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Some results on M-fuzzy metric space

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ABSTRACT : In this paper some common fixed point theorems are proved by using the property (E) and different implicit relations. Also, the number of weak compatible mappings are extended from four to six and Fuzzy metric space is replaced by M-Fuzzy metric space.

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I. INTRODUCTION

In 1965, Zadeh [1] introduced the famous theory of fuzzy sets and used it as a tool for dealing with uncertainty arising out of lack of information about certain complex system. Since then, to use this concept in topology and analysis many authors have extensively developed the theory of fuzzy sets and applications. Fixed point theorems in fuzzy mathematics are emerging with vigorous hope and vital trust. It appears that Kramosil and Michalek's study of fuzzy metric spaces paves a way for very soothing machinery to develop fixed point theorems for contractive type maps. George and Veeramany [2], Kramosil and Michalek [3] have introduced the concept of fuzzy topological space induced by fuzzy metric space which has very important application in quantum physics.In due course of time many researchers have defined fuzzy metric space in different ways. Vashuki [4] obtained the fuzzy version of common fixed point theorem for using extra conditions. Grabiec [5], Subrahmanyam [6], and Vasuki [4] used this concept to generalize some metric fixed point results; one such generalization is generalized metric space or D-metric space initiated by Dhage [7] in 1992. He proved some regulation fixed points for a self map satisfying a contraction for complete and bounded D-metric spaces. Rhoades [8] generalized Dhage's contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self-map in D-metric space. Recently Sedghi and Shobhe [9] introduced D*metric space as a probable modification of the definition of Dmetric introduced by Dhage, and proved some basic properties in D*metric spaces. Using D*metric concepts, Sedghi and Shobhe defined M–fuzzy metric space and proved a common fixed point theorem in it. Using this concept, many researchers worked on it and proved many results. One of such result was proved by Chauhan [10] using four weak compatible mappings.

The main objective of this paper is to obtain some common fixed point theorems for six weak compatible mappings using implicit relation with property (E), which improve and extend the results of Chauhan and Nidhi [11] ,Rathore et.al [12], Pant [13], Singh and Jain [14] and Popa [15,16] by

(i) replacing the fuzzy metric space by M-fuzzy metric space.

(ii) increase the number of self maps from four to six .

(iii) use the property (E)

First we give some known definitions and results in M-fuzzy metric space given by Sedghi and Shobe [13] and then prove our main result.

Definition 1.1 ([17]) A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t- norm if it satisfies the following conditions

(1) * is associative and commutative,

(2) * is continuous,

(3) $a * 1 = a$ for all $a \in [0, 1]$,

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each a, b, c, $d \leq [0, 1]$.

Two typical examples of continuous t-norm are $a*b = a\,b$ and $a*b = \min\{a, b\}$.

Definition 1.2 ([9]) A 3-tuple (X, M, *) is called a M-fuzzy metric space if X is an arbitrary (non-empty) set, * is a continuous t-norm, and M is a fuzzy set , satisfying the following conditions for each x, y, z, a on $X^3 \times (0, \infty) \in X$ and t, s > 0,

(1) $M(x, y, z, t) > 0$,

(2) $M(x, y, z, t) = 1$ if and only if $x = y = z$,

(3) $M(x, y, z, t) = M(p\{x, y, z\}, t)$, (symmetry) where p is a permutation function,

(4) $M(x, y, a, t) * M(a, z, z, s) \leq M(x, y, z, t + s)$,

(5) $M(x, y, z, .): (0, \infty) \rightarrow [0, 1]$ is continuous.?

Remark 1.3 ([9]) Let (X, M, *) be a M-fuzzy metric space. Then for every $t > 0$ and for every $x, y \in X$ we have $M(x, x, y, t) = M(x, y, y, t)$.

Definition 1.3 ([9]) Let (X, M, *) be a M-fuzzy metric space. For $t > 0$, the open ball BM(x, r, t) with center $x \in X$ and radius $0 < r < 1$ is defined by $BM(x, r, t) = \{y \in X: M(x, y, y, t) > 1 - r\}$.

A subset A of X is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$such that $BM(x, r, t) \subseteq A$.

Definition 1.4 ([9]) A sequence $\{x_n\}$ in X converges to x if and only if $M(x, x, x_n, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < e < 1$ and $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x_n, x_m, t) > 1 - e$ for each $n, m \geq n_0$.The M-fuzzy metric space (X, M, *) is said to be complete if every Cauchy sequence is convergent.

Lemma 1.1 ([9]). Let (X, M, *) be a M-fuzzy metric space. Then M(x, y, z, t) is non-decreasing with respect to t, for all x, y, z in X.

Lemma 1.2 ([9]). Let (X, M, *) be a M-fuzzy metric space. Then M is continuous function on $X^3 \times (0,\infty)$.

In 1998, Jungck and Rhoades introduced the concept of weakly compatibility of pair of self mappings in a metric space.

Definition 1.5 Let f and g be two self maps of (X, M, *). Then *f* and *g* are said to be weakly compatible if there exists *u* in X with fu = gu implies fgu = gfu.

Here we introduce new implicit relation and an example in support of our main result.

A class of implicit relation. Let F be the set of all real continuous functions f : $(R+)^4 \to R$, non decreasing and satisfying the following conditions.

$(R+)^4 \to R$, nondecreasing and satisfying the following conditions.

(i) For $u ,v \geq 0$, $f(u, u ,u, u) \geq 0$ implies that $u \geq 1$.

(ii) $f(u,1, 1, 1) \geq 0$ implies that $u \geq 1$.

Example. De?ne $f(t_1,t_2,t_3,t_4) = 6t_1$ “ $6t_2 + 3t_3$ “ $2t_4$-1 .

Then $f \in F$.

Deûnition 1.6 . Let A and B be two self-mappings of a M-fuzzy metric space (X, M, ∗). We say that A and B satisfy the property (E), if there exists a sequence {xn} such that

Lim $n \to \infty$ M(Axn, u, u, t) = lim $n \to \infty$ M(Bxn, u, u, t) = 1 for some $u \in X$ and $t > 0$ for every $t > 0$. Then A and B satisfying in the property (E).

Example Let X = R and $M(x, y, z, t) = \exp\{t / (t + |x – y| + |y – z| + |x – z|)\}$for every $x, y, z \in X$ and $t > 0$. Let A and B defined by

Ax = x + 1, Bx = x + 2.

Consider the sequence xn = n+ 1, n = 1, 2, · · · . Thus we have

lim $n \to \infty$ M (A xn, 4, 4, t) = lim $n \to \infty$ M (Bxn, 4, 4, t) = 1for every $t > 0$.

Then A and B satisfying in the property (E).

II. MAIN RESULTS

Theorem 2.1 Let S and T be two continuous self mappings of a complete M-fuzzy metric space (X,M,∈).Let A and B be two self mappings of X satisfying

(1) $AB(X) \subseteq J(X)$, $ST(X) \subseteq I(X)$.

(2) {AB, I} and {ST, J} are weakly compatible pairs, and

(3) {AB, I} or {ST, J} satisfy the property (E).

(4) for some $f \in \Phi$,x ,y ,z ∈ X, $t > 0$

f { M(ABx ,STy , Iz ,t), M(Ix, Jy, Jz, t), M(Jy, STy, STz, t),M(STy, Jy, Jz,t)} ≥ 0.

Then AB, ST, I and J have unique common fixed point.

Proof. Let the pair (ST, J) satisfy the property (E), then there exist a sequence {xn} such that M(STxn, v, v, t) = lim n $\to \infty$ M(Jxn, v, v, t) = 1 . As, $ST(X) \subseteq I(X)$,then there exist a sequence {yn} s.t. STxn = Iyn .Then lim $n \to \infty$ M(Jxn, v, v, t) = 1.

Now , Put x = yn, y = xn, z = xn+1 in (4) ,we get

f { M(AB yn ,ST xn , ST xn+1, t), M(I yn , J xn , J xn+1, t), M(J xn , ST xn , ST xn+1, t),

M(ST xn , J xn , J xn+1 ,t)} ≥ 0,

Taking $n \to \infty$

f { M(AB yn ,v, v ,t), M(v , v , v , t), M(v, v, v , t),M(v, v , v ,t)} ≥ 0

f { M(AB y_n ,v, v ,t), 1, 1,1}≥ 0

lim $n \to \infty$ M(AB y_n ,v, v ,t) e” 1

Therefore, lim $n \to \infty$ AB y_n = lim $n \to \infty$ I y_n = lim $n \to \infty$ STy_n = lim $n \to \infty$ J y_n = v.

Since $AB(X) \subseteq J(X)$, $ST(X) \subseteq I(X)$ then there must exist u, v ∈ X s.t. Jq= v, Ip = v

Put x = p, y = x_n , z = x_n in condition (4)

f {M(AB p,STx_n ,STx_n ,t), M(Ip , J x_n , J x_n , t),M(J x_n ,ST x_n , ST x_n , t),M(ST x_n ,J x_n ,J x_n,t)}≥ 0

f { M(AB p , v, v, t), M(Ip, v, v, t), M(v, v, v, t),M(v, v , v ,t)}≥ 0

f { M(AB p ,v, v ,t), M(v , v , v , t), M(v, v, v , t),M(v, v , v ,t)}≥ 0

AB p = v , which implies AB p = v = Ip.

Again put x = p, y = x_n, z = q in condition (4)

f { M(AB p, ST x_n, STq, t), M(Ip, Jx_n, Jq, t), M(J x_n, STx_n, STq , t),M(ST x_n ,Jx_n , Jq ,t)}≥ 0

f{ M(v , v, STq, t), M(v, v, Jq, t), M(v, v, STq, t), M(v, v, Jq , t)}≥ 0 f{ M(v, v, STq, t), M(v, v, v, t), M(v, v, v, t), M(v, v, v, t)} ≥ 0

f{ M(v ,v, STq ,t), 1,1 ,1}≥ 0

M(v ,v, STq ,t) $\geq$ 1.

Therefore , STq = v. ,which implies Jq = STq = v .

Thus, AB p = Ip= Jq = STq

Since, (AB, I) and (ST, I) are weak compatible, therefore ABIp = IABp implies

ABv = Jv .Also, STJq = JSTq implies STv = Jv .

Therefore, v is a coincident point of AB, ST, I, and J.

Now, we shall prove that v is a fixed point of AB, ST, I and J

Put x = p, y = x_n, z = v in (4) ,we get

f{M(AB p, ST x_n, STv, t), M(Ip, Jx_n, Jv, t), M(Jx_n, STx_n, STv, t), M(STx_n, Jx_n, Jv, t)}$\geq$ 0

f{M(v, v, STv, t), M(v, v, Jv, t), M(v, v, STv, t), M(v, v, Jv, t)}$\geq$ 0

f{M (v,v, Jv ,t), M(v, v, Jv, t), M(v, v, Jv, t), M(v, v, Jv, t)} ≥ 0

M(v, v, Jv, t) $\geq$ 1 ,which implies Jv = v.

AB v = Iv= Jv = STv = v.

Consequently, v is a common fixed point of AB, ST, I and J.

Uniqueness

Let w be a fixed point other than v of AB, ST, I and J. Then, ABw = STw = Iw = Jw = w

Put x = v, y = v, z = w in (4), we get

f {M(ABv, STv, STw, t), M(Iv, Jv, Jw, t), M(Jv, STv, STw, t), M(STv, Jv, Jw, t)}$\geq$ 0,implies

f {M(v, v, w, t), M(v, v, w, t), M(v, v, w, t) ,M(v, v, w t)}$\geq$ 0

M(v, v, w, t) $\geq$ 1 ,which implies v = w. Hence, v is a unique fixed point of AB, ST, I and J.

Corollary 2.1.

Let S and T be two continuous self mappings of a complete M-fuzzy metric space

(X, M, *). Let A and B be two self mappings of X satisfying

(1) A(X) $\subseteq$ J(X), S(X) $\subseteq$ I(X).

(2) {A, I} and {S, J} are weakly compatible pairs, and

(3) {A, I} or {S, J} satisfy the property (E).

(4) for some f$\in$F,x ,y ,z $\in$ X , t > 0

f { M(Ax ,Sy, Iz ,t), M(Ix, Jy, Jz, t), M(Jy, Sy, Sz, t),M(Sy, Jy, Jz,t)}$\geq$ 0.

Then A, S, I and J have unique common fixed point.

Proof. Taking B = T = Id =Identity mapping in above theorem, we get the required result.

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