International Journal of Theoretical & Applied Sciences, **2**(2)**:** 54-57(2010) **ISSN : 0975-1718**

Integration of phystoplankton and mussel in fish farm : Mathematical model of analysis

V.S. Dhakad, D.V.S. Kushwah and Neetu Sharma

Institute of Information Technology and Management, Gwalior (M.P.) INDIA

ABSTRACT : The paper deals with the dynamical behavior of fish and mussel population in a fish farm where external food and phytoplankton are supplied. The ecosystem of the fish farm is represented by a set of nonlinear differential equations involving the nutrient (food), phytoplankton, fish and mussels. We have studied the boundedness, local stability and global stability of the model system.

Keywords : Nutrient (food), phytoplankton, fish and mussels

I. INTRODUCTION

There is a necessity for sustainable development in aquaculture for both individual livelihoods and the economy; most of the European countries have taken some strategy towards the development of aquaculture. There are several constraints. One, the most important, being the environmental concern linked to the location of the fish farm and the impact of their effluents on the surrounding environment. Nutrient pollution form aqua cultural waste exceeds the assimilation capacity of receiving water deterioration water quality [1].

There are different categories of fish farming. The brackish water ponds depend on tidal water flow. Several ponds are connected in a series to allow the fish to move farm several species of fish are cultured. The main foods of such a system are agricultural by-products and commercial feeds. In fresh water pens or cages, the species are dependent on natural food and phytoplankton. In open coastal water, seaweed, oysters and mussels are cultivated. The cultured macro algae are present in this system, External food in sometimes a threat to the fish farm. The heavy organic load causes depletion of dissolved oxygen in the water column. The water Current speed in sometimes decreased for the uncontrolled proliferation of milkfish cages that causes the depletion of dissolved oxygen in the water column with organic loading from fish waster and excess feeds [10]

We have studied a mathematical model of mussel in fish farm [18]. They have considered the deterministic model of a fish farm with three interacting comments: food (nutrient), fish and mussel population. They have stated and proved several results giving criteria for the existence of several equilibrium points and stability. The major, as well as, important results they have obtained for stability (local and global) of the differential equation.

Further in this aquatic ecosystem phytoplankton plays a big role for the growth of the fish population. In this chapter, we proposed a deterministic model of a fish firm with four interacting components, namely, (*i*) nutrient, (*ii*) phytoplankton, (*iii*) fish and (*iv*) mussel population.

MATHEMATICAL MODEL

We consider

x = Nutrient density

p = Phytoplankton density

y = Fish population density

z = Mussel population density

and their interactions are similar as in the previous chapter. Then the model is being formulated with the help of following system of non-linear ordinary differential equation :

$$\frac{dx}{dt} = \phi - (\mu_1 + \alpha p)x$$

$$\frac{dp}{dt} = \alpha_1 px - (\mu_2 + \alpha_2 y + \rho_2 z)p \qquad ...(1)$$

$$\frac{dy}{dt} = -(\delta_1 + \gamma y - \beta p)y$$

$$\frac{dz}{dt} = -(\delta_2 - \eta p)z$$

With the initial conditions

$x(0) > 0,\ p(0) > 0,\ y(0) > 0,\ z(0) > 0$

Where

ϕ = The constant external input of nutrient

α = Harvesting rate of nutrient by the phytoplankton

μ_1 = Outflow or sedimentation rate of nutrient

α_1 = Corresponding the growth rate of Phytoplankton

μ_2 = Death rate of the phytoplankton

α_2 = Decay rate of phytoplankton in the presence of Fish population

ρ_2 = Decay rate of phytoplankton in the presence of mussel population

δ_1 = Death rate of the fish

γ = The intra-specific competition

β = The proportion of phytoplankton contributing to the biomass of fish

δ_2 = Death rate of the mussel population

η = The conversion efficiency of the mussel Population

Here all the parameters are positive and

$\alpha_1 \le \alpha$, $\beta \le \alpha_2$, $\eta \le \rho_2$

EQUILIBRIUM POINTS

The several equilibria are :

1. s_1 $(x_1, 0, 0, 0)$, $x_1 = \dfrac{\phi}{\mu_1}$

2. s_2 $(x_2, p_2, 0, 0)$, where $x_2 = \dfrac{\mu_2}{\alpha_2}$, $p_2 = \dfrac{1}{\alpha}\left(\dfrac{\phi\alpha_1}{\mu_2} - \mu_1\right)$

3. s_3 $(x_3, p_3, y_3, 0)$,

where

$$x_3 = \frac{-\left(\dfrac{\alpha_2\beta}{\alpha\gamma}\mu_1 + \dfrac{\alpha_2\delta_1}{\gamma} + \mu_2\right) + \sqrt{\left(\dfrac{\alpha_2\beta}{\alpha\gamma}\mu_1 + \dfrac{\alpha_2\delta_1}{\gamma} + \mu_2\right)^2 + 4\dfrac{\alpha_1\alpha_2\beta}{\alpha\gamma}\phi}}{2\alpha_1},$$

$$y_3 = \frac{1}{\gamma}(\beta p_3 - \delta_1), \quad p_3 = \frac{1}{\alpha}\left(\frac{\phi}{x_3} - \mu_1\right)$$

4. s_4 $(x_4, p_4, 0, z_4)$,

where

$$x_4 = \frac{\phi}{\left(\mu_1 + \dfrac{\alpha\gamma_2}{\eta}\right)}, \quad p_4 = \frac{\delta_2}{\eta}, \quad z_4 = \left(\frac{\alpha_1 x_4 - \mu_2}{\rho_2}\right)$$

4. s_5 (x_5, p_5, y_5, z_5),

where

$$x_5 = \frac{\phi}{(\mu_1 + \alpha p_5)}, \quad p_5 = \frac{\delta_2}{\eta}, \quad y_5 = \frac{1}{\gamma}(\beta p_5 - \delta_1),$$

$$z_5 = \frac{\alpha_1 x_5 - (\mu_2 + \alpha_2 y_5)}{\rho_2}$$

THEOREM : All the solutions of system (1) with the positive initial condition are uniformly bounded within the region B where

$$B = \{(x, p, y, z) \in R^4_+ : 0 \in x + p + y + z \in \phi/v + \in\}$$

For any $\in\ \succ\ 0$

LOCAL BEHAVIOR

Now we study the stability of model system (1) around several equilibria. We went to study the behavior of the model system in the neighborhood of the equilibria. We winterize model equations the equilibria and determine the associated variational matrix U. if all roots of the characteristic equation of the variation matrix and determine the associated variation matrix U about a point $s_*(x_*, p_*, y_*, z_*)$ have a negative real part, then the model system is stable around the equilibrium point s_* we employ Rough-Hurwitz [11], [12] criteria to determine the conditions for negativity of the real of the roots of the characteristic equation. Here

$$U(s(x_*, p_*, y_*, z_*)) = \begin{bmatrix} -(\mu_1 + \alpha p_*) & -\alpha x_* & 0 & 0 \\ \alpha_1 p_* & \alpha_1 x_* - (\mu_2 + \alpha_2 y_* + \rho_2 z_*) & -\alpha_2 p_* & -\rho_2 p_* \\ 0 & \beta y_* & -(\delta_1 + 2\gamma y_* - \beta p_*) & 0 \\ 0 & \eta z_* & 0 & -(\delta_2 - \eta p_*) \end{bmatrix} \quad ...(2)$$

The characteristic equation in of the variational matrix $U(s(x_*, p_*, y_*, z_*))$ is

$$\det[U(x_*, p_*, y_*, z_*)] - \lambda I_4 = 0 \quad ...(3)$$

Where I_4 the third order identity matrix we is will determine the negativity criteria of the roots of the characteristic equations corresponding to be variational matrix at different equilibria.

Thus the variational matrix system for (1) about $s_1(x_1, 0, 0, 0)$ is

$$U(s_1(x_1, 0, 0, 0)) = \begin{bmatrix} -\mu_1 & -\alpha x_1 & 0 & 0 \\ 0 & \alpha_1 x_1 - \mu_2 & 0 & 0 \\ 0 & 0 & -\delta_1 & 0 \\ 0 & 0 & 0 & -\delta_2 \end{bmatrix} \quad ...(4)$$

and the real parts of the roots of the characteristic equation $\det(U(x_1, 0, 0, 0) - \lambda I_4) = 0$ are negative if

$$\mu_2 - \alpha_1 x_1 < 0 \ \textit{i.e.}, \ \alpha < \frac{\mu_1\mu_2}{\alpha_1}$$

Similarly, the variational matrix of the linearized system about $s_2(x_2, p_2, 0, 0)$ is

$U(s_2(x_2, p_2, 0, 0))$

$$= \begin{bmatrix} -\dfrac{\phi}{x_2} & -\alpha x_2 & 0 & 0 \\ \alpha_1 p_2 & \alpha_1 x_2 - \mu_2 & -\alpha_2 p_2 & -\rho_2 p_2 \\ 0 & 0 & -(\delta_1 - \beta p_2) & 0 \\ 0 & 0 & 0 & -(\delta_2 - \eta p_2) \end{bmatrix} \quad ...(5)$$

and real parts of the roots of the characteristic equation $\det(U(x_2, p_2, 0, 0) - \lambda I_4) = 0$ are negative

Similarly the variational matrix of the linearized system about $s_3(x_3, p_3, y_3, 0)$ is

$U(s_3(x_3, p_3, y_3, 0))$

$$= \begin{bmatrix} -\dfrac{\phi}{x_3} & -\alpha p_3 & 0 & 0 \\ \alpha_1 p_3 & 0 & -\alpha_2 p_3 & -\rho_2 p_3 \\ 0 & \beta y_3 & -\gamma y_3 & 0 \\ 0 & 0 & 0 & -(\delta_2 - \eta p_3) \end{bmatrix} \quad ...(6)$$

and real parts of the roots of the characteristic equation det $Us_3(x_3, p_3, y_3, 0) - \lambda I_4) = 0$ are negative if $\rho/\eta < \delta/\beta$, that is,

$\phi_2 < \phi_1$.

Similarly, the variational matrix associated with the linearized system of (1) about $s_4(x_4, p_4, 0, z_4)$ is

$U(s_4(x_4, p_4, 0, z_4))$

$$= \begin{bmatrix} -\frac{\phi}{x_4} & -\alpha x_4 & 0 & 0 \\ \alpha_1 p_4 & 0 & -\alpha_2 p_4 & -\rho_2 p_4 \\ 0 & 0 & -(\delta_1 - \beta p_4) & 0 \\ 0 & \eta z_4 & 0 & 0 \end{bmatrix} \quad ...(7)$$

and real parts of the roots of the characteristic equation det $(U(x_4, p_4, 0, z_4) - \lambda I_4) = 0$ ...(8)

The real parts of the roots of the Equation (8) are negative if it satisfies the Routh-Hurwitz criteria. Applying the criteria we obtain that the roots are always negative.

And similarly, the variational matrix associated with the linearized system of (1) about $s_5(x_5, p_5, p_5, z_5)$ is

$$U(s_5(x_5, p_5, p_5, z_5)) = \begin{bmatrix} -\frac{\phi}{x_5} & -\alpha x_5 & 0 & 0 \\ \alpha_1 p_5 & 0 & -\alpha_2 p_5 & -\rho_2 p_5 \\ 0 & \beta y_5 & -\gamma y_5 & 0 \\ 0 & \eta z_5 & 0 & 0 \end{bmatrix} \quad ...(9)$$

and real parts of the roots of the characteristic equation det $(U(x_5, p_5, p_5, z_5) - \lambda I_4) = 0$

$$\Delta(\lambda, \tau) \equiv \lambda^4 + \left(\frac{\phi}{x_5} + \gamma y_5\right)\lambda^3 + \left(\frac{\phi}{x_5}\gamma y_5 + \alpha_2 p_5 \beta y_5 + \eta z_5 \rho_2 p_5 + \alpha x_5 \alpha_1 p_5\right)\lambda^2 + \left(\eta z_5 \rho_2 p_5 \gamma y_5 + \frac{\phi}{x_5}\eta z_5 \rho_2 p_5 + \frac{\phi}{x_5}\alpha_2 p_5 \beta y_5 + \alpha x_5 \alpha_1 p_5 \gamma y_5\right) + \frac{\phi}{x_5}\eta z_5 \rho_2 p_5 \gamma y_5 = 0 \quad ...(10)$$

The real parts of the roots of the Equation (10) are negative if it satisfies the Routh-Hurwitz criteria. Applying the criteria we obtain that the roots are always negative.

GLOBAL BEHAVIOR

We want to study the global asymptotic stability of subsystem of (1) with nutrient and Phytoplankton. The equilibrium point of (1) (with fish and mussel population at zero level) is $S_2(x_2, p_2, 0, 0)$. Now S_2 exists and is locally stable if $\phi_1 < \phi_2 < \phi_3$, then the sub system is globally asymptotically stable about S_2 in the $x - p$ plane.

We define the Liapunov function V(x, p) following Harrison [27] as follows :

$$V(x, p) = \int_{x_2}^{x} \frac{s - x_2}{s} ds + \frac{\alpha}{\alpha_2} \int_{p_2}^{p} \frac{s - p_2}{s} ds \quad ...(10)$$

Here $V(x_2, p_2) = 0$ and $V(x, p)$ 1 0 in the $x - p$ plane. Differentiating V along the solution of the subsystem of (1) with r. t. to t, we get

$$= \frac{x - x_2}{x}[\phi - (\mu_1 + \alpha p)x] + \frac{\alpha}{\alpha_1}(p - p_2)(\alpha_1 x - \mu_2)$$

$$= (x - x_2)\left[\frac{\phi - \mu x}{x} - \frac{\phi - \mu x_2}{x_2} - \alpha p + \alpha p_2\right] + \frac{\alpha}{\alpha_1}(p - p_2)(\alpha_1 x - \mu_2)$$

$$= (x - x_2)\left[\frac{\phi(x_2 - x)}{x x_2} - \alpha(p - p_2)\right] - \frac{\alpha}{\alpha_1}(p - p_2)\left[(p - p_2) - \alpha_1(x - x_2)\right]$$

$$= -\frac{\phi(x - x_2)^2}{x x_2} - \frac{\alpha}{\alpha_2}(p - p_2)^2 \leq 0$$

$$\frac{dV}{dt} = \frac{x - x_2}{x}\frac{dx}{dt} + \frac{\alpha}{\alpha_1}\frac{p - p_2}{p}\frac{dp}{dt}$$

$$\frac{dV}{dt} = -\frac{\phi(x - x_2)^2}{x x_2} - \frac{\alpha}{\alpha_2}(p - p_2)^2 \leq 0$$

If $x = x_2$ and $p = p_2$, $\frac{dV}{dt} = 0$. Thus the largest invariant set at which $\frac{dV}{dt} = 0$ is equilibrium $S_2(x_2, p_2, 0, 0)$. Hence, by LaSalle's theorem [14], model subsystem of (1) is globally stable about equilibrium point $S_2(x_2, p_2, 0, 0)$ in the $x - p$ plane.

Now we will study the globally asymptotically stability of model system (1) around

$S_5(x_5, p_5, y_5, z_5)$. We define a Liapunov function $V_1(t) = V_1(x(t), p(t), y(t), z(t))$ defined in

$c = \{(x, p, y, z) \in B, t \in R_+\}$

$$V_1(t) = c_1\left(x - x_5 - x_5 \ln\frac{x}{x_5}\right) + c_2\left(p - p_5 - p_5 \ln\frac{p}{p_5}\right) + c_3\left(y - y_5 - y_5 \ln\frac{y}{y_5}\right) + c_4\left(z - z_5 - z_5 \ln\frac{z}{z_5}\right) \quad ...(11)$$

Where c_1, c_2, c_3, c_4 are positive arbitrary constants to the chosen later? Here $V_1(x, p, y, z) \geq 0$ and $V_1(x_5, p_5, y_5, z_5) = 0$. Differentiating (11) along the trajectories of the model system (1), we get

$$\frac{dV_1}{dt} = c_1 \frac{x - x_5}{x}\left[\phi - (\mu_1 + \alpha p)x\right] + c_2 \frac{p - p_5}{p}$$

$$\left[\alpha_1 px - (\mu_2 + \alpha_2 y + \rho_2 z)p\right] + c_3 \frac{y - y_5}{y}$$

$$\left[-(\delta_1 + \gamma y - \beta p)\, y\right] + c_4 \frac{z - z_5}{z}\left[-(\delta_2 - \eta p)\, z\right]$$

Since $s_5(x_5, p_5, y_5, z_5)$ is an equilibrium point, the above expression reduces to

we choose $c_1 = 1$, $c_2 = 1$, $c_3 = \frac{\alpha_2}{\beta}$ and $c_4 = \frac{\rho_2}{\eta}$ then

$$\frac{dV_1}{dt} = -c_1 \frac{(x - x_5)^2}{x.x_5} - c_3\gamma (y - y_5)^2 \le 0$$

At, $s_5(x_5, p_5, y_5, z_5)$, $\frac{dV_1}{dt} = 0$. Thus the largest invariant subset at which $\frac{dV_1}{dt} = 0$ is the equilibrium $s_5(x_5, p_5, y_5, z_5)$. Hence, by LaSalle's theorem [14], the model system (1) is globally asymptotically stable about the coexisting equilibrium $s_5(x_5, p_5, y_5, z_5)$.

CONCLUSION

In this paper we have considered the deterministic model of a fish farm with four interacting comments : food (nutrient), Phytoplankton, fish and mussel population. We have stated and proved several results giving criteria for the existence of several equilibrium points and stability. The major, as well as, important results we have obtained for stability (local and global) of the differential equation.

REFERENCES

[1] R.L. Naylor, R.J. Goldburg, J.H. Primavera, N. Kautsky, M.C.M. Beveridge, J. Clay, C. Folke, *J. lubchenco, H.Mooney, m. Troell, Effect of aquaculture on world fish supplies, Nature,* **405:** 117-124 (2000).

[2] P. Pitta, I. Karassis, M. Tsapkis, S. Zivanovic, Natural vs mariculture induced variability in nutrients and plankton in the *Eastern Mediterranean. Hydrobiologia,* **391:** 181-194 (1999).

[3] I. Karassis, M. Tsapkis, E. Hatziyanni, P. Pitta, Diel variation of nutrients and chlorophyll in sea beam and sea bass cages in the Mediterranean. *Fresenius Environment Bulletin,* **10**(221)**:** 278-283.

[4] O. Delgado. J. Ruiz, M. perez, J. Romero, E. Ballestreros, Effects of fish farming on sea grass (Posidonia Oceanica) in a Mediterranean bay: sea grass decline after loading cessation. *Oceanol. Acta,* **22:** 109-117.

[5] J. M. Ruiz, m. perez, J. Romero, Effect of fish farm loading on seagrass (Posidonia Oceania) distribution, growth and photosynthsi, marine *Pollution Bulletin,* **42:** 749-760 (2001).

[6] I. Karakassis, P. Pitta, M.D.Krom, Contribution of fish farming to the nutrient loading of the *Mediterranean, Scientia marina,* **69**(2)**:** 313-321 (2005).

[7] S. Ray, M. Strakraba, The impact of detritivorous fishes on the mangrove estuarine system, *Ecological modeling,* **140:** 207-218 (2001).

[8] G.P. Samanta, D. Manna, A. Maiti, Bioeconomic modilling of a three-species fishery with switching effect. The Korean journal of *Computational & Applied Mathematics,* **12**(1-2)**:** 219-231 (2003).

[9] M. Bandyopadhyay, R. Bhattacharyya, B. Mukhopadhyay, Dynamics of autotroph herbivore ecosystem with nutrient recycling, *Ecological Modelling,* **176:** 201-209 (2004).

[10] G. S. Jacinto, fish cage farming in coastal waters-environmental, biological, social and governance issues, pp. 48-56, in : B.V.L. Querijero, C.R. Pagdilao and S.V. Ilagan (Eds) Guidelines in the Establishment of Fish Cages and other structures in lakes and Coastal Waters. PCAMRD Book series NO. 36/2006, *Los Baos, Laguna, Philippines*, 174 2006.

[11] *J.D. Murray mathematical biology springer* (2002).

[12] R. M May, stability and complexity in model eco system

[13] G.W Harrison, N.D. Global stability of predator-prey interaction, *Journal of mathematical Biology,* **8:** 159-171 (1979).

[14] J. LaSalle, S. Lefschetz, stability by Liapunov's Derect Method, Academic press *New York,* (1961).

[15] B.D. Hassard,N.D. Kazarinoff,Y.H. Wan, Theory and Application of Hopf-bifurcation, *Cambridge university press, Cambridge,* (1981).

[16] J.E. Marsard, M.Ma Krachen, Hopf-bifurcation and its application, springer, *New York,* (1974).

[17] M.Kot, Elements of Mathematical Ecology, *Cambridge university press, Cambridge,* (2001).

[18] Nurul Huda Gazi, Safiur Rahaman,Khan,Charu Gopal Chakrarti ,integration of mussel in fish farm : *Mathematical model and analysis,* (2008).