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A note on inequalities between the moments

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ABSTRACT : Some alternative proofs of the inequalities between the moments of the discrete probability distributions are given here. We use method of Lagrange multipliers to show that the power mean is an increasing function of its argument. This also suggests the possibility of investigating more alternative non-linear programming proofs of the inequalities that have been studied extensively in the literature for means, moments and variance.

I. INTRODUCTION

Let $x_1, x_2, ..., x_n$ denote n real numbers. The r^{th} order moment m'_r about origin of these n real numbers $x_i (i = 1, 2, ..., n)$ is defined as

$$m'_r = \frac{1}{n}\Sigma_{i=1}^{n} x_i^r . \qquad ...(1.1)$$

The power mean M_r of order r is defined for the positive real numbers $x_i (i = 1, 2, ..., n)$ as

$$M_r = (m'_r)^{1/r}, \quad r \neq 0 \qquad ...(1.2)$$

and

$$M_r = \lim_{r \to 0} (m'_r)^{1/r}, \quad r = 0 \qquad ...(1.3)$$

It may be noted that M_{-1}, M_0, M_1 respectively define harmonic mean, geometric mean and arithmetic mean. The general properties of means are studied extensively in literature [1]. In the field of theory of inequalities a number of inequalities are based on the means or involve means of the various kinds. It is well known that the power mean M_r is an increasing function of r. The different methods have been used to prove the inequalities for the moments [1-4]. If r is a positive real number and s is any real number with $r > s$ then [4]

$$m'_r \geq (m'_s)^{r/s}. \qquad ...(1.4)$$

If r and s are negative real numbers with $r > s$ then

$$m'_r \leq (m'_s)^{r/s}. \qquad ...(1.5)$$

The inequalities between the moments are important in the theory of mathematical statistics, entropy and have applications in the theory of polynomial equations. These inequalities have also been generalized and extended to the higher abstract spaces such as inner product spaces and Hilbert spaces. Such extensions give inequalities for operators and functional [6-9].

It is instructive to consider the several alternative proofs of the inequalities. The alternative proofs often suggest different possible extensions [5]. In the present paper, we discuss nonlinear programming proofs of the inequalities of the type (1.4) and (1.5). We first give proofs of the inequalities (1.4) and (1.5) for the case when $s = 1$ (Theorem 1, below) and deduce (1.4) and (1.5) from it. We then give the direct proofs of the inequalities (1.4) and (1.5) using the methods of the Lagrange multipliers, more general (Theorem 2, below). The inequalities for the power means are proved (Theorem 3, below) which also help us to give a proof of the monotonicity of power means M_r (Corollary 1, below). We show that the method can be extended to study the inequalities between the moments of discrete probabilities distribution (Theorem 4, below).

II. MAIN RESULTS

Theorem 1. Let m'_r be the r^{th} order moment of n positive real numbers $x_i (i = 1, 2,..., n)$ defined by (1.1). Let $r \geq 1$, then

$$m'_r \geq (m'_1)^r. \qquad ...(2.1)$$

The inequality (2.1) also holds good when $r < 0$ and reverses its order when $0 < r < 1$. The equality sign holds in the inequality (2.1) when all the x_i's are equal, that is $x_1 = x_2 = ... = x_n$.

Proof : We optimize

$$f(x) = \frac{1}{n}\Sigma_{i=1}^{n} x_i^r , \qquad ...(2.2)$$

subject to equality constraint

$$\Sigma_{i=1}^{n} x_i = k. \qquad ...(2.3)$$

Forming the Lagrangian function

$$L(x_1, x_2, ..., x_n, \lambda) = \Sigma_{i=1}^{n} x_i^r - \lambda \left(\Sigma_{i=1}^{n} x_i - k\right) \qquad ...(2.4)$$

The necessary conditions for the extremum of $f(x)$ are

$$\frac{\partial L}{\partial x_i} = 0 \Rightarrow x_i = \left(\frac{\lambda}{r}\right)^{\frac{1}{r-1}}, i = 1, 2, ..., n \qquad ...(2.5)$$

and

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow \Sigma_{i=1}^{n} x_i = k. \qquad ...(2.6)$$

From (2.5) and (2.6), we get that

$$\lambda = r\left(\frac{k}{n}\right)^{r-1} \qquad ...(2.7)$$

Combining (2.5) and (2.7), we find that

$$x_i = \frac{k}{n}, \; i = 1, 2, ..., n. \qquad (2.8)$$

The Hessian matrix of $f(x)$ is

$$H(x) = \begin{bmatrix} r(r-1)x_1^{r-2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r(r-1)x_n^{r-2} \end{bmatrix} ...(2.9)$$

For $x_i \geq 0$ ($i = 1, 2, ..., n$) and $r \geq 1$ the Hessian matrix $H(x)$ is positive semi-definite, the function $f(x)$ is therefore convex and the necessary conditions (2.8) also become sufficient conditions. Hence

$$\Sigma_{i=1}^{n} x_i^r \;\geq\; \Sigma_{i=1}^{n}\left(\frac{k}{n}\right)^{r} \qquad ...(2.10)$$

The inequality (2.10) immediately gives the inequality (2.1), for $r \geq 1$. The Hessian matrix $H(x)$ is also positive semi-definite for $r \leq 0$. The Hessian matrix becomes negative semi-definite for $0 \leq r \leq 1$, therefore $f(x)$ is concave and in this case the necessary condition (2.8) gives the maximum of the function $f(x)$. So, the inequality (2.1) will reverse its order. It also follows from the derivation that the inequality (2.1) becomes equality when

$$x_i = \frac{k}{n}, \; i = 1, 2, ..., n.$$

Theorem 2. Let r be a positive real number and be a non-zero number such that $r > s$ then

$$m'_r \geq (m'_s)^{r/s}. \qquad ...(2.11)$$

If r and s are negative real numbers with $r > s$ then reverse inequality holds. Equality holds in (2.11) if and only if all the x_i, $i = 1, 2, ..., n$ are equal.

Proof : We optimize $f(x)$ defined in (2.2) subject to constraint

$$\Sigma_{i=1}^{n} x_i^s = c. \qquad ...(2.12)$$

The Lagrangian is

$$L(x_1, x_2, ..., x_n, \lambda) = \Sigma_{i=1}^{n} x_i^r - \lambda\left(\Sigma_{i=1}^{n} x_i^s - c\right). \qquad ...(2.13)$$

The necessary conditions for the extremum are

$$\frac{\partial L}{\partial x_i} = 0 \Rightarrow \quad x_i = \left(\frac{s\lambda}{r}\right)^{\frac{1}{r-s}}, \; i = 1, 2, ..., n \qquad ...(2.14)$$

and

$$\frac{\partial L}{\partial \lambda} = 0 \quad \Rightarrow \quad \Sigma_{i=1}^{n} x_i^s = c. \qquad ...(2.15)$$

From (2.14) and (2.15),

$$\lambda = \frac{r}{s}\left(\frac{c}{n}\right)^{\frac{r-s}{s}}. \qquad ...(2.16)$$

Combining (2.14) and (2.16), we get

$$x_i = \left(\frac{c}{n}\right)^{1/s}, \quad i = 1, 2, ..., n. \qquad ...(2.17)$$

The Hessian matrix for the Langrangian function $L(x_1, x_2, ..., x_n, \lambda)$ is

$$H(x_i, \lambda) = \lambda s(r-s)\begin{bmatrix} x_1^{s-2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n^{s-2} \end{bmatrix} \qquad ...(2.18)$$

For $r > 0$ and $r > s$ the Hessian matrix $H(x_i, \lambda)$ is positive semi-definite and so the Lagrangian $L(x_i, \lambda)$ is convex. It follows from Karush-Kuhn- Tucker conditions see [10], the necessary conditions (2.17) are also sufficient for minimization of $f(x)$. We therefore have

$$\Sigma_{i=1}^{n} x_i^r \;\geq\; \Sigma_{i=1}^{n}\left(\frac{c}{n}\right)^{r/s} \qquad ...(2.19)$$

The inequality (2.11) follows easily from (2.19). Further, if $r < 0$, $s < 0$ and $r > s$ then $\lambda > 0$ and therefore Hessian matrix $H(x_i, \lambda)$ is negative semi-definite. The Lagrangian is therefore concave and the conditions (2.17) become the sufficient conditions for the maximization of the function $f(x)$. Hence

$$m_r' \leq (m'_s)^{r/s} \qquad ...(2.20)$$

It follows from (2.17) that equality holds in (2.11) when $x_1 = x_2 = ... = x_n$. Alternatively, we can deduce the theorem (2) from Theorem1. Suppose $r > 0$ and s is a non-zero real number, then we have to prove that

$$\frac{1}{n}\Sigma_{i=1}^{n} x_i^r \;\geq\; \left(\frac{1}{n}\Sigma_{i=1}^{n} x_i^s\right)^{r/s}. \qquad ...(2.21)$$

The inequality (2.21) can also be written as

$$\frac{1}{n}\Sigma_{i=1}^{n} y_i^{r/s} \;\geq\; \left(\frac{1}{n}\Sigma_{i=1}^{n} y_i\right)^{r/s} \qquad ...(2.22)$$

where $y_i = x_i^s$, $i = 1, 2, ..., n$. The inequality (2.22) follows from Theorem 1, as in one case $\frac{r}{s} > 1$ and in other case $\frac{r}{s} < 0$. For $0 < R < 1$, it follows from Theorem (1) that

$$\frac{1}{n}\Sigma_{i=1}^{n} x_i^R \;\leq\; \left(\frac{1}{n}\Sigma_{i=1}^{n} x_i\right)^{R}. \qquad ...(2.23)$$

Let $r < 0$, $s < 0$ and $r > s$, then $0 < \frac{r}{s} < 1$. Put $R = \frac{r}{s}$ and $y_i = x_i^s$, we find from (2.23) that

$$\frac{1}{n}\Sigma_{i=1}^{n}\ x_i^{r/s} = \left(\frac{1}{n}\Sigma_{i=1}^{n}\ x_i\right)^{r/s} \qquad ...(2.24)$$

This proves that $m'_r \le (m'_s)^{r/s}$.

Theorem 3. Let r be a positive real number, then

$$(m'_r)^{1/r} = (M_0)^r \qquad ...(2.25)$$

where m'_r and M_0 are defined in (1.1) and (1.3), respectively. If $r < 0$, the reverse inequality holds. Equality holds in (2.25) if and only if all the x_i are equal.

Proof : We optimize

$$f(x) = \Sigma_{i=1}^{n} \log x_i \qquad ...(2.26)$$

subject to constraint

$$\Sigma_{i=1}^{n}\ x_i^r = l. \qquad ...(2.27)$$

The Lagrangian is

$$L(x_1, x_2, ..., x_n, \lambda) = \Sigma_{i=1}^{n} \log x_i - \lambda\left(\Sigma_{i=1}^{n} x_i^r - l\right). \qquad ...(2.28)$$

The necessary conditions for extremum are

$$\frac{\partial L}{\partial x_i} = 0 \quad \Rightarrow \quad x_i = \left(\frac{1}{\lambda r}\right)^{1/r} \qquad ...(2.29)$$

and
$$\frac{\partial L}{\partial \lambda} = 0 \quad \Rightarrow \quad \Sigma_{i=1}^{n}\ x_i^r = l\,. \qquad ...(2.30)$$

From (2.29) and (2.30),

$$\lambda = \frac{n}{lr}. \qquad ...(2.31)$$

Combining (2.29) and (2.31), we get

$$x_i = \left(\frac{l}{n}\right)^{1/r}. \qquad ...(2.32)$$

The Hessian matrix for $L(x_1, x_2,..., x_n, \lambda)$ is

$$H(x_i, \lambda) = -r\begin{bmatrix} \frac{1}{x_1^2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{x_n^2} \end{bmatrix} \qquad ...(2.33)$$

For $r > 0$ the Hessian matrix $H(x_i, \lambda)$ is negative semi-definite and so the Lagrangian $L(x_i, \lambda)$ is concave. It follows that the necessary conditions (2.31) are also sufficient for maximization of $f(x)$. Thus,

$$\Sigma_{i=1}^{n} \log x_i \le n \log\left(\frac{l}{n}\right)^{1/r} \qquad ...(2.34)$$

The inequality (2.25) now follows easily from (2.34). Further, if $r < 0$ the Hessian matrix in (2.33) is positive semi-definite and hence the inequality (2.25) reverses its order when $r < 0$. It is evident from (2.32) that the equality sign holds in (2.25) when $x_1 = x_2 = ... = x_n$.

Alternatively, the inequality (2.25) follows from (2.11) on using the continuity arguments.

Corollary 1. The power mean M_r is an increasing function of r.

Proof: If r is a positive real number and s is a non-zero real number such that $r > s$, then from the inequality (2.11), we have

$$(m'_r)^{1/r} \ge (m'_s)^{1/s}. \qquad ...(2.35)$$

Therefore, M_r is an increasing function when r is a positive real number. It follows from Theorem 2, that (2.35) also holds good when $s < r < 0$, therefore M_r also increases when r is a negative real number. The case $r = 0$, follows from Theorem 3.

Theorem 4. Let

$$\mu'_r = \Sigma_{i=1}^{n}\ P_i x_i^r \qquad ...(2.36)$$

where all P_i are positive real numbers such that

$$\Sigma_{i=1}^{n}\ P_i = 1. \qquad ...(2.37)$$

If r is a positive real number and s is any non-zero real number such that $r > s$, then

$$\mu'_r \ge (\mu'_s)^{r/s}. \qquad ...(2.38)$$

If r is negative real number with $r > s$, then inequality (2.38) reverses its order. Further, if r is a positive real number,

$$(\mu'_r)^{1/r} \ge (\mu_0)^r \qquad ...(2.39)$$

where

$$\mu_0 = x_1^{p_1} x_2^{p_2} ... x_n^{p_n}. \qquad ...(2.40)$$

The inequality (2.39) reverse their order for $r < 0$. The inequalities (2.38) and (2.39) become equalities when all the x_i, $i = 1, 2, ..., n$ are equal.

Proof : The inequalities in this theorem may be proved in much the same way as the inequalities are proved in Theorem 2. We only write the proof of the inequality (2.38).

We optimize the function

$$f(x) = \Sigma_{i=1}^{n} P_i x_i^r \qquad ...(2.41)$$

subject to constraint

$$\Sigma_{i=1}^{n} P_i x_i^s = c. \qquad ...(2.42)$$

The Lagrangian is

$$L(x_1, x_2, ..., x_n, \lambda) = \Sigma_{i=1}^{n}\ P_i x_i^r - \lambda\left(\Sigma_{i=1}^{n} P_i x_i^s - c\right). \qquad ...(2.43)$$

The necessary conditions for the extremum are

$$\frac{\partial L}{\partial x_i} = 0 \Rightarrow x_i = \left(\frac{s\lambda}{r}\right)^{\frac{1}{r-s}}, i = 1, 2, ..., n \qquad ...(2.44)$$

and
$$\frac{\partial L}{\partial \lambda} = 0 \quad \Rightarrow \quad \Sigma_{i=1}^{n} P_i x_i^s = c. \qquad ...(2.45)$$

From (2.44) and (2.45),

$$\lambda = \frac{r}{s}(c)^{\frac{r-s}{s}}. \quad ...(2.46)$$

Combining (2.44) and (2.46), we get

$$x_i = (c)^{1/s}, \; i = 1, 2, ..., n. \quad ...(2.47)$$

The Hessian matrix for the Langrangian function $L(x_1, x_2, ..., x_n, \lambda)$ is

$$H(x_i, \lambda) = \lambda s(r-s)\begin{bmatrix} P_1 x_1^{s-2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_n x_n^{s-2} \end{bmatrix} \quad ...(2.48)$$

For $r > 0$ and $r > s$ the Hessian matrix $H(x_i, \lambda)$ is positive semi-definite and so the Lagrangian $L(x_i, \lambda)$ is convex. It follows from Karush-Kuhn- Tucker conditions see [10], the necessary conditions (2.47) are also sufficient for minimization of $f(x)$. We therefore have

$$\Sigma_{i=1}^{n} P_i x_i^r \geq \Sigma_{i=1}^{n} (c)^{r/s} \quad ...(2.49)$$

The inequality (2.38) follows easily from (2.49). Further, if $r < 0$, $s < 0$ and $r > s$ then $\lambda > 0$ and therefore Hessian matrix $H(x_i, \lambda)$ is negative semi-definite. The Lagrangian is therefore concave and the conditions (2.47) become the sufficient conditions for the maximization of the function $f(x)$. Hence

$$\mu'_r \leq (\mu'_s)^{r/s}. \quad ...(2.50)$$

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