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Qualitative aspects of population interaction with dispersal

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ABSTRACT : A system of differential equation of dispersion between two populations in habitats separated by a barrier with a predator feeding indiscriminately on this population is considered. In this paper, we have extend the result of [2] in term of adding more nonlinearty. A region in the $e_1 - e_2$ plane where equilibrium points exist is studied. The stability properties of these equilibrium points are investigated.

Keywords : Ecology, Prey-Predator, Habited, Stability.

I. INTRODUCTION

The subject of the effect of dispersal of population is a topic of considerable ecological interest. Holt [6] has considered a two patch model and a migrating predator from an optimal habitat selection point of view. Hasting [1] focused on spatial diffusion but also had a two patch model and showed the stabilizing effect of dispersal rates .The view of the problem taken in this paper, as in Freedman and Waltman[4] and Freedman[3], is that "pressure" to disperse is given as a (monotone increasing)function of population size but that dispersal is inhibited by the difficulty of leaving the habitat which we, in turn, think of as surmounting a "barrier". It turns out that the more reasonable parameter is inverse barrier strength. This view appeared in Freedman and Waltman[4] When this (vector) parameter is zero, dispersal is impossible (the barrier is infinite) and each population grows to its carrying capacity.

By the implicit function theorem, Freedman and Waltman[4] showed that, for Small values of this parameter, the equilibrium was continuous and they approximated this equilibrium as an expansion in the parameter. This was done in Freedman and Waltman [4] for two habitats and common barrier strength.

Both the two habitats and n-habitats cases each permitted to have a different level of difficulty in its "escape" barrier. In addition, once the population has left its present habitat it may not successfully reach a new one (predation harvesting, or for other reasons). In this analysis, by Freedman [3] they regarded the probability of a successful transition between habitats as given and analyzed the question of the existence of the equilibrium as a function of the inverse barrier strength. Under reasonable biological hypotheses and one technical hypothesis, they determine the region exactly for two habitats and, in general case of n habitats [3].

II. THE MODEL EQILIBIRIA

In this section we shall consider the case where the population is able to disperse among two different habitats at some cost to the population in the sense that the probability of survival during a change habitat may be less than one. This situation is described by a system of two preys and one predator of the form

$$x_1' = \alpha_1 x_1\left(1-\frac{x_1}{k_1}\right) - \beta_1 x_1 y - \varepsilon_1 x_1 + \varepsilon_2 p_{21} x_2$$

$$x_2' = \alpha_2 x_2\left(1-\frac{x_2}{k_2}\right) - \beta_2 x_2 y - \varepsilon_2 x_2 + \varepsilon_1 p_{12} x_1$$

$$y' = y(-\gamma + \delta_1 x_1 + \delta_2 x_2)$$

with $p_{12} + p_{21} \le 1$ and $x_i(0) > 0$, $i = 1, 2$ and $y(0) > 0$.

x_i represent the same population (prey)in the two habitats; y is a predator feeding indiscriminately on the two prey x_1 and x_2 ; β_1 and β_2 measure the feeding rates of the predator on the two prey x_1 and x_2 respectively ; γ is the death rate of predator ; δ_1 and δ_2 the conversion rates of prey to predator; ε_1 and ε_2 not necessarily small but positive and represent inverse barrier strength in going out of the first habitat and second habitat ; and p_{ij} is the probability of successful transition from i^{th} habitat to j^{th} habitat (where).

Above system has been discussed by H.EL-OWAIDY AND A.A.AMMAR [2] in the special case in which ε_1 and ε_1 are positive but not necessarily small and also determine its stability properties. In our paper, we extend the result of H.EL-OWAIDY AND A.A.AMMAR [2] in term of adding more nonlinearity. The modified model now takes the form:

$$x_1' = \alpha_1 x_1\left(1-\frac{x_1}{k_1}\right) - \beta_1 x_1 y - \varepsilon_1 x_1^2 + \varepsilon_2 p_{21} x_2$$

$$x_2' = \alpha_2 x_2\left(1-\frac{x_2}{k_2}\right) - \beta_2 x_2 y - \varepsilon_2 x_2^2 + \varepsilon_1 p_{12} x_1 \qquad \text{... (2.1)}$$

$$y' = y(-\gamma + \delta_1 x_1 + \delta_2 x_2^2)$$

Let $E^*(\varepsilon_1, \varepsilon_2) \equiv (x_1^*(\varepsilon_1,\varepsilon_2),\ x_2^*(\varepsilon_1,\varepsilon_2),\ y^*(\varepsilon_1,\varepsilon_2))$ denote an equilibrium point in positive octant, if it exists. Then, we have following theorems.

Theorem 2.1. E^* (0, 0) exists if:

$$\frac{\alpha_1}{\beta_1}-\frac{\alpha_2}{\beta_2}>\frac{\alpha_1\gamma}{\beta_1 k_1\delta_1}\ ,\ k_2^2>\frac{\gamma}{\delta_2}$$

and $$\delta_1 k_1+\delta_2 k_2^2+\frac{\delta_2\beta_2^2 k_2^2 y^2}{\alpha_2^2}>\gamma \qquad ...(2.2)$$

where

$x_1^*(0,0)$

$$=\frac{\alpha_2^2\beta_1^2k_1^2}{2k_2^2\alpha_1^2\beta_2^2}\left[\frac{-\delta_1}{\delta_2}+\frac{2\alpha_1\beta_2k_2^2}{k_1\beta_1^2\alpha_2^2}(\beta_2\alpha_1-\alpha_2\beta_1)\pm\sqrt{\left(\frac{\delta_1}{\delta_2}-\frac{2\alpha_1\beta_2k_2^2}{k_1\beta_1^2\alpha_2^2}(\beta_2\alpha_1-\alpha_2\beta_1)\right)^2-\frac{4k_2^2\alpha_1^2}{k_1^2\beta_1^2\alpha_2^2}\left\{\frac{k_2^2}{\alpha_2^2\beta_1^2}(\beta_2\alpha_1-\alpha_2\beta_1)^2-\frac{\gamma}{\delta_2}\right\}}\right]$$

$x_2^*(0,0)$

$$=\frac{k_1\delta_1}{2\alpha_1\beta_2\delta_2}\left[\frac{-\alpha_2\beta_1}{k_2}\pm\sqrt{\frac{\alpha_2^2\beta_1^2}{k_2^2}-\frac{4\alpha_1^2\beta_2^2\delta_2}{k_1\delta_1}+\frac{4\alpha_1\beta_2\delta_2\beta_1\alpha_2}{k_1\delta_1}+\frac{4\alpha_1^2\beta_2^2\delta_2\gamma}{k_1^2\delta_1^2}}\right]$$

y^* (0, 0)

$$\frac{k_1\delta_1\alpha_2}{2\alpha_1\beta_2^2\delta_2k_2}\left[\frac{-\alpha_2\beta_1}{k_2}\pm\sqrt{\frac{\alpha_2^2\beta_1^2}{k_2^2}-\frac{4\alpha_1^2\beta_2^2\delta_2}{k_1\delta_1}+\frac{4\alpha_1\beta_2\delta_2\beta_1\alpha_2}{k_1\delta_1}+\frac{4\alpha_1^2\beta_2^2\delta_2\gamma}{k_1^2\delta_1^2}}\right]$$

...(2.3)

Proof. At $\varepsilon_1=\varepsilon_2=0$, equilibria are solution of following system of equations is,

$$\alpha_1x_1(1-\frac{x_1}{k_1})-\beta_1x_1y=0$$

$$\alpha_2x_2(1-\frac{x_2}{k_2})-\beta_2x_2y=0 \qquad ...(2.4)$$

$$\delta_1x_1+\delta_2x_2^2-\gamma=0$$

$$\alpha_1\ (1-\frac{x_1}{k_1})-\beta_1y=0$$

or $$\alpha_2(1-\frac{x_2}{k_2})-\beta_2y=0$$ ...(2.5) eliminating y

$$\delta_1x_1+\delta_2x_2^2y-\gamma=0$$

from first two equation of (2.5)

we get $$\beta_2\alpha_1-\alpha_2\beta_1-\frac{\alpha_1\beta_2}{k_1}x_1+\frac{\alpha_2\beta_1}{k_2}x_2=0 \qquad ...(2.6)$$

and from third equation of (2.5)

$$x_1=\frac{\gamma-\delta_2x_2^2}{\delta_1}$$

put in (2.6) we get

$$\beta_2\alpha_1-\alpha_2\beta_1-\frac{\alpha_1\beta_2}{k_1}(\frac{\gamma-\delta_2x_2^2}{\delta_1})+\frac{\alpha_2\beta_1}{k_2}x_2=0$$

this takes of the form

$$ax_2^2+bx_2+c=0 \qquad ...(2.7)$$

where $a=\frac{\alpha_1\beta_2\delta_2}{k_1\delta_1}$; $\quad b=\frac{\alpha_2\beta_1}{k_2}$

and $c=\beta_2\alpha_1-\alpha_2\beta_1-\frac{\alpha_1\beta_2\gamma}{k_1\delta_1}$

from (2.7) $x_2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

put the value of a, b and c

we get

$$x_2=\frac{k_1\delta_1}{2\alpha_1\beta_2\delta_2}\left[\frac{-\alpha_2\beta_1}{k_2}\pm\sqrt{\frac{\alpha_2^2\beta_1^2}{k_2^2}-\frac{4\alpha_1^2\beta_2^2\delta_2}{k_1\delta_1}+\frac{4\alpha_1\beta_2\delta_2\beta_1\alpha_2}{k_1\delta_1}+\frac{4\alpha_1^2\beta_2^2\delta_2\gamma}{k_1^2\delta_1^2}}\right]$$

now from equation third of (2.5) is

$$\delta_1x_1+\delta_2x_2^2y=\gamma$$

$$\Rightarrow \quad x_2=\sqrt{\frac{\gamma-\delta_2x_2}{\delta_1}}$$

and from (2.6) $\beta_2\alpha_1-\alpha_2\beta_1-\frac{\alpha_1\beta_2}{k_1}x_1+\frac{\alpha_2\beta_1}{k_2}x_2=0$

put the value of x_2 in (2.6)

$$\beta_2\alpha_1-\alpha_2\beta_1-\frac{\alpha_1\beta_2}{k_1}x_1+\frac{\alpha_2\beta_1}{k_2}\sqrt{\frac{\gamma-\delta_1x_1}{\delta_2}}=0$$

$$\frac{\alpha_2\beta_1}{k_2}\sqrt{\frac{\gamma-\delta_1x_1}{\delta_2}}=-(\beta_2\alpha_1-\alpha_2\beta_1-\frac{\alpha_1\beta_2}{k_1}x_1)$$

squaring both the side and solve we get

$$ax_1^2+bx_1+c=0 \qquad ...(2.8)$$

where $a=\frac{k_2^2\alpha_1^2\beta_2^2}{\alpha_2^2\beta_1^2k_1^2}$; $b=\frac{\delta_1}{\delta_2}-\frac{2\beta_2\alpha_1k_2^2}{k_1\beta_1^2\alpha_2^2}(\beta_2\alpha_1-\beta_1\alpha_2)$ and

$$c=\frac{k_2^2}{\alpha_2^2\beta_1^2}(\beta_2\alpha_1-\beta_1\alpha_2)^2-\frac{\gamma}{\delta_2}$$

from (2.8) $x_1=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

put the value of a, b and c

we get

$$x_1 = \frac{\alpha_2^{\,2}\beta_1^{\,2}k_1^{\,2}}{2k_2^{\,2}\alpha_1^{\,2}\beta_2^{\,2}}\left[\begin{array}{l}\dfrac{-\delta_1}{\delta_2} + \dfrac{2\alpha_1\beta_2 k_2^{\,2}}{k_1\beta_1^{\,2}\alpha_2^{\,2}}(\beta_2\alpha_1 - \alpha_2\beta_1) \pm \\ \sqrt{\left(\dfrac{\delta_1}{\delta_2} - \dfrac{2\alpha_1\beta_2 k_2^{\,2}}{k_1\beta_1^{\,2}\alpha_2^{\,2}}(\beta_2\alpha_1 - \alpha_2\beta_1)\right)^2 - \dfrac{4k_2^{\,2}\alpha_1^{\,2}}{k_1^{\,2}\beta_1^{\,2}\alpha_2^{\,2}}\left\{\dfrac{k_2^{\,2}}{\alpha_2^{\,2}\beta_1^{\,2}}(\beta_2\alpha_1 - \alpha_2\beta_1)^2 - \dfrac{\gamma}{\delta_2}\right\}}\end{array}\right]$$

now from second equation of (2.5)

$$\alpha_2\left(1 - \frac{x_2}{k_2}\right) - \beta_2 y = 0$$

$$y = \frac{\alpha_2}{\beta_2} - \frac{\alpha_2 x_2}{k_2\beta_2}$$ Put the value of x_2 we get

$$y = \frac{\alpha_2}{\beta_2} - \frac{k_1\delta_1\alpha_2}{2\alpha_1\beta_2^{\,2}\delta_2 k_2}\left[\frac{-\alpha_2\beta_1}{k_2} \pm \sqrt{\frac{\alpha_2^{\,2}\beta_1^{\,2}}{k_2^{\,2}} - \frac{4\alpha_1^{\,2}\beta_2^{\,2}\delta_2}{k_1\delta_1} + \frac{4\alpha_1\beta_2\delta_2\beta_1\alpha_2}{k_1\delta_1} + \frac{4\alpha_1^{\,2}\beta_2^{\,2}\delta_2\gamma}{k_1^{\,2}\delta_1^{\,2}}}\right]$$

now from first and second equation of (2.5)

$$x_1 = k_1 - \frac{\beta_1 k_1 y}{\alpha_1} \quad \text{and} \quad x_2 = k_2 - \frac{\beta_2 k_2 y}{\alpha_2}$$

put the value of x_1 and x_2 in third equation of (2.5) we get

$$\delta_1\left(k_1 - \frac{\beta_1 k_1 y}{\alpha_1}\right) + \delta_2\left(k_2 - \frac{\beta_2 k_2 y}{\alpha_2}\right)^2 = \gamma$$

After solving we get

$$\delta_1 k_1 + \delta_2 k_2^{\,2} + \frac{\delta_2\beta_2^{\,2}k_2^{\,2}y^2}{\alpha_2^{\,2}} = \gamma + \frac{\delta_1\beta_1 k_1 y}{\alpha_1} + \frac{2\delta_2\beta_2 k_2^{\,2}y}{\alpha_2}$$

or $$\delta_1 k_1 + \delta_2 k_2^{\,2} + \frac{\delta_2\beta_2^{\,2}k_2^{\,2}y^2}{\alpha_2^{\,2}} > \gamma$$

again from second equation of (2.5)

$$y = \frac{\alpha_2}{\beta_2} - \frac{\alpha_2 x_2}{k_2\beta_2}$$

and from third equation of (2.5)

$$x_1 = \frac{\gamma - \delta_2 x_2^{\,2}}{\delta_1}$$

put in first equation of (2.5) we get

$$\alpha_1 - \frac{\alpha_1}{k_1}\left(\frac{\gamma - \delta_2 x_2^{\,2}}{\delta_1}\right) - \beta_1\left(\frac{\alpha_2}{\beta_2} - \frac{\alpha_2}{\beta_2}\frac{x_2}{k_2}\right) = 0$$

after solving we get, $ax_2^2 + bx_2 + c = 0$

where $a = \dfrac{\alpha_1\delta_2}{\delta_1 k_1}$ $b = \dfrac{\beta_1\alpha_2}{\beta_2 k_2}$ and $c = \alpha_1 - \dfrac{\beta_1\alpha_2}{\beta_2} - \dfrac{\gamma\alpha_1}{k_1\delta_1}$

If $x_1 > 0$ then $x_2 < \sqrt{\dfrac{\gamma}{\delta_2}}$

let $f(x_2) = ax_2^{\,2} + bx_2 + c$

$$f\left(\sqrt{\frac{\gamma}{\delta_2}}\right) = a\left(\sqrt{\frac{\gamma}{\delta_2}}\right)^2 + b\sqrt{\frac{\gamma}{\delta_2}} + c$$ after solving we get

$$f\left(\sqrt{\frac{\gamma}{\delta_2}}\right) = \frac{\beta_1\alpha_2}{\beta_2 k_2}\sqrt{\frac{\gamma}{\delta_2}} - \alpha_1 - \frac{\beta_1\alpha_2}{\beta_2}$$

and if $c > 0$

then $$\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} > \frac{\alpha_1\gamma}{\beta_1 k_1\delta_1}$$

and if $y > 0$

then $$\frac{\alpha_2}{\beta_2} - \frac{x_2\alpha_2}{\beta_2 k_2} > 0$$

or $$\frac{\alpha_2}{\beta_2} > \frac{\alpha_2}{\beta_2 k_2}\sqrt{\frac{\gamma}{\delta_2}}$$

or $$k_2^{\,2} > \frac{\gamma}{\delta_2}$$

Hence proved theorem 2.1.

Theorem 2.2. (a) $E^*(0, \varepsilon_2)$ exists if

$$\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2} < \frac{\alpha_1\gamma}{\beta_1 k_1\delta_1}\ ;\ \frac{\alpha_2 k_2}{\alpha_2 + k_2\varepsilon_2} > x_2$$

(b) $E^*(\varepsilon_1, 0)$ Exists if

$$\varepsilon_1 p_{12}\frac{\gamma}{\delta_1} < 0\ ;\ \frac{\alpha_1 k_1}{\alpha_1 + k_1\varepsilon_1} > x_1$$

Proof. (a) At $\varepsilon_1 = 0$, equilibria are solution of following system of equation

$$\alpha_1 x_1\left(1 - \frac{x_1}{k_1}\right) - \beta_1 x_1 y + \varepsilon_2 p_{21}x_2 = 0$$

$$\alpha_2 x_2\left(1 - \frac{x_2}{k_2}\right) - \beta_2 x_2 y - \varepsilon_2 x_2^{\,2} = 0 \qquad \text{...(2.9)}$$

$$-\gamma + \delta_1 x_1 + \delta_2 x_2^{\,2} = 0$$

from second equation of (2.9) is

$$y = \frac{\alpha_2}{\beta_2} - \frac{\varepsilon_2 x_2}{\beta_2} - \frac{\alpha_2 x_2}{k_2\beta_2} \qquad \text{...(2.10)}$$

and from third equation of (2.9) is

$$x_1 = \frac{\gamma - \delta_2 x_2^{\,2}}{\delta_1} \qquad \text{...(2.11)}$$

put in first equation of (2.9)

$$\alpha_1\left(\frac{\gamma - \delta_2 x_2^{\,2}}{\delta_1}\right)\left(1 - \frac{\gamma - \delta_2 x_2^{\,2}}{k_1\delta_1}\right) - \beta_1\left(\frac{\gamma - \delta_2 x_2^{\,2}}{k_1\delta_1}\right)\left(\frac{\alpha_2}{\beta_2} - \frac{\varepsilon_2 x_2}{\beta_2} - \frac{\alpha_2 x_2}{k_2\beta_2}\right)$$
$$-\varepsilon_2 p_{21}x_2 = 0$$

after solving we get, $ax_2^4 + bx_2^3 + cx_2^2 + dx_2 + e = 0$ ...(2.12)

where $a = \dfrac{\delta_2{}^2\alpha_1}{\delta_1{}^2 k_1}$; $b = \dfrac{\delta_2\varepsilon_2\beta_1}{\beta_2\delta_1} + \dfrac{\delta_2\alpha_2\beta_1}{\beta_2\delta_1 k_2}$

$$c = \frac{\alpha_1\delta_2}{\delta_1} - \frac{2\alpha_1\gamma\delta_2}{k_1\delta_1{}^2} - \frac{\beta_1\delta_2\alpha_2}{\beta_2\delta_1};$$

$$d = \frac{-\beta_1\varepsilon_2\gamma}{\delta_1\beta_2} - \frac{\beta_1\alpha_2\gamma}{\delta_1\beta_2 k_2} - \varepsilon_2 p_{21}$$

and $e = \dfrac{\alpha_1\gamma^2}{\delta_1{}^2 k_1} - \dfrac{\alpha_1\gamma}{\delta_1} - \dfrac{\beta_1\gamma\alpha_2}{\beta_2\delta_1}$

Let $f(x_2) = ax_2^4 + bx_2^3 + cx_2^2 + dx_2 + e$

If $x_1 > 0$ then $x_2 < \sqrt{\dfrac{\gamma}{\delta_2}}$

$$f\left(\sqrt{\frac{\gamma}{\delta_2}}\right) = a\left(\sqrt{\frac{\gamma}{\delta_2}}\right)^4 + b\left(\sqrt{\frac{\gamma}{\delta_2}}\right)^3 + c\left(\sqrt{\frac{\gamma}{\delta_2}}\right)^2 + d\left(\sqrt{\frac{\gamma}{\delta_2}}\right) + e$$

after solving we get, $f\left(\sqrt{\dfrac{\gamma}{\delta_2}}\right) = -\varepsilon_2 p_{21}\sqrt{\dfrac{\gamma}{\delta_2}} < 0$

and if $e > 0$

$$\frac{-\alpha_1\gamma}{\delta_1} + \frac{\alpha_1\gamma^2}{\delta_1{}^2 k_1} + \frac{\beta_1\gamma\alpha_2}{\beta_2\delta_1} > 0$$

or $\dfrac{-\alpha_1}{\beta_1} + \dfrac{\alpha_2}{\beta_2} > \dfrac{-\alpha_1\gamma}{\beta_1 k_1\delta_1}$

or $\dfrac{\alpha_1}{\beta_1} - \dfrac{\alpha_2}{\beta_2} < \dfrac{\alpha_1\gamma}{\beta_1 k_1\delta_1}$

and if $y > 0$

$$\frac{\alpha_2}{\beta_2} - \frac{\varepsilon_2 x_2}{\beta_2} - \frac{\alpha_2 x_2}{k_2\beta_2} > 0$$

or $\dfrac{\alpha_2 k_2}{\alpha_2 + k_2\varepsilon_2} > x_2$

Hene proved theorem 2.2.(*a*).

Proof. (*b*) At $\varepsilon_2 = 0$, equilibria are solution of following system of equations

$$\alpha_1 x_1(1 - \frac{x_1}{k_1}) - \beta_1 x_1 y + \varepsilon_1 x_1{}^2 = 0$$

$$\alpha_2 x_2(1 - \frac{x_2}{k_2}) - \beta_2 x_2 y - \varepsilon_1 x_1 p_{12} = 0$$

$$-\gamma + \delta_1 x_1 + \delta_2 x_2{}^2 = 0$$

$$\alpha_1(1 - \frac{x_1}{k_1}) - \beta_1 y + \varepsilon_1 x_1 = 0$$

or $\alpha_2 x_2(1 - \dfrac{x_2}{k_2}) - \beta_2 x_2 y - \varepsilon_1 x_1 p_{12} = 0$ ...(2.13)

$$-\gamma + \delta_1 x_1 + \delta_2 x_2{}^2 = 0$$

from first equation of (2.13)

$$\beta_1 y = \alpha_1 - \frac{\alpha_1 x_1}{k_1} - \varepsilon_1 x_1$$

or $y = \dfrac{\alpha_1}{\beta_1} - \dfrac{\alpha_1 x_1}{k_1\beta_1} - \dfrac{\varepsilon_1 x_1}{\beta_1}$ ...(2.14)

from third equation of (2.13)

$x_1 = \dfrac{\gamma - \delta_2 x_2{}^2}{\delta_1}$ ...(2.15)

put in second equation of (2.13)

$$\alpha_2 x_2(1 - \frac{x_2}{k_2}) - \beta_2 x_2(\frac{\alpha_1}{\beta_1} - \frac{\alpha_1 x_1}{k_1\beta_1} - \frac{\varepsilon_1 x_1}{\beta_1}) - \varepsilon_1 x_1 p_{12} = 0$$

or

$$\alpha_2 x_2(1 - \frac{x_2}{k_2}) - \beta_2 x_2\{\frac{\alpha_1}{\beta_1} - \frac{\alpha_1}{k_1\beta_1}\left(\left(\frac{\gamma - \delta_2 x_2{}^2}{\delta_1}\right) - \frac{\varepsilon_1}{\beta_1}\left(\frac{\gamma - \delta_2 x_2{}^2}{\delta_1}\right)\right)$$

$$-\varepsilon_1\left(\frac{\gamma - \delta_2 x_2{}^2}{\delta_1}\right)p_{12} = 0$$

after solving we get

$ax_2^3 + bx_2^2 + cx_2 + d = 0$ ...(2.16)

where $a = \dfrac{\alpha_1\beta_2\delta_2}{\beta_1 k_1\delta_1} + \dfrac{\varepsilon_1\beta_2\delta_2}{\delta_1\beta_1}$; $b = \dfrac{\alpha_2}{k_2} + \dfrac{\varepsilon_1 p_{12}\delta_2}{\delta_1}$

$c = \dfrac{\beta_2\alpha_1}{\beta_1} - \alpha_2 - \dfrac{\alpha_1\beta_2\gamma}{\beta_1 k_1\delta_1} - \dfrac{\varepsilon_1\beta_2\gamma}{\delta_1\beta_1}$ and $d = -\varepsilon_1 p_{12}\dfrac{\gamma}{\delta_1}$

Let $f(x_2) = ax_2^3 + bx_2^2 + cx_2 + d$

If $x_1 > 0$ then $x_2 < \sqrt{\dfrac{\gamma}{\delta_2}}$

then $f\left(\sqrt{\dfrac{\gamma}{\delta_2}}\right) = a\left(\sqrt{\dfrac{\gamma}{\delta_2}}\right)^3 + b\left(\sqrt{\dfrac{\gamma}{\delta_2}}\right)^2 + c\left(\sqrt{\dfrac{\gamma}{\delta_2}}\right) + d$

after solving we get

$$f\left(\sqrt{\frac{\gamma}{\delta_2}}\right) = \frac{\alpha_2\gamma}{k_2\delta_2} + \frac{\varepsilon_1 p_{12}\gamma}{\delta_1} - \varepsilon_1 p_{12}\frac{\gamma}{\delta_1} + \frac{\alpha_1\beta_2}{\beta_1}\sqrt{\frac{\gamma}{\delta_2}} - \alpha_2\sqrt{\frac{\gamma}{\delta_2}}$$

If $d > 0$ then $-\varepsilon_1 p_{12}\dfrac{\gamma}{\delta_1} > 0$

or $\varepsilon_1 p_{12}\dfrac{\gamma}{\delta_1} < 0$

and if $y > 0$

$$\frac{\alpha_1}{\beta_1} - \frac{\alpha_1 x_1}{k_1\beta_1} - \frac{\varepsilon_1 x_1}{\beta_1} > 0$$

or $\dfrac{\alpha_1 k_1}{\alpha_1 + k_1\varepsilon_1} > x_1$

Hence proved theorem (2.2)(b)

Theorem 2.3. $E^*(\varepsilon_1, \varepsilon_2)$ exists if

$$\frac{-\beta_1\varepsilon_1 p_{12}\gamma^2}{\delta_1^{\,2}} < 0$$

Proof. This is the case where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ equilibria of the following system of equations

$$\alpha_1 x_1(1-\frac{x_1}{k_1}) - \varepsilon_1 x_1^{\,2} + \varepsilon_2 p_{21} x_2 = \beta_1 x_1 y$$

$$\alpha_2 x_2(1-\frac{x_2}{k_2}) - \varepsilon_2 x_2^{\,2} + \varepsilon_1 p_{12} x_1 = \beta_2 x_2 y \qquad \text{...(2.17)}$$

$$\delta_1 x_1 + \delta_2 x_2^{\,2} = \gamma$$

eliminating y from first and second equation of (2.17) we get

$$\beta_2\alpha_1 x_2 x_1 - \frac{\alpha_1 x_1^{\,2}\beta_2 x_2}{k_1} - \beta_2 x_2 x_1^{\,2} + \beta_2 x_2^{\,2}\varepsilon_2 p_{21} - \beta_1 x_1 \alpha_2 x_2$$

$$+\frac{\beta_1\alpha_2 x_2^{\,2} x_1}{k_2} + \beta_1 x_2^{\,2} x_1 \varepsilon_2 - \beta_1 x_1^{\,2}\varepsilon_1 p_{12} = 0 \qquad \text{...(2.18)}$$

by third equation of (2.17)

$$x_1 = \frac{\gamma - \delta_2 x_2^{\,2}}{\delta_1}$$

put in (2.18) we get

$$a x_2^5 + b x_2^{\,4} + c x_2^{\,3} + d x_2^{\,2} + e x_2 + f = 0$$

where

$$a = \frac{\alpha_1\beta_2\delta_2^{\,2}}{k_1\delta_1^{\,2}} + \frac{\beta_2\delta_2^{\,2}}{\delta_1^{\,2}}; \quad b = \frac{\beta_1\alpha_2}{k_2\delta_1} + \frac{\beta_1\delta_2\varepsilon_2}{\delta_1} + \frac{\beta_1\delta_2^{\,2}\varepsilon_1 p_{12}}{\delta_1^{\,2}}$$

$$c = \frac{\beta_2\alpha_1\delta_2}{\delta_1} - \frac{2\beta_2\alpha_1\delta_2}{\delta_1 k_1} - \frac{2\beta_2\delta_2\gamma}{\delta_1} - \frac{\beta_1\alpha_2\delta_2}{\delta_1};$$

$$d = -\varepsilon_2\beta_2 p_{21} - \frac{\beta_1\alpha_2\gamma}{\delta_1 k_2} - \frac{\beta_1\varepsilon_2\gamma}{\delta_1} - \frac{2\beta_1\varepsilon_1 p_{12}\delta_2\gamma}{\delta_1^{\,2}};$$

$$e = \frac{-\beta_2\alpha_1\gamma}{\delta_1} + \frac{\beta_2\alpha_1\gamma^2}{\delta_1^{\,2}k_1} + \frac{\beta_2\gamma^2}{\delta_1^{\,2}} + \frac{\beta_1\alpha_2\gamma}{\delta_1}$$

and $f = \dfrac{\beta_1\varepsilon_1 p_{12}\gamma^2}{\delta_1^{\,2}}$

Let $x_1 > 0$ then $x_2 < \sqrt{\dfrac{\gamma}{\delta_2}}$

then
$$f\left(\sqrt{\frac{\gamma}{\delta_2}}\right) = a\left(\sqrt{\frac{\gamma}{\delta_2}}\right)\frac{\gamma^2}{\delta_2^2} + b\left(\frac{\gamma^2}{\delta_2^2}\right) + c\left(\sqrt{\frac{\gamma}{\delta_2}}\right)\frac{\gamma}{\delta_2} + d\frac{\gamma}{\delta_2} + e\sqrt{\frac{\gamma}{\delta_2}} + f$$

after solving we get

$$f\left(\sqrt{\frac{\gamma}{\delta_2}}\right) = \frac{2\beta_2\alpha_1\gamma^2}{k_1\delta_1^{\,2}}\sqrt{\frac{\gamma}{\delta_2}} + \frac{2\beta_2\gamma^2}{\delta_1^{\,2}}\sqrt{\frac{\gamma}{\delta_2}} + \frac{\beta_1\alpha_2\gamma^2}{\delta_1 k_1\delta_2^{\,2}} - \frac{2\beta_1\alpha_1\gamma}{k_1\delta_1}\sqrt{\frac{\gamma}{\delta_2}} - \frac{2\beta_2\gamma^2}{\delta_1}\sqrt{\frac{\gamma}{\delta_2}} - \frac{\beta_2\varepsilon_2 p_{21}\gamma}{\delta_2} - \frac{\beta_1\alpha_2\gamma^2}{\delta_1 k_2}$$

If $f > 0$ then $\dfrac{\beta_1\varepsilon_1 p_{12}\gamma^2}{\delta_1^{\,2}} > 0$

or $\dfrac{-\beta_1\varepsilon_1 p_{12}\gamma^2}{\delta_1^{\,2}} < 0$

III. STABILITY

Having established the existence of equilibrium $E^*(\varepsilon_1, \varepsilon_2)$, we proceed to examine its stability properties .In fact we have the following theorem

Theorem 3.1. $E^*(0, 0)$ is asymptotically stable.

Theorem 3.2. (a) $E^*(0, \varepsilon_2)$ is asymptotically stable if $\beta_1 \geq \beta_2$

(b) $E^*(\varepsilon_1, 0)$ is asymptotically stable if $\beta_2 \geq \beta_1$

Theorem 3.3. $E^*(\varepsilon_1, \varepsilon_2)$ is asymptotically stable if $\beta_1 = \beta_2$

Proof. The first step is to compute the variational matrix of $E^*(\varepsilon_1, \varepsilon_2)$ which takes the form

$v(\varepsilon_1, \varepsilon_2)$

$$= \begin{bmatrix} \alpha_1(1-\frac{2x_1}{k_1}) - \beta_1 y - 2\varepsilon_1 x_1 & \varepsilon_2 p_{21} & -\beta_1 x_1 \\ \varepsilon_1 p_{12} & \alpha_2(1-\frac{x_2}{k_2}) - \beta_2 y - 2\varepsilon_1 x_1 & -\beta_2 x_2 \\ \delta_1 y & 2x_2\delta_2 y & 0 \end{bmatrix}$$

using (2.17)

$v(\varepsilon_1, \varepsilon_2)$

$$= \begin{bmatrix} \frac{-\alpha_1 x_1}{k_1} - \frac{\varepsilon_2 p_{21} x_2}{x_1} - \varepsilon_1 x_1 & \varepsilon_2 p_{21} & -\beta_1 x_1 \\ \varepsilon_1 p_{12} & \frac{-\alpha_2 x_2}{k_2} - \frac{\varepsilon_1 p_{12} x_1}{x_2} - \varepsilon_2 x_2 & -\beta_2 x_2 \\ \delta_1 y & 2\delta_2 x_2 y & 0 \end{bmatrix}$$

the characteristic equation of above matrix is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where $a_1 = \dfrac{\varepsilon_2 p_{21} x_2}{x_1} + \varepsilon_1 x_1 + \dfrac{\alpha_1 x_1}{k_1} + \dfrac{\varepsilon_1 p_{12} x_1}{x_2} + \varepsilon_2 x_2 + \dfrac{\alpha_2 x_2}{k_2}$

$$a_2 = \frac{\varepsilon_2^{\,2} p_{21} x_2^{\,2}}{x_1} + \varepsilon_1 x_1 \varepsilon_2 x_2 + \frac{\varepsilon_2 p_{21} \alpha_2 x_2^{\,2}}{k_2 x_1} + \frac{\varepsilon_1^{\,2} p_{12} x_1^{\,2}}{x_2}$$
$$+ \frac{\varepsilon_1 x_1 \alpha_2 x_2}{k_2} + \frac{\alpha_1 \varepsilon_1 \; p_{12} x_1^{\,2}}{k_1 x_2} + \frac{\alpha_1 x_1 \varepsilon_2 x_2}{k_1}$$
$$+ \frac{\alpha_1 x_1 \alpha_2 x_2}{k_1 k_2} + 2\beta_2 x_2^{\,2} \delta_2 y + \beta_1 x_1 \delta_1 y$$

$$a_3 = \frac{2\varepsilon_2 \; p_{21} x_2^{\,3} \beta_2 \delta_2 y}{x_1} + \varepsilon_1 x_1^{\,2} \beta_2 x_2^{\,2} \delta_2 y + \frac{2\alpha_1 x_1 \delta_2 \beta_2 x_2^{\,2} y}{k_1}$$
$$+ \varepsilon_2 p_{21} \beta_2 x_2 \delta_1 y + 2\beta_1 x_1 \delta_2 x_2 \varepsilon_1 p_{12} y$$
$$+ \frac{\beta_1 x_1^{\,2} \delta_1 y \varepsilon_1 p_{12}}{x_2} + \beta_1 x_1 \delta_1 \varepsilon_2 x_2 y + \frac{\beta_1 x_1 \delta_1 \alpha_2 x_2 \; y}{k_2}$$

here $a_1 > 0$; $a_2 > 0$ and $a_3 > 0$

now $a_1 a_2 - a_3 > 0$ if $\varepsilon_1 = \varepsilon_2 = 0$

Hence E^* (0, 0) is asymptotically stable. This proves theorem 3.1.

also $a_1 a_2 - a_3 > 0$ if $\varepsilon_1 = 0$ and $\beta_1 \geq \beta_2$

this proves theorem 3.2.(a).

also $a_1 a_2 - a_3 > 0$ if $\varepsilon_2 = 0$ and $\beta_1 \geq \beta_2$

this proves theorem 3.2.(b)

finally $a_1 a_2 - a_3 > 0$ if $\beta_1 = \beta_2$ for every positive values of ε_1 and ε_2 this proves theorem 3.3.

Second method: System of equations is

$$\begin{aligned} x_1' &= \alpha_1 x_1 (1 - \frac{x_1}{k_1}) - \beta_1 x_1 y - \varepsilon_1 x_1^{\,2} + \varepsilon_2 p_{21} x_2 \\ x_2' &= \alpha_2 x_2 (1 - \frac{x_2}{k_2}) - \beta_2 x_2 y - \varepsilon_2 x_2^{\,2} + \varepsilon_1 p_{12} x_1 \qquad \text{...(2.19)} \\ y' &= y(-\gamma + \delta_1 x_1 + \delta_2 x_2^{\,2}) \end{aligned}$$

$$f(x_1, x_2, y) = \alpha_1 x_1 (1 - \frac{x_1}{k_1}) - \beta_1 x_1 y - \varepsilon_1 x_1^{\,2} + \varepsilon_2 p_{21} x_2$$

Let $f(x_1, x_2, y) = \alpha_2 x_2 (1 - \frac{x_2}{k_2}) - \beta_2 x_2 y - \varepsilon_2 x_2^{\,2} + \varepsilon_1 p_{12} x_1$

$$f(x_1, x_2, y) = y(-\gamma + \delta_1 x_1 + \delta_2 x_2^{\,2})$$

we linearized the system by Taylor's theorem

we get

$$\begin{aligned} x_1' &= \alpha_1 x_1 + \varepsilon_2 p_{21} x_2 \\ x_2' &= \varepsilon_1 p_{12} x_1 + \alpha_2 x_2 \qquad \text{...(2.20)} \\ y' &= \delta_1 x_1 - \gamma \end{aligned}$$

variational matrix of (2.20) is

$$v(\varepsilon_1, \varepsilon_2) = \begin{bmatrix} \alpha_1 & \varepsilon_2 p_{21} & 0 \\ \varepsilon_1 p_{12} & \alpha_2 & 0 \\ \delta_1 & 0 & 0 \end{bmatrix}$$

characteristic equation of this is

$$\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0 \qquad \text{...(2.21)}$$

where

$$a_1 = -(\alpha_1 + \alpha_2); \quad a_2 = \alpha_1 \alpha_2 - p_{21} p_{12} \varepsilon_2 \varepsilon_1$$

and $a_3 = 0$

If both a_1 and a_2 are negative and p_{21} is very- very small then

$a_1 > 0$; $a_2 > 0$ and $a_3 > 0$

and $a_1 a_2 - a_3 = -(\alpha_1 + \alpha_2)\, \alpha_2 \alpha_1 > 0$

Hence system (2.19) is stable at the point (0, 0, 0).

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