Quaternionic Formulation of Dirac Equation

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ABSTRACT: Quaternion Dirac equation has been obtained from the square root of Klein-Gordon equation in compact and consistent way. Dirac matrices are described as quaternion valued and the Dirac Hamiltonian is considered as Hermitian with real eigenvalues of energy. Dirac spinors and free particle energy solution has been obtained in terms of one component, two-component and four-component Dirac spinors.

Keywords: Quaternions, Dirac Equation, Dirac Hamiltonian.

1. INTRODUCTION

Quaternions were very first example of hyper-complex numbers having the significant impact of mathematics and physics [1]. Because of their beautiful and unique properties quaternions attracted many to study the laws of nature over the fields of these numbers. Quaternions were already used in the context of special relativity [2], electrodynamics [3], Maxwell’s equations [4], quantum mechanics[5], quaternion oscillator [6]and gauge theories [7]. The two-component formulation and the non-commutative algebra of quaternions are the two important discoveries in theoretical physics. Relativistic quantum mechanics is the theory of quantum mechanics that is consistent with the Einstein’s theory of relativity. Dirac [8] was the first who attempted in this field and then Feshbach and Villars [9]. Since relativistic quantum mechanics in 3+1 space-time dimension becomes difficult because of different dimensionality of time and space and the use of quaternions has become essential because quaternion algebra [10] has certain advantages and provides 4-dimensional structure to relativistic quantum mechanics and also in terms of quaternions compact representation and simple theory is obtained. Quaternions are expressed in terms of Pauli spin matrices and accordingly the spin [11-12]. Pioneer work in the field of relativistic quaternionic quantum mechanics was done by Adler [11], Rotelli [13] and Leo et al [14] who obtained quaternionic wave equation. Gürsey [15] and Hestens [16] among others have reformulated the Dirac equation in quaternionic valued term. In order to be algebraic equivalent to Dirac equation, their equations are forced to break the automorphism group of quaternions. A modified Dirac equation has been described by Fredsted [17] as the square root of Klein-Gordon equation with mass advantages.

Keeping in view the advantages the applications of quaternionic algebra in this paper we made an attempt to develop the quaternionic Dirac equation. We have developed the quaternion Dirac equation from the square root of Klein-Gordon equation in compact and consistent way. Dirac matrices are described as quaternion valued and the Dirac Hamiltonian is considered as hermitian with real eigenvalues of energy. We have obtained the Dirac spinors and free particle energy solution in terms of one component, two-component and four-component Dirac spinors. Though the one component and two component solutions are not obtained in ordinary quantum mechanics, the quaternion quantum mechanics that provides the consistent one and two component Dirac spinors. All the spinors incorporate the spin. The first spinor is associated with positive energy spin up spinors, second describes positive energy spin down, third gives negative energy spin up while forth one describes the negative energy spin down for all the cases of spinors (i.e. one component, two component and four component). Thus the minimal representation for quaternion Dirac equation is described in N = 1 quaternionic, N = 2 complex and N = 4 real representation. It has been shown that one-component spinor amplitudes are isomorphic to two component complex spinor amplitudes and four component real spinor amplitudes.

II. QUATERNIONIC KLEIN-GORDON EQUATION

Let us define space-time four vector \( \{x_\mu\} \) in Euclidean representation in terms of natural units \( \hbar = c = 1 \). The space coordinate \( x_\mu \), momentum component \( \{P_\mu\} \), differential operator \( \{e_\mu\} \) and four-potentials \( \{A_\mu\} \) and \( \{B_\mu\} \) in this representation are defined as

\[
\{x_\mu\} = (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, it), \quad \{P_\mu\} = (P_1, P_2, P_3, iE), \quad \{e_\mu\} = (e_1, e_2, e_3, it)
\]

\[
\{A_\mu\} = (A_1, A_2, A_3, i\phi) \quad \text{and} \quad \{B_\mu\} = (B_1, B_2, B_3, i\phi')
\]

...(1)

Here we have adopted \((-1, +1, +1, +1)\) metric, and then
space coordinates are defined as \((x_0, x_1, x_2, x_3)\). The space coordinates \(\{x_\mu\}\), momentum coordinate \(\{P_\mu\}\), differential operator \(\{\partial_\mu\}\) and potentials \(\{\vec{A}_\mu\}\) & \(\{\vec{B}_\mu\}\) are now defined as equations (2). The covariant and contra-variant coordinates are related in terms of \(x_\mu = g_{\mu \nu} x^\nu\), where \(g_{\mu \nu}\) is the metric (or signature) defined as:

\[ g_{\mu \nu} = (1, -1, -1, -1) \forall \mu, \nu = 0, 1, 2, 3, \]

Thus the contra-variant vectors are defined as coordinates are defined as

\[ P_\mu = (E, P_1, P_2, P_3), \quad \partial_\mu = (-\partial_t, \partial_1, \partial_2, \partial_3) \]

\[ \vec{A}_\mu = (\phi A_1, A_2, A_3) & \quad \vec{B}_\mu = (\phi B_1, B_2, B_3, i\phi) \]

We may now write the space–time, energy momentum and differential operator four vectors in terms of following quaternion representation i.e.

\[ x = x_0 + e_1x_1 + e_2x_2 + e_3x_3 \approx \{x_\mu\} \]

\[ P = P_0 + e_1P_1 + e_2P_2 + e_3P_3 \approx \{P_\mu\} \]

\[ \partial = \partial_0 + e_1\partial_1 + e_2\partial_2 + e_3\partial_3 \approx \{\partial_\mu\} \]

Their quaternion conjugate is defined as

\[ \overline{x} = x_0 - e_1x_1 - e_2x_2 - e_3x_3 \]

\[ \overline{P} = P_0 + e_1P_1 + e_2P_2 + e_3P_3 \]

\[ \overline{\partial} = \partial_0 + e_1\partial_1 + e_2\partial_2 + e_3\partial_3 \]

Using the multiplication rule applied for quaternion basis elements, we get

\[ \overline{x} = x_0 + x^2 + x^2 + x^2 = x_\mu^\mu \]

\[ \overline{P} = P_0 + P_1 + P_2 + P_3 \]

\[ \overline{\partial} = \partial_0 + \partial_1^2 + \partial_2^2 + \partial_3^2 \]

We can now establish the relations between the covariant formalism and quaternion formalism to reformulate the relativistic quantum mechanics by means of quaternions. The relativistic relation between energy and momentum is given by

\[ E^2 = P^2 + m^2 & E = \pm \sqrt{P^2 + m^2} \]

where

\[ E = i \frac{\partial}{\partial t} & \quad \overline{\partial} = -i\nabla = -i \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \]

by writing the relations between classical observables and quantum mechanical operators. Substituting the value of \(E\) and \(P\) in equation (7) we get the following relation

\[ \left( -\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2 \right) \Psi(x, t) = 0 \]

where \(\square\) is D’Alembertian operator and is defined as

\[ \square = -\frac{\partial^2}{\partial t^2} + \nabla^2 \]

Equation (8) is described as well known Klein –Gordon equation. From equation (2c) and (4c) and using (6c), we get

\[ \partial_\mu^\mu = (-\partial_t^2 + \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2) = -\partial_t^2 + \nabla^2 = \Theta \overline{\partial} \]

Substituting the value of \(-\partial_t^2 + \nabla^2\) in equation (8a) we get the following covariant form of Klein –Gordon equation

\[ (\partial_\mu^\mu - m^2) \Psi(x, t) = 0 \]

Here the wave function \(\Psi(x, t)\) is defined as

\[ \Psi(x, t) = \Psi e^{ip_\mu \partial_\mu} \]

Let us define the wave function \(\Psi\) in terms of quaternion component as follows

\[ \phi = \phi_0 + e_1\phi_1 + e_2\phi_2 + e_3\phi_3 = \phi_0 + \sum_{j=1}^{3} e_j \phi_j \]

and let us consider the real part of the quaternion only and letting pure quaternionic part to zero. Equation (8a) can be written as

\[ \left( -\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2 \right) \phi_0 = 0 \]
Taking the quaternion conjugate of equation (12)

\[ \phi_0 \left( -\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2 \right) = 0 \]  

...(13)

Pre-multiplying eqn (12) by \( \phi_0^\dagger \) and post-multiplying eqn(13) by \( \phi_0 \) and subtracting we get

\[ \left[ \nabla (\phi_0^\dagger \nabla \phi_0) + \frac{\partial}{\partial t} \left( -\phi_0^\dagger \frac{\partial \phi_0}{\partial t} + \frac{\partial \phi_0^\dagger}{\partial t} \phi_0 \right) \right] = 0 \]  

...(14)

Substituting \( j_j = \phi_0^\dagger \nabla \phi_0 - (\nabla \phi_0) \phi_0 \) and

\[ j_0 = \rho - \frac{\partial \phi_0}{\partial t} \phi_0 - \phi_0^\dagger \frac{\partial \phi_0}{\partial t} \]

Equation (14) reduces to following continuity equation.

\[ \nabla j_j + \frac{\partial \rho}{\partial t} = 0 \quad \text{or} \quad \frac{\partial}{\partial t} \rho_{jm} = 0 \]  

...(15)

Klein-Gordon equation (8) holds good for each and every component of quaternion \( \phi \) and satisfy the continuity equation for all component of wave function i.e. \( \phi_1, \phi_2 \) and \( \phi_3 \). Here also the K.G. equation faces all the difficulties like the usual K.G. equation for relativistic quantum mechanics and cannot be treated as the wave equation of electron (or particle).

**III. QUATERNIONIC DIRAC EQUATION**

Let us discuss the Dirac equation, which is described as particle equation and accordingly we generalize it in terms of quaternions. So we start, as usual, by linearizing equation (7) as follows

\[ \sqrt{p_1^2 + m_0^2} = \sqrt{p_1^2 + p_2^2 + p_3^2 + m^2} = \sum_{i=1}^{3} (\alpha_i P_i + \beta m) \]  

...(17)

where \( \alpha_i \) and \( \beta \) are the Dirac coefficients . Squaring equation (17) gives

\[ \left( \sum_{j=1}^{3} \alpha_j P_j + \beta m \right) \left( \sum_{m=1}^{3} \alpha_m P_m + \beta m \right) = \left( \sum_{i=1}^{3} \alpha_i^2 P_i^2 + \sum_{i=m} \left( \alpha_i \alpha_m + \alpha_m \alpha_i \right) P_i P_m + \sum_{i=1}^{3} \left( \alpha_i \beta + \beta \alpha_i \right) \beta m^2 \right) \]  

...(18)

This equation leads to following relations for \( \alpha_i \) and \( \beta \)

\[ \alpha_i^2 = \alpha_m^2 = \alpha_s^2 = 1, \quad \beta^2 = 1, \quad \alpha_i \alpha_m + \alpha_m \alpha_i = 0 \quad \text{and} \quad \alpha_i \beta + \beta \alpha_i = 0 \]  

...(19)

So Dirac Hamiltonian can be written as

\[ \hat{H}_D = \sqrt{p^2 + m^2} = \sum_{i=1}^{3} \alpha_i P_i + \beta m. \]  

...(20)

Which may be written as Schrodinger equation i.e.

\[ \hat{H}_D \psi = i \frac{\partial \psi}{\partial t}. \]  

...(21)

Hence Dirac equation can be written as

\[ i \frac{\partial \psi}{\partial t} = (\alpha_i P_i + \beta m) \psi \]  

...(22)

Since \( \hat{H}_D \) should be real and quaternion Hermitian we get

\[ \alpha_i = \alpha^*_i, \beta = \beta, \quad \rho = P \]  

...(23)

So we can construct \( 2 \times 2 \) quaternion valued \( \alpha_i \) and \( \beta \) matrices in the following matrix realization of quaternion basis elements

\[ \alpha_i = \begin{bmatrix} 0 & i \epsilon_i \\ -i \epsilon_i & 0 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]  

...(24)

We may now consider the plain wave solution of Dirac equation. Equation (21) gives,

\[ \hat{H}_D \psi = \left( \sum_{i=1}^{3} \alpha_i P_i + \beta m \right) \psi = E \psi \]  

...(25)

where \( \Psi \) is described as the quaternionic Dirac spinor defined by

\[ \psi = \psi_0 + e_1 \psi_1 + e_2 \psi_2 + e_3 \psi_3 \]  

...(26a)

and can be decompose as

\[ \psi = \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ -\psi_3 \end{pmatrix} \]  

...(26b)

in terms of two and four components Dirac spinors associated with symplectic representation of quaternions as \( \psi = \psi_a + e_2 \psi_b \) in terms of complex and accordingly \( \psi_a = \psi_0 + e_1 \psi_1 \) and \( \psi_a = \psi_2 - e_1 \psi_3 \) over the field of real number representation. In other words, we can write one component quaternion valued Dirac spinor which is isomorphic to two component complex spinor or four component real representation. In describing the theory of Dirac equation, the Dirac spinor is taken as the four-component spinor with complex (or real) coefficients. Hence in equation (26) the spinor \( \psi \) may be described as bi-quaternion valued where all components

\[ i.e. \psi_0, \psi_1, \psi_2, \psi_3 \]  

are complex ones and the complex quantity \( i = \sqrt{-1} \) commutes with all the quaternion basis elements. The equations (24), (25) and (26a) may then be written as
and accordingly we get two coupled equations in terms of quaternionic component

\[(E + m) \psi_a = -ie P_1 \psi_b \]  
\[(E + m) \psi_b = ie P_1 \psi_a \]  \(\text{(28a)}\)

We have obtained two equations in terms of quaternion component \(\psi_a\) and \(\psi_b\). Equation (28b) is described for positive energy and equation (28a) is for negative energy. If we break \(\psi_a\) and \(\psi_b\) into further two components, we get four-component equation. So by using quaternions we obtain the Dirac equation in compact form. Let us write the solution of free particle Dirac equation in terms of spinor components

(i) One component spinor solutions

Using equations (28) and the symplectic representation

\[\psi = \psi_a + e_2 \psi_b,\] where \(\psi_a\) and \(\psi_b\) are defined by \(\psi_a = \psi_0 + e_1 \psi_1\) & \(\psi_b = \psi_2 - e_1 \psi_3\) we get the solutions in the following ways

(a) \[\psi^I = 1 + e_2 \frac{ie P_1}{(E + m)}\] 
\text{(Energy = positive, spin = \(\uparrow\))}, \(\psi_a\)  \(\text{(29a)}\)

(b) \[\psi^II = \left(1 + e_2 \frac{ie P_1}{(E + m)}\right) e_1\] 
\text{(Energy = positive, spin = \(\downarrow\))}, \(\psi_a\) \(\text{(29b)}\)

(c) \[\psi^III = - \frac{ie P_1}{(E + m)} + e_2\] 
\text{(Energy = negative, spin = \(\uparrow\))}, \(\psi_a\) \(\text{(29c)}\)

(d) \[\psi^IV = - \frac{ie P_1}{(E + m)} + e_2\] 
\text{(Energy = negative, spin = \(\downarrow\))}. \(\psi_a\) \(\text{(29d)}\)

(ii) For Two component spinor amplitude

Let us consider \(E = +E = E_+\) and hence the wave function component is defined as \(\psi_a = \left[ \begin{array}{c} x_+ \\ 1 \end{array} \right]\), where \(x_+ = \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]\) corresponds to spin up state and \(x_- = \left[ \begin{array}{c} 0 \\ 1 \end{array} \right]\) corresponds to spin down state. Thus we obtain the following cases to obtain the solution of free particle Dirac equation.

\(a\) Positive energy and spin up \((\uparrow)\) solution \(\psi_a\)

\[x_+ = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(\psi_0 = 1, \psi_1 = 0),\text{ we get}\]

\[\psi_a = \psi_0 + e_1 \psi_1 = 1 & \psi_b = \frac{ie P_1}{(E_+ + m)} \] \(\text{(30a)}\)

Hence we obtain the Dirac spinor for positive energy spin up as

\[\psi^I(E, \uparrow) = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \] \(\text{(30b)}\)

\(b\) Positive energy and spin down \((\downarrow)\) solution \(\psi_a\)

\[\psi^II = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} i.e. \psi_a = e_1, \psi_b = \frac{ie P_1}{(E_+ + m)} e_1 \] \(\text{(31a)}\)

and we get the solution for positive energy spin down \(\psi_a\)

\[x_+ = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \] \(\text{i.e.} \psi_a = 1 & \psi_b = \frac{ie P_1}{(E_+ + m)} \] \(\text{(31b)}\)

\(c\) Negative energy and spin up \((E = -E = E_-)\) solution

\[x_- = \begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} i.e. \psi_b = \frac{ie P_1}{(E_+ + m)} \] \(\text{(32a)}\)

Hence we get the solution for negative energy spin up \(\psi_a\)

\[\psi^II = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \] \(\text{(32b)}\)

\(d\) Similarly we may obtain the solution for negative energy and spin down \(\psi_a\)

\[\psi^IV = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} - \frac{ie P_1}{(E_+ + m)} \\ 1 \end{pmatrix} \] \(\text{(33)}\)

(iii) Four-component spinor amplitudes

These are obtained by restricting the propagation along Z-axis \(\text{i.e.} P_x = P_y = 0\) (direction of propagation) and on substituting
As such we have obtain the solution of quaternion Dirac equation in terms of one component quaternion, two-component complex and four-component real spinor amplitudes. Equation (35) is same as obtained in the case of usual Dirac equation. Thus we may interpret that the $N = 1$ quaternion spinor amplitude is isomorphic to $N = 2$ complex and $N = 4$ real spinor amplitude solution of Dirac equation. We can accordingly interpret the minimum dimensional representation for Dirac equation is $N = 1$ in quaternionic case, $N = 2$ in complex case and $N = 4$ for the case of real number field.

**REFERENCES**