{\wedge}_1}\int_{\tilde{\wedge}_2}\int_{\tilde{R}}\int_{\tilde{S}} |\det\tilde{\wedge}_1|^{b-m} |\det\tilde{\wedge}_2|^{b^{'}-m}$$

$$.|\det(R - \tilde{\wedge}_1)|^{d-b-m} |\det(\tilde{S} - \tilde{\wedge}_2)|^{d^{'}-b^{'}-m}$$

$$.|\det(\tilde{I} - \tilde{R})|^{g-d-m} |\det(I - \tilde{S})|^{g^{'}-d^{'}-m}$$

$$.|\det(\tilde{I} - \tilde{X}\tilde{\wedge}_1 - \tilde{Y}\tilde{\wedge}_2)|^{-a} d\tilde{\wedge}_1 d\tilde{\wedge}_2 d\tilde{R}d\tilde{S} \quad ...(3.11)$$

The integration is taken over $\tilde{\wedge}_1, \tilde{\wedge}_2 \tilde{R}$ and $\tilde{S}$ such that $0 < \tilde{\wedge}_1 < \tilde{R}, 0 < \tilde{\wedge}_2 < \tilde{S}, 0 < \tilde{R} < \tilde{I}$ and $0 < \tilde{S} < 1$ respectively

Here $.|\det(\tilde{R} - \tilde{\wedge}_1)| = |\det\{(I - \tilde{\wedge}_1) - (I - \tilde{R})\}|,$

$.|\det(\tilde{S} - \tilde{\wedge}_2)| = |\det\{(I - \tilde{\wedge}_2) - (I - \tilde{S})\}|$

Put $I - \tilde{R} = (I - \tilde{\wedge}_1)^{1/2}\tilde{Z}_1(I - \tilde{\wedge}_1)^{1/2}$

$d\tilde{R} = (I - \tilde{\wedge}_1)^m d\tilde{Z}_1$

$I - \tilde{S} = (I - \tilde{\wedge}_2)^{1/2}\tilde{Z}_2(I - \tilde{\wedge}_2)^{1/2}, d\tilde{S} = (I - \tilde{\wedge}_2)^m d\tilde{Z}_2,$

The above substitution in (3.10) yield

$$\sum_1 = X_1 \int_0^I\int_0^I |\det\tilde{\wedge}_1|^{b-m}|\det\tilde{\wedge}_2|^{b'-m}|\det(I-\tilde{\wedge}_1)|^{g-b-m}$$

$$.|\det(I-\tilde{\wedge}_2)|^{g'-b'-m}|\det(I-\tilde{X}^{1/2}\tilde{\wedge}_1\tilde{X}^{1/2}-\tilde{Y}^{1/2}\tilde{\wedge}_1\tilde{Y}^{1/2})|d\tilde{\wedge}_1 d\tilde{\wedge}_2$$

$$.\int_0^I\int_0^I |\det(\tilde{Z}_1)|^{g-d-m}|\det\tilde{Z}_2|^{g'-d-m}|\det(I-\tilde{Z}_1)|^{d-b-m}$$

$$|\det(I-\tilde{Z}_2)|^{g'-b'-m} d\tilde{Z}_1 d\tilde{Z}_2 \qquad ...(3.12)$$

Now interpreting (3.10) with the help of the result due Herz (1955)

$$b_m(a,b) = \int_0^I |\det(\tilde{X})|^{d-m}|\det I-\tilde{X})|^{b-m}d\tilde{X}$$

$$= \frac{\tilde{\Gamma}_m(d)\tilde{\Gamma}_m(b)}{\tilde{\Gamma}_m(a+b)} \qquad ...(3.13)$$

Vaild for $\mathrm{Re}(a) > (m-1)$ $\mathrm{Re}(b) > (m-1)$ the theorem is established.

**Alternative Proof :**

As pointed out by the refree, a simple proof of theorem (6) on be worked out as follow. Note that in theorem (3). since $_1F_1$(like $_1F_0$, $_2F_1$ etc.) is a function of the latent roots of the argument. We may also take $\tilde{R}^{1/2}\tilde{X}\tilde{R}^{1/2}, \tilde{R}^{1/2}\tilde{Y}\tilde{R}^{1/2}$ instead $\tilde{X}^{1/2}\tilde{R}\tilde{X}^{1/2}$ of and $\tilde{Y}^{1/2}\tilde{R}\tilde{Y}^{1/2}$. Hence the result (3.9) follows simply from the integral

$$\int_0^I |\det(\tilde{U})|^{d-m}|\det(I-\tilde{U})|^{g-d-m}\ {}_1F_1(b;d;\tilde{R}^{1/2}\tilde{X}^{1/2}\tilde{U}\tilde{X}^{1/2}\tilde{R}^{1/2})d\tilde{U}$$

$$= \frac{\tilde{\Gamma}_m(d)\tilde{\Gamma}_m(g-d)}{\tilde{\Gamma}_m(g)}\ {}_1F_1(b;g;\tilde{R}^{1/2}\tilde{X}\tilde{R}^{1/2}) \qquad ...(3.14)$$

where $\mathrm{Re}(d) > (m-1)$ $\mathrm{Re}(g-d) > (m-1)$

## IV. APPELL FUNCTION $F_3$ OF MATRIX ARGUMENT

The Appell function $F_3$ of matrix argeiments will be denoted by $F_3(a,a',b,b';g;\tilde{X},\tilde{Y})$ analogous to the corresponding scalar case and is defined by the integral

$$F_3(a,a',b,b';g;\tilde{X},\tilde{Y}) = \frac{\tilde{\Gamma}_m(g)}{\tilde{\Gamma}_m(a)\tilde{\Gamma}_m(a')\tilde{\Gamma}_m(g-a-a')}$$

$$.\int_{0<\tilde{U}<1}^{I}\int_{0<\tilde{V}<1}^{I} |\det(\tilde{U})|^{a-m}|\det\tilde{V}|^{a'-m}$$

$$0<\tilde{U}+\tilde{V}<1$$

$$.|\det(I-\tilde{U}-\tilde{V})|^{g-a-a'-m}|\det(I-\tilde{X}^{1/2}\tilde{U}\tilde{X}^{1/2})|^{-b}$$

$$.|\det(I-\tilde{Y}^{1/2}\tilde{V}\tilde{Y}^{1/2})|^{-b}\ d\tilde{U}d\tilde{V} \qquad ...(3.14)$$

for $\tilde{X}=\tilde{X}'>0, \tilde{Y}=\tilde{Y}'>0 \quad 0<\tilde{X}<I \quad 0<\tilde{Y}<I$

The integration is taken oher $0<\tilde{U}<I, 0<\tilde{V}<I$ and $0<\tilde{U}+\tilde{V}<I$ (4.1) follows by an appeal to type -I Dirchlet integral given by Mathai (1993).

**Theorem 7 :** If

$\mathrm{Re}(a), \mathrm{Re}(a') > (m-1)$, $\mathrm{Re}(g) > (m-1)$

$\mathrm{Re}(g-a-a') > (m-1)$ $\tilde{X}=\tilde{X}'>0, \tilde{Y}=\tilde{Y}'>0$ $0<\tilde{X}+\tilde{Y}<1$

$\|\tilde{X}\|<1$ and $\|\tilde{Y}\|<1$ the following result holds

$$F_3(a,a',b,b';g;\tilde{X},\tilde{Y}) = \frac{1}{\tilde{\Gamma}_m(b)\tilde{\Gamma}_m(b')}$$

$$\int_{\tilde{R}>0}\int_{\tilde{S}>0} \exp[-\mathrm{tr}(\tilde{R}+\tilde{S})]\ |\det\tilde{R}|^{b-m}|\det\tilde{S}|^{b'-m}$$

$$.\Phi(a,a';g;\tilde{X}^{1/2}\tilde{R}\tilde{X}^{1/2},\tilde{Y}^{1/2}\tilde{S}\tilde{Y}^{1/2})d\tilde{R}d\tilde{S} \qquad ...(4.2)$$

Where $\Phi$ function is given by

$$\Phi(b,b';g;\tilde{A},\tilde{B}) = \frac{\tilde{\Gamma}_m(g)}{\tilde{\Gamma}_m(b)\tilde{\Gamma}_m(b')\tilde{\Gamma}_m(g-b-b')}$$

$$\int_{\tilde{U}}\int_{\tilde{V}} |\det\tilde{U}|^{b-m}\ .|\det\tilde{V}|^{b'-m}|\det(I-\tilde{U}-\tilde{V})|^{g-b-b'-m}$$

$$.\exp[\mathrm{tr}(\tilde{U}^{1/2}\tilde{A}\tilde{U}^{1/2}+\tilde{V}^{1/2}\tilde{B}\tilde{V}^{1/2})d\tilde{U}d\tilde{V}$$

The integration is taken oher $0<\tilde{U}<I$, $0<\tilde{V}<I$ and $0<\tilde{U}+\tilde{V}<I$

**Proof :** If we substitute

$${}_1F_0(a;\tilde{U}\tilde{X})\ |\det(I-\tilde{U}\tilde{X})|^{-a}\ \int_{\tilde{Z}>0}|\det\tilde{Z}|^{a-m}$$

$$= \frac{1}{\tilde{\Gamma}_m(g)}\exp[-\mathrm{tr}(I-\tilde{X}^{1/2}\tilde{U}\tilde{X}^{1/2})\tilde{Z}]d\tilde{Z}$$

for $\mathrm{Re}(a) > m-1$ $(I-\tilde{U}\tilde{X})>0$ $\tilde{Z}=\tilde{Z}'>0$

In (4.1) and invert the order of the integral which is premissible under the conditions stated with the theorem. It is found that

$$F_3(a,a',b;b',g,\tilde{X},\tilde{Y})$$

$$= \frac{\tilde{\Gamma}_m(g)}{\tilde{\Gamma}_m(a)\tilde{\Gamma}_m(a')\tilde{\Gamma}_m(g-a-a')\tilde{\Gamma}_m(b)\tilde{\Gamma}_m(b')}$$

$$\int_{\tilde{R}>0}\int_{\tilde{S}>0} |\det\tilde{R}|^{b-m}|\det\tilde{S}|^{b'-m}\ \exp[-\mathrm{tr}(\tilde{R}+\tilde{S}]$$

$$. \int_{0<\tilde{U}<1} \int_{0<\tilde{V}<1} \left|\det \tilde{U}\right|^{a-m} \left|\det \tilde{V}\right|^{a'-m} \left|\det(I - \tilde{U} - \tilde{V})\right|^{a-a'-m}$$

$$0 < \tilde{U} + \tilde{V} < 1$$

$$. \exp \operatorname{tr}[\tilde{U}^{1/2}(\tilde{X}^{1/2}\tilde{R}\tilde{X}^{1/2})\tilde{U}^{1/2} + \tilde{V}^{1/2}(\tilde{Y}^{1/2}\tilde{S}\tilde{Y}^{1/2})\tilde{V}^{1/2}]\, d\tilde{U}\, d\tilde{V}\, d\tilde{R}\, d\tilde{S} \cdots (4.3)$$

Equation (4.2) now follow the authors definition of $\Phi$.

## V. CONCLUSION & FUTURE TREND

This paper Provided integral representation of Appell's function of matrix argument in complex case. The further process is to be continued.

**Acknowledgement :** The author are highely thankful to Prof. P. L. Sethi, Jai Narayan Vyas University, Jodhpur for their valuable suggestions and encouragement throughout the work.

## REFERENCES


[1] Exton, H. "Multiple hypergeometric function and applications" *Ellins Horwood, London,* (1976).

[2] Herz, C.S. Bessel functions of matrix argument, *Ann. of Math,* **61**(2)**:** 474-523 (1955).

[3] HUA, L.K. "Harmonic analysis of function of serveral complex variables in the classical domains" *Transl.Math. Soc. Providence,* **R-1:** (1963).

[4] James, A.T. Distributions of matrix Variables and latent roots derived form normal samples. ann. math. *Statist.,* **35:** 475-501 (1964).

[5] Mathai, A.M. and Saxena, R.K. A generalized probability distribution, *University Nac. Tucuman Rev. Ser. A,* **21:** 193-202 (1971).

[6] Mathai A.M. and saxena, R.K. Various partical problems in probability and statistics where Lauricella's confluent hypergeometric functions appears naturally, *Rajasthan Ganite Parishad,* **I:** 41-48 (1987).

[7] Mathai, A.M. Special function of Matrix arguments and Statistical distributions, *Indian J.Pure Appl. Math,* **22**(11)**:** 887-903 (1991).

[8] Mathai, A.M. : Appell's and Humbert's Functions of Matrix Arguments, Linear *Algebra and its Applications,* **183:** 201.22 (1993).

[9] Muirhead, R.J. Aspects of multivariate Statistical Theory, *Wiley, New York,* (1983).

[10] Saxena R.K. and Sethi, P.L. "Certain properties of function distributions associated with generalized hypergeo-metric functions, *canadian j. Statis.,* **1**(2)**:** 171-180 (1973).

[11] TAKEMURA, A. Zonal polynomials, *Inst. math. Statist Monnograph Series,* **14:** California, (1985).

[12] Vyas, Mahesh and Sethi P.L. : Four varibles associated with hypergeometric functions, Ph.D. Thesis, J.N.V. University Jodhpur (2001).