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Fixed point theorem for fuzzy 2-metric spaces

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ABSTRACT : In the present paper we are proving a common fixed point theorem for fuzzy 2- metric spaces for rational expressions.

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I. INTRODUCTION

In 1965, the concept of fuzzy sets was introduced by Zadeh [36]. After that many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [8], Erceg [10], Kaleva and Seikhala [23], Karamosil and Michalek [25], have introduced the concept of fuzzy metric spaces in different ways. Recently, many authors [1, 6, 11, 17, 20, 21, 22, 27, 28, 31, 32] have also studied the fixed point theory in the fuzzy metric spaces and [2, 3, 4, 5, 19, 26, 33] have studied for fuzzy mappings which opened an avenue for further development of analysis in such spaces and such mappings. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors.

Gahler in a series of papers [13, 14, and 15] investigated 2-metric spaces. Sharma, Sharma and Iseki [30] studied for the first time contraction type mappings in 2-metic space. We [34, 35] have also worked on 2-Metric spaces and 2- Banach spaces for rational expressions.

Definition (1.1 A) : A binary operation *: [0, 1] × [0,1] × [0,1] → [0,1] is called a continuous t-norm if ([0,1],$_*$) is an abelian topological monodies with unit 1 such that $a_1 * b_1 * c_1 \geq a_2 * b_2 * c_2$

whenever

$a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2$ for all a_1, a_2, b_1, b_2 and c_1, c_2 are in [0,1].

Definition (1.1 B): The 3-tuple (X, M, *) is called a fuzzy 2-metric space if X is an arbitrary set, * is continuous t-norm and M is fuzzy set in $X^3 \times [0,\infty)$ satisfying the followings

$(FM' - 1): M(x,y,z,0) = 0$

$(FM' - 2): M(x,y,z,t) = 1, \forall t \succ 0, \Leftrightarrow x = y$

$(FM' - 3): M(x,y,t) = M(x,z,y,t) = M(y,z,x,t)$, symmetryaboutthreevarriable

$(FM' - 4): M(x,y,z,t_1, t_2, t_3) \geq M(x,y,u,t_1) * M(x,u,z,t_2) * M(u,y,z,t_3)$

$(FM' - 5): M(x,y,z): [0,1) \to [0,1]$ is leftcontinuous, $\forall x,y,z,u \in X, t_1, t_2, t_3 \succ 0$

Definition (1.1 C) : Let (X, M, $_*$) be a fuzzy 2-metric space. A sequence $\{x_n\}$ in fuzzy 2-metric space X is said to be convergent to a point x ∈ X,

$$\lim_{n\to\infty} M(x_n, x, a, t) = 1, \text{for all } a \in X \text{ and } t \succ 0$$

(2) A sequence $\{x_n\}$ in fuzzy 2-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} M(x_{n+p}, x_n, a, t) = 1, \text{for all } a \in X \text{ and } t, p \succ 0$$

(3) A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition (2.1 D): A function M is continuous in fuzzy 2-metric space, iff whenever

For all a ∈ X and t > 0.

$x_n \to x, y_n \to y,$

then $\lim_{n\to\infty} M(x_n, y_n, a, t) = M(x,y,a,t), \forall a \in X \text{ and } t \succ 0$

Definition (1.1 E): Two mappings A and S on fuzzy 2-metri space X are weakly commuting iff

$$M(ASu, SAu, a, t) \geq M(Au, Su, a, t), \forall u, a \in X \text{ and } t \succ 0$$

1.2 Some Basic Results

Lemma (i) [17] for all x, y,∈ X, M(x, y) is non -decreasing.

Lemma (ii) [7] Let $\{y_n\}$ be a sequence in a fuzzy metric space (X, M,*) with the condition

(FM -6) If there exists a number q ∈ (0,1) such that

$M(y_{n+2}, y_{n+1}, qt) \geq M(y_{n+1}, y_n, t), \forall t \succ 0 \text{ and } n = 1,2,3...,$ then $\{y_n\}$ isacauchysequenceinX.

Lemma (iii) [28] If, for all x, y ∈ X, t > 0 and for a number q ∈ (0,1),

$$M(x,y,qt) \geq M(x,y,t), \text{then } x = y$$

Lemma (i,ii and iii) are also true for fuzzy 2- metric spaces.

II. MAIN RESULT

THEOREM (2.1) : Let (X,M,*) be a complete fuzzy 2-metric space and let S and T be continuous mappings of X in X, then S and T have a common fixed point in X if there exists continuous mapping A of X into S (X) $\cap$ T (X) which commute weakly with S and T and,

(3.1a)

$$M(Ax, Ay, a, qt) \geq \min \begin{Bmatrix} M(Ty, Ay, a, t), M(Sx, Ax, a, t), \\ M(Sx, Ty, a, t), \dfrac{M(Sx, Ty, a, t)}{M(Ax, Ty, a, t)}, \end{Bmatrix}$$

for all x, y, a $\in$ X, t > 0, and 0 < q < 1And

(3.1b) $\lim_{n\to\infty} M(x, y, z, t) = 1$ for all x, y, z in X

Then S, T and A have a unique common fixed point

PROOF: We define a sequence $\{x_n\}$ such that $Ax_{2n} = Sx_{2n-1}$ and $Ax_{2n-1} = Tx_{2n}$, n = 1, 2,

We shall prove that $\{Ax_n\}$ is a Cauchy sequence. For this suppose $x = x_{2n}$ and $y = x_{2n+1}$ in (3.1a), we write

$$M(Ax_{2n}, Ax_{2n+1}, a, qt)$$

$$\geq \min \begin{Bmatrix} M(Tx_{2n+1}, Ax_{2n+1}, a, t), M(Sx_{2n}, Ax_{2n}, a, t), \\ M(Sx_{2n}, Tx_{2n+1}, a, t), \dfrac{M(Sx_{2n}, Tx_{2n+1}, a, t)}{M(Ax_{2n}, Tx_{2n+1}, a, t)}, \end{Bmatrix}$$

$$M(Ax_{2n}, Ax_{2n+1}, a, qt)$$

$$\geq \min \begin{Bmatrix} M(Ax_{2n}, Ax_{2n+1}, a, t), M(Ax_{2n+1}, Ax_{2n}, a, t), \\ M(Ax_{2n+1}, Ax_{2n}, a, t), \dfrac{M(Ax_{2n+1}, Ax_{2n}, A, t)}{M(Ax_{2n}, A_{2n}, a, t)} \end{Bmatrix}$$

$$= \min \begin{Bmatrix} M(Ax_{2n}, Ax_{2n+1}, a, t), M(Ax_{2n+1}, Ax_{2n}, a, t), \\ M(Ax_{2n+1}, Ax_{2n}, a, t), 1 \end{Bmatrix}$$

$$\geq \min \left\{ M\left(Ax_{2n-1}, Ax_{2n}, a, \frac{t}{q}\right), M\left(Ax_{2n}, Ax_{2n-1}, a, \frac{t}{q}\right) \right\}$$

There fore

$$M(Ax_{2n}, Ax_{2n+1}, a, qt) \geq M\left(Ax_{2n-1}, Ax_{2n}, a, \frac{t}{q}\right)$$

By induction

$$M(Ax_{2k}, Ax_{2m+1}, a, qt) \geq M\left(Ax_{2m}, Ax_{2k-1}, a, \frac{t}{q}\right)$$

for every *k* and *m* in N, Further if 2m +1 > 2k, then

$$M(Ax_{2k}, Ax_{2m+1}, a, qt) \geq M\left(Ax_{2k-1}, Ax_{2m}, a, \frac{t}{q}\right) \ldots\ldots$$

$$\geq M\left(Ax_0, Ax_{2m+1-2k}, a, \frac{t}{q^{2k}}\right) \quad \ldots(3.1c)$$

If 2k > 2m+1, then

$$M(Ax_{2k}, Ax_{2m+1}, a, qt) \geq M\left(Ax_{2k-1}, Ax_{2m}, a, \frac{t}{q}\right) \ldots\ldots$$

$$\geq M\left(Ax_{2k-(2m+1)}, Ax_0, a, \frac{t}{q^{2m+1}}\right) \quad \ldots(3.1d)$$

By simple induction with (3.1c) and (3.1d) we have

$$M(Ax_n, Ax_{n+p}, a, qt) \geq M\left(Ax_0, Ax_p, a, \frac{t}{q^n}\right).$$

For n = 2k, p = 2m+1 or n = 2k+1, p = 2m +1 and by (FM-4)

$$M(Ax_n, Ax_{n+p}, a, qt) \geq M\left(Ax_0, Ax_1, a, \frac{t}{2q^n}\right).$$

$$* M\left(Ax_1, Ax_p, a, \frac{t}{q^n}\right). \quad \ldots(3.1e)$$

If n = 2k, p = 2m or n = 2k + 1, p = 2m

For every positive integer *p* and *n* in N, by nothing that

$$M\left(Ax_0, Ax_p, a, \frac{t}{q^n}\right). \to 1 \text{ as } n \to \infty$$

Thus $\{Ax_n\}$ is a Cauchy sequence. Since the space X is complete there exists z $\in$ X, such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} Tx_{2n} = z$$

It follows that Az = Sz = Tz and

Therefore

$$M(Az, AAz, a, qt) \geq \min \begin{Bmatrix} M(TAz, AAz, a, t), M(Sz, Az, a, t), \\ M(Sz, TAz, a, t), \dfrac{M(Sz, TAz, a, t)}{M(Az, TAz, a, t)} \end{Bmatrix}$$

$M(Az, A^2z, a, qt) \geq M(Sz, TAz, at)$

$\geq M(Sz, ATz, a, t)$

$\geq M(Az, A^2z, a, t)$

$$\geq M\left(Az, A^2z, a, \frac{t}{q^n}\right).$$

Since, $\lim_{n\to\infty} M\left(Az, A^2z, a, \frac{t}{q^n}\right) = 1 \Rightarrow Az = A^2z$

Thus z is common fixed point of A, S and T.

For uniqueness, let v (v ≠ z) be another common fixed point of S, T and A.

By (3.1a) we write

$$M(Az, Av, a, qt) \geq \min \begin{Bmatrix} M(Tv, Av, a, t), M(Sz, Az, a, t), \\ M(Sz, Tv, a, t), \dfrac{M(Sz, Tv, a, t)}{M(Az, Tv, a, t)} \end{Bmatrix}$$

$$\mathrm{M}(\mathrm{Az}, \mathrm{Av}, \mathrm{a}, \mathrm{qt}) \geq \min\{\mathrm{M}(\mathrm{z}, \mathrm{v}, \mathrm{a}, \mathrm{t})\}$$

This implies that

$$\mathrm{M}(\mathrm{z}, \mathrm{v}, \mathrm{a}, \mathrm{qt}) \geq \min\{\mathrm{M}(\mathrm{z}, \mathrm{v}, \mathrm{a}, \mathrm{t})\}$$

Therefore by lemma (iii) of (1.2) we write z = v

This completes the proof of Theorem (2.1)

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