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Common fixed point theorem in fuzzy metric space

Jyoti Nema and K. Qureshi

NRI Institute of Information Sci. & Techn. Bhopal (MP)

Abstract : The purpose of this paper is to prove a unique common fixed point theorem for six mappings on a fuzzy metric space using notion of semi compatibility and the continuity of self maps. Our result generalizes the results of Bijendra Singh, Shishir Jain and Shobha Jain[5].

Keywords : Semi compatibility, Fuzzy metric space, Common fixed point

I. INTRODUCTION

After introduction of Fuzzy sets by Zadeh [8], many researchers have defined fuzzy metric space in different ways. In this paper we are using the definition of Schweizer and Sklar [7] and Kramosil and Michalek [3]. In 1998 Grabiec [2] extended the well known contraction principal to fuzzy metric space .Singh and Chauhan [4] introduced the concept of compatible mappings on fuzzy metric spaces and proved a common fixed point theorem. Recently Singh and Jain [6] have introduced semi compatibility of maps in fuzzy metric space. In this paper, we prove a fixed point theorem for six self maps using the concept of semi compatibility of pair of self maps in fuzzy metric space .

Following definitions are known.

Definition 1 : (Schweizer and Sklar [7])

$* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if it satisfies the following conditions,

(*a*) $*$ is associative and commutative,

(*b*) $*$ is continuous,

(*c*) $a * 1 = a, \ \forall \ a \in [0,1]$,

(*d*) $a * b \leq c * d$,

whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Definition 2 : (Kramosil and Michalek [3])

A triplet $(X, M, *)$ is a fuzzy metric space if X is an arbitrary set, $*$ is continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty) \to [0,1]$ satisfying, $\forall x, y \in X$, the following conditions :

(1) $M(x, y, 0) = 0$,

(2) $M(x, y, t) = 1, \ \forall \ t > 0$ iff $x = y$,

(3) $M(x, y, t) = M(y, x, t)$,

(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s), \ t, s \in [0,1]$,

(5) $M(x, y, .): (0, \infty) \to [0,1]$ is left continuous.

Definition 3 : (George and Veeramani [1])

Let $(X, M, *)$ be a Fuzzy metric space. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all $t > 0$ and $p > 0$.

A Fuzzy metric space X is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 4 : (Singh and Chauhan [4])

Self mappings S and T of a Fuzzy metric space $(X, M,*)$ are said to be compatible if and only if $M(STx_n, TSx_n, t) \to 1$ for all $t > 0$, when ever $\{x_n\}$ is a sequence in X such that $Sx_n, Tx_n \to u$ for some u in X as $n \to \infty$.

Definition 5 : A pair (A,S) of self mappings of a fuzzy metric space is said to be semi compatible if $ASx_n = Sx$ when ever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x.$$

To prove our main theorem, we will use following proposition and the theorem of Bijendra Singh, Shishir Jain and Shobha Jain [5].

Proposition : In a fuzzy metric space $(X, M,*)$ limit of a sequence is unique.

Theorem A : Let A, B, S and T be self mapping of a complete Fuzzy metric space $(X, M, *)$ with continuous t-norm defined by $a * b = \min\{a, b\}$, $a, b \in [0, 1]$ satisfying :

(*i*) $A(X) \subset T(X)$, $B(X) \subset S(X)$,

(*ii*) One of A, B, S and T is continuous,

(*iii*) Pairs (A, S) and (B, T) are semi compatible.

(*iv*) There exists some $k \in (0,1)$ such that for all $x, y \in X$, $t > 0$

$$M(Ax, By, kt) \geq \text{Min}\{M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t), M(Sx, By, 2t), M(Ty, Ax, t)\}$$

(*v*) $\forall x, y \in X$, $M(x, y, t) \to 1$ as $t \infty$

Then A, B, S and T have a unique common fixed point.

II. MAIN THEOREM

Theorem B : Let A, B, S, T, I and J be six self maps of a complete Fuzzy metric space $(X, M, *)$ with continuous t-norm $*$ defined by $a * b = \min\{a, b\}$, $a, b \in [0, 1]$ satisfying :

(*i*) $AB(X) \subset J(X)$, $ST(X) \subset I(X)$,

(*ii*) One of I or AB is continuous,

(*iii*) $AB = BA$, $ST = TS$, $TJ = JT$, $BI = IB$, (*iv*) Pairs (AB, I) and (ST, J) are semi compatible.

(*v*) There exists some $k \in (0,1)$ such that for all $x, y \in X$, $t > 0$

$$M(ABx, STy, kt) \geq \text{Min}\{M(Ix, Jy, t), M(Ix, ABx, t), M(Jy, STy, t), M(Ix, STy, 2t), M(Jy, ABx, t)\}$$

(*vi*) $\forall x, y \in X$, $M(x, y, t) \to 1$ as $t \to \infty$.

Then A, B, S, T, J and I have a unique common fixed point.

Proof : Let $x_0 \in X$ be any point. As $AB(X) \subset J(X)$, $ST(X) \subset I(X)$, $\exists\ x_1 \in X$ and $x_2 \in X$ such that $ABx_0 = Jx_1$ and $STx_1 = Ix_2$. Inductively we construct a sequence $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n-1} = Jx_{2n-1} = ABx_{2n-2}$, $y_{2n} = Ix_{2n} = STx_{2n-1}$ $n = 1,2,3\ldots..$

Now using (*v*), we have

$$\begin{aligned}M(ABx_{2n}, STx_{2n+1}, kt) &\geq \text{Min}\{M(Ix_{2n}, Jx_{2n+1}, t), M(Ix_{2n}, ABx_{2n}, t), M(Jx_{2n+1}, STx_{2n+1}, t), M(Ix_{2n}, STx_{2n+1}, 2t), M(Jx_{2n+1}, ABx_{2n}, t)\}\\ &\geq \text{Min}\{M(y_{2n}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n}, y_{2n+2}, 2t), M(y_{2n+1}, y_{2n+1}, t)\}\\ &= \text{Min}\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n}, y_{2n+2}, t) * M(y_{2n+1}, y_{2n+2}, t), 1\}\\ &= \text{Min}\{M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+2}, t)\}\\ &= M(y_{2n+1}, y_{2n}, t),\end{aligned}$$

As $M(x, y, t)$ is non decreasing (Geroge and Veeramani [1], 1994) .

Similarly $M(y_{2n+1}, y_{2n}, kt) \geq M(y_{2n}, y_{2n-1}, t)$.

Hence $M(y_{n+1}, y_n, kt) \geq M(y_n, y_{n-1}, t)$, $\forall n$

We show that $\lim_{n\to\infty} M(y_{n+p}, y_n, t) = 1$ for all p and $t > 0$

$$\begin{aligned}\text{Now,}\quad M(y_{n+1}, y_n, kt) &\geq M(y_n, y_{n-1}, t/k)\\ &\quad M(y_{n-1}, y_{n-2}, t/k^2)\\ &\quad \ldots\ldots\ldots\ldots\ldots\\ &> M(y_1, y_0, t/k^n) \to 1\end{aligned}$$

as $t/k^n \to \infty$ as $n \to \infty$.

Thus the result holds for $p = 1$. By induction hypothesis suppose that the result holds for $p = r$. Now,

$$\begin{aligned}M(y_n, y_{n+r+1}, t) &\geq M(y_n, y_{n+r}, t/2)*\\ &\quad M(y_{n+r}, y_{n+r+1}, t/2)\\ &\to 1 * 1 = 1\end{aligned}$$

Thus the result holds for $p = r + 1$.

Hence $\{y_n\}$ is a Cauchy sequence in X and as X is complete we get $\{y_n\} \to z \in X$. Hence

$$ABx_{2n} \to z,\ Ix_{2n} \to z$$
$$Jx_{2n+1} \to z,\ STx_{2n+1} \to z$$

Case I : Let I is continuous

$$IABx_{2n} \to Iz,$$
$$I^2x_{2n} \to Iz,$$

As (AB, I) is semi compatible

$$ABIx_{2n} \to Iz$$

Put $x = Ix_{2n}$ and $y = x_{2n+1}$ in (*v*) we have

$$\begin{aligned}M(ABIx_{2n}, STx_{2n+1}, kt) &\geq \text{Min}\{M(I^2x_{2n}, Jx_{2n+1}, t), M(I^2x_{2n}, AB\,Ix_{2n}, t), M(Jx_{2n+1}, STx_{2n+1}, t), M(I^2x_{2n}, STx_{2n+1}, 2t), M(Jx_{2n+1}, ABIx_{2n}, t)\}\\ &\geq \text{Min}\{M(Iz, z, t), M(Iz, Iz, t), M(z, z, t), M(Iz, z, 2t), M(z, Iz, t)\}\\ &\geq M(Iz, z, t)\end{aligned}$$

Therefore, $Iz = z$.

Put $x = z$ and $y = x_{2n+1}$ in (*v*) we have

$$\begin{aligned}M(ABz, STx_{2n+1}, kt) &\geq \text{Min}\{M(Iz, Jx_{2n+1}, t), M(Iz, ABz, t), M(Jx_{2n+1}, STx_{2n+1}, t), M(Iz, STx_{2n+1}, 2t), M(Jx_{2n+1}, ABz, t)\}\\ &\geq \text{Min}\{M(z, z, t), M(z, ABz, t), M(z, z, t), M(z, z, 2t), M(z, ABz, t)\}\\ &\geq M(ABz, z, t)\end{aligned}$$

$\therefore$ $ABz = z$.

Thus $ABz = z = Iz$.

Put $x = Bz$ and $y = x_{2n+1}$ in (*v*) we have

$$\begin{aligned}M(AB(Bz), STx_{2n+1}, kt) &\geq \text{Min}\{M(I(Bz), Jx_{2n+1}, t), M(I(Bz), AB(Bz), t), M(Jx_{2n+1}, STx_{2n+1}, t),\\ &\quad M(I(Bz), STx_{2n+1}, 2t), M(Jx_{2n+1}, AB(Bz), t)\}\end{aligned}$$

As $BI = IB$, $AB = BA$,

So we have $I(Bz) = B(Iz) = Bz$

and

$$\begin{aligned}AB(Bz) &= BA(Bz) = B(ABz) = Bz\\ &\geq \text{Min}\{M(Bz, z, t), M(Bz, Bz, t), M(z, z, t), M(Bz, z, 2t), M(z, Bz, t)\}\\ &\geq M(z, Bz, t).\end{aligned}$$

We get Bz = z and ABz = z implies Az = z .

Therefore $Az = Bz = Iz = z$. ...(1)

As $AB(X) \subset J(X)$, $\exists\ w \in X$ such that

$$ABz = Jw = z.$$

Put $x = x_{2n}$ and $y = w$ using (*v*) we have

$$\begin{aligned}M(ABx_{2n}, STw, kt) &\geq \text{Min}\{M(Ix_{2n}, Jw, t), M(Ix_{2n}, ABx_{2n}, t), M(Jw, STw, t), M(Ix_{2n}, STw, 2t), M(Jw, ABx_{2n}, t)\}\\ &\geq \text{Min}\{M(z, z, t), M(z, z, t), M(z, STw, t), M(z, STw, 2t), M(z, z, t)\}\end{aligned}$$

$\geq$ M(z, STw, t)

$\therefore$ $Jw = STw = z$ and as (ST, J) is semi compatible

$$STJx_{2n+1} \to Jz$$

Since in fuzzy metric space limit of the sequence is unique therefore $STz = Jz$.

Take $x = z$ and $y = z$ in (v)

$$\begin{aligned} M(ABz, STz, kt) &\geq \text{Min }\{M(Iz, Jz, t), M(Iz, ABz, t), \\ &\quad M(Jz, STz, t), M(Iz, STz, 2t), \\ &\quad M(Jz, ABz, t)\} \\ &\geq \text{Min }\{M(z, Jz, t), M(z, z, t), M(Jz, z, t), \\ &\quad M(z, Jz, 2t), M(Jz, z, t)\} \\ &\geq M(z, Jz, t) \end{aligned}$$

which gives $Jz = z$. $\therefore$ $ABz = STz = Jz = Iz = z$.

Put $x = x_{2n+1}$ and $y = Tz$ using (v) we have

$$\begin{aligned} M(ABx_{2n+1}, ST(Tz), kt) &\geq \text{Min }\{M(Ix_{2n+1}, J(Tz), t), \\ &\quad M(Ix_{2n+1}, ABx_{2n+1}, t), \\ &\quad M(J(Tz), ST(Tz), t), \\ &\quad M(Ix_{2n+1}, ST(Tz), 2t), \\ &\quad M(J(Tz), ABx_{2n+1}, t)\} \end{aligned}$$

As $ST = TS$, $TJ = JT$ we have $ST(Tz) = T(STz) = Tz$ and $JTz = TJz = Tz$. Letting $n \to \infty$, we get

$$\begin{aligned} &\geq \text{Min }\{M(z, Tz, t), M(z, z, t), \\ &\quad M(Tz, Tz, t), \\ &\quad M(z, Tz, 2t), M(Tz, z, t)\} \\ &\geq M(z, Tz, t) \end{aligned}$$

$\therefore$ $Tz = z$.

Further $STz = Tz = z$ implies $Sz = z$.

Hence $Sz = Tz = Jz = z$. ...(2)

Combining (1) and (2), we get

$$Az = Bz = Iz = Sz = Tz = Jz = z.$$

Hence z is a common fixed point of A, B, S, T, J and I.

Case II : If AB is continuous, the proof follows by Case I.

Remark : If in theorem B, we take $B = T =$ the identity map on X and $J = T$, $I = S$, $S = B$, we obtain the theorem A.

Uniqueness : Let z' be another fixed point of A, B, S, T, I and J. Then

$$z' = Iz' = Jz' = Sz' = Tz' = Az' = Bz'.$$

Using (v) we get

$$\begin{aligned} M(ABz, STz', kt) &\geq \text{Min }\{M(Iz, Jz', t), M(Iz, ABz, t), \\ &\quad M(Jz', STz', t), \\ &\quad M(Iz, STz', 2t), M(Jz', ABz, t)\} \\ &\geq \text{Min }\{M(z, z', t), \\ &\quad M(z, z, t), M(z', z', t), \\ &\quad M(z, z', 2t), M(z', z, t) \\ &\geq M(z', z, t) \end{aligned}$$

Thus $z = z'$.

Hence z is a unique common fixed point of mapping A, B, S, T, I and J.

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