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Quaternionic Formulation of Dirac Equation

A.S. Rawat*, Seema Rawat** and O.P.S. Negi ***
Deparment of Physics,
**H.N.B. Garhwal University Campus Pauri, Garhwal Uttarakhand, India.*
***Zakir Hussain College New Delhi, India.*
**** SSJ Kumaun University Campus Almora, Uttarakhand, India*

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ABSTRACT : Quaternion Dirac equation has been obtained from the square root of Klein-Gordon equation in compact and consistent way. Dirac matrices are described as quaternion valued and the Dirac Hamiltonian is considered as Hermitian with real eigenvalues of energy. Dirac spinors and free particle energy solution has been obtained in terms of one component, two-component and four-component Dirac spinors.

Keywords: Quaternions, Dirac Equation, Dirac Hamiltonion.

1. INTRODUCTION

Quaternions were very first example of hyper-complex numbers having the significant impact of mathematics and physics [1]. Because of their beautiful and unique properties quaternions attracted many to study the laws of nature over the fields of these numbers. Quaternions were already used in the context of special relativity [2], electrodynamics [3], Maxwell's equations [4], quantum mechanics[5], quaternion oscillator [6]and gauge theories [7]. The two-component formulation and the non-commutative algebra of quaternions are the two important discoveries in theoretical physics. Relativistic quantum mechanics is the theory of quantum mechanics that is consistent with the Einstein's theory of relativity. Dirac [8] was the first who attempted in this field and then Feshback and Villars [9]. Since relativistic quantum mechanics in 3+1 space-time dimension becomes difficult because of different dimensionality of time and space and the use of quaternions has become essential because quaternion algebra [10] has certain advantages and provides 4-dimensional structure to relativistic quantum mechanics and also in terms of quaternions compact representation and simple theory is obtained. Quaternions are expressed in terms of Pauli spin matrices and accordingly the spin [11-12]. Pioneer work in the field of relativistic quaternionic quantum mechanics was done by Adler [11], Rotelli [13] and Leo et al [14] who obtained quaternionic wave equation. Gürsey [15] and Hestens [16] among others have reformulated the Dirac equation in quaternionic valued term. In order to be algebraic equivalent to Dirac equation, their equations are forced to break the automorphism group of quaternions. A modified Dirac equation has been described by Fredsted [17] as the square root of Klein-Gordon equation with mass advantages.

Keeping in view the advantages the applications of quaternionic algebra in this paper we made an attempt to develop the quaternionic Dirac equation. We have developed the quaternion Dirac equation from the square root of Klein-Gordon equation in compact and consistent way. Dirac matrices are described as quaternion valued and the Dirac Hamiltonian is considered as hermitian with real eigenvalues of energy. We have obtained the Dirac spinors and free particle energy solution in terms of one component, two-component and four-component Dirac spinors. Though the one component and two component solutions are not obtained in ordinary quantum mechanics, the quaternion quantum mechanics that provides the consistent one and two component Dirac spinors. All the spinors incorporate the spin. The first spinor is associated with positive energy spin up spinors, second describes positive energy spin down, third gives negative energy spin up while forth one describes the negative energy spin down for all the cases of spinors (*i.e.* one component, two component and four component). Thus the minimal representation for quaternion Dirac equation is described in N = 1 quaternionic, N = 2 complex and N = 4 real representation. It has been shown that one-component spinor amplitudes are isomorphic to two component complex spinor amplitudes and four component real spinor amplitudes.

II. QUATERNIONIC KLEIN-GORDON EQUATION

Let us define space-time four vector $\{x_\mu\}$in Euclidean representation in terms of natural units $\hbar = c = 1$. The space coordinate x_μ, momentum component $\{P_\mu\}$, differential operator $\{\partial_\mu\}$ and four-potentials $\left\{\overrightarrow{A}_\mu\right\}$ and $\left\{\overrightarrow{B}_\mu\right\}$ in this representation are defined as

$$\{x_\mu\} = (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, it),\ \{P_\mu\} = (P_1, P_2, P_3, iE),\ \{\partial_\mu\} = (-\partial x_1, \partial x_2, \partial x_3, i\partial)$$

$$\left\{\overrightarrow{A}_\mu\right\} = (A_1, A_2, A_3, i\phi)\ \&\ \left\{\overrightarrow{B}_\mu\right\} = (B_1, B_2, B_3, i\phi') \quad ...(1)$$

Here we have adopted (–1, +1, +1, +1) metric, and then space coordinates are defined as (x_0, x_1, x_2, x_3). The space coordinates $\{x_\mu\}$, momentum coordinate $\{P_\mu\}$, differential operator$\{\partial_\mu\}$ and potentials $\{\vec{A}_\mu\}$ & $\{\vec{B}_\mu\}$ are now defined as follows

$$\{x_\mu\} = (-x_0, x_1, x_2, x_3) = (-t, x_1, x_2, x_3),\ \{P_\mu\} = (-E, P_1, P_2, P_3),\ \{\partial_\mu\} = (-\partial_0, \partial x_1, \partial x_2, \partial x_3)$$

$$\{\vec{A}_\mu\} = (-\phi, A_1, A_2, A_3)\ \&\ \{\vec{B}_\mu\} = (-\phi', B_1, B_2, B_3, i\phi') \quad ...(2)$$

The covariant and contra-variant coordinates are related in terms of $x_\mu = g_{\mu\nu} x^\nu$, where $g_{\mu\nu}$ is the metric (or signature) defined as

$$g_{\mu\nu} = (-1, +1, +1, +1)\ \forall\ \mu,\nu = 0, 1, 2, 3,$$

$$= 0\ \forall\ \mu \neq \nu$$

Thus the contra-variant vectors are defined as coordinates are defined as

$$x^\mu = (t, x_1, x_2, x_3),\ p^\mu = (E, P_1, P_2, P_3),\ \partial^\mu = (-\partial_t, \partial_{x1}, \partial_{x2}, \partial_{x3})$$

$$\vec{A}^\mu = (\phi, A_1, A_2, A_3)\ \&\ \vec{B}^\mu = (\phi', B_1, B_2, B_3,) \quad ...(3)$$

We may now write the space–time, energy momentum and differential operator four vectors in terms of following quaternion representation *i.e.*

$$x = x_0 + e_1 x_1 + e_2 x_2 + e_3 x_3 \approx \{x_\mu\} \quad ...(4a)$$

$$P = P_0 + e_1 P_1 + e_2 P_2 + e_3 P_3 \approx \{P_\mu\} \quad ...(4b)$$

$$\Theta = \partial_0 + e_1\partial_1 + e_2\partial_2 + e_3\partial_3 \approx \{\partial_\mu\} \quad ...(4c)$$

Their quaternion conjugate is defined as

$$\bar{x} = x_0 - e_1 x_1 - e_2 x_2 - e_3 x_3 \quad ...(5e)$$

$$\bar{P} = P_0 + e_1 P_1 + e_2 P_2 + e_3 P_3 \quad ...(5f)$$

$$\bar{\Theta} = \partial_0 + e_1\partial_1 + e_2\partial_2 + e_3\partial_3 \quad ...(5g)$$

Using the multiplication rule applied for quaternion basis elements, we get

$$\bar{x}x = x\bar{x} = x^2_0 + x^2_1 + x^2_2 + x^2_3 = x^2 + y^2 + z^2 - c^2t^2 = x_\mu x^\mu \quad ...(6a)$$

$$\bar{P}P = P\bar{P} = P^2_0 + P^2_1 + P^2_2 + P^2_3 = P^2_x + P^2_y + P^2_z - E^2 = P_\mu P^\mu \quad ...(6b)$$

$$\bar{\Theta}\Theta = \Theta\bar{\Theta} = \partial_0^{\,2} + \partial_1^{\,2} + \partial_2^{\,2} + \partial_3^{\,2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} = \left|\vec{\nabla}\right|^2 - \frac{\partial^2}{\partial t^2} = \partial_\mu\partial^\mu \quad ...(6c)$$

We can now establish the relations between the covariant formalism and quaternion formalism to reformulate the relativistic quantum mechanics by means of quaternions. The relativistic relation between energy and momentum is given by

$$E^2 = P^2 + m^2\ \&\ E = \pm\sqrt{P^2 + m^2} \quad ...(7)$$

where $E = i\frac{\partial}{\partial t}$ & $\vec{P} = -i\vec{\nabla} = -i\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)$

by writing the relations between classical observables and quantum mechanical operators. Substituting the value of E and P in equation (7) we get the following relation

$$\left(-\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2 - m^2\right)\Psi(x, t) = 0 \text{ or } (\Box - m^2)\Psi(x, t) = 0 \quad ...(8)$$

where $\Box$ is D'Alembertian operator and is defined as

$$\Box = -\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2 \quad ...(8a)$$

Equation (8) is described as well known Klein –Gordon equation. From equation (2c) and (4c) and using (6c), we get

$$\partial_\mu\partial^\mu = (-\partial_t^{\,2} + \partial_{x1}^{\,2} + \partial_{x2}^{\,2} + \partial_{x3}^{\,2}) = -\partial_t^{\,2} + \vec{\nabla}^2 = \Theta\bar{\Theta} \quad ...(9)$$

Substituting the value of $-\partial_t^{\,2} + \vec{\nabla}^2$ in equation (8a) we get the following covariant form of Klein –Gordon equation

$$(\partial_\mu\partial^\mu - m^2)\Psi(x, t) = 0 \quad ...(10a)$$

Here the wave function Ψ (x, t) is defined as

$$\Psi(x, t) = \Psi e^{iP_\mu x^\mu} = \Psi e^{i(P_l x^l - Et)} \quad ...(10b)$$

Let us define the wave function ψ in terms of quaternion component as follows

$$\phi = \phi_0 + e_1\phi_1 + e_2\phi_2 + e_3\phi_3 = \phi_0 + \sum_{j=1}^{3} e_j\,\phi_j \quad ...(11b)$$

and let us consider the real part of the quaternion only and letting pure quaternionic part to zero. Equation (8a) can be written as

$$\left(-\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2 - m^2\right)\phi_0 = 0 \quad ...(12)$$

Taking the quaternion conjugate of equation (12)

$$\phi_0{}^{\dagger}\left(-\frac{\partial^2}{\partial t^2}+\vec{\nabla}^2-m^2\right) = 0 \qquad ...(13)$$

Pre-multiplying eqn (12) by $\phi_0{}^{\dagger}$ and post-multiplying eqn(13) by ϕ_0 and subtracting we get

$$[\nabla(\phi_0{}^{\dagger}\nabla\phi_0) + \frac{\partial}{\partial t}\left(-\phi_0{}^{\dagger}\frac{\partial\phi_0}{\partial t}+\frac{\partial\phi_0{}^{\dagger}}{\partial t}\phi_0\right) = 0 \qquad ...(14)$$

Substituting $j_j = \phi_0{}^{\dagger}\nabla\phi_0 - (\nabla\phi_0)\phi_0$ and

$$j_0 = \rho = \frac{\partial\phi_0}{\partial t}\phi_0 - \phi_0{}^{\dagger}\frac{\partial\phi_0}{\partial t}$$

Equation (14) reduces to following continuity equation.

$$\nabla_j . j_j + \frac{\partial\rho}{\partial t} = 0 \text{ or } \partial_\mu . j_\mu = 0 \qquad ...(15)$$

Klein-Gordon equation (8) holds good for each and every component of quaternion ϕ and satisfy the continuity equation for all component of wave function *i.e.* ϕ_1, ϕ_2 and ϕ_3. Here also the K.G. equation faces all the difficulties like the usual K.G. equation for relativistic quantum mechanics and cannot be treated as the wave equation of electron (or particle).

III. QUATERNIONIC DIRAC EQUATION

Let us discuss the Dirac equation, which is described as particle equation and accordingly we generalize it in terms of quaternions. So we start, as usual, by linearizing equation (7) as follows

$$\sqrt{p_1{}^2+m_0{}^2} = \sqrt{p_1{}^2+p_2{}^2+p_3{}^2+m^2} = \sum_{l=1}^{3}(\alpha_1 P_1+\beta m) \qquad ...(17)$$

where α_1 and β are the Dirac coefficients . Squaring equation (17) gives

$$\left(\sum_{l=1}^{3}\alpha_1 P_1+\beta m\right)\left(\sum_{m=1}^{3}\alpha_m P_m+\beta m\right)$$

$$= \left(\sum_{l=1}^{3}\alpha_l{}^2 P_l^2+\sum_{l\neq m}(\alpha_l\alpha_m+\alpha_m\alpha_l)P_l P_m+\sum_{l=1}(\alpha_l\beta+\beta\alpha_l)\beta^2 m^2\right) \qquad ...(18)$$

This equation leads to following relations for α_l and β

$$\alpha_l{}^2 = \alpha_m{}^2 = \alpha_3{}^2 = 1,\ \beta^2 = 1,\ \alpha_l\alpha_m + \alpha_m\alpha_l = 0 \text{ and } \alpha_l\beta + \beta\alpha_l = 0 \qquad ...(19)$$

So Dirac Hamiltonian can be written as

$$\hat{H}_D = \sqrt{P^2+m^2} = \sum_{l=1}^{3}\alpha_1 . p_1+\beta m. \qquad ...(20)$$

Which may be written as Schrodinger equation *i.e.*

$$\hat{H}_D\,\psi = i\frac{\partial\psi}{\partial t}. \qquad ...(21)$$

Hence Dirac equation can be written as

$$i\frac{\partial\psi}{\partial t} = (\alpha_l P_l+\beta m)\psi \qquad ...(22)$$

Since $\hat{H}_D$ should be real and quaternion Hermitian we get

$$\alpha_l{}^{\dagger} = \alpha_l,\ \beta^{\dagger} = \beta,\ P^{\dagger} = P \qquad ...(23)$$

So we can construct 2 × 2 quaternion valued α_1 and β matrices in the following matrix realization of quaternion basis elements

$$\alpha_l = \begin{bmatrix} 0 & ie_l \\ ie_l & 0 \end{bmatrix} \text{ and } \beta = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \qquad ...(24)$$

We may now consider the plain wave solution of Dirac equation. Equation (21) gives,

$$\hat{H}_D\psi = \left(\sum_{l=1}^{3}\alpha_l P_l+\beta m\right)\psi = E\psi \qquad ...(25)$$

where Ψ is described as the quaternionic Dirac spinor defined by

$$\psi = \psi_0 + e_1\psi_1 + e_2\psi_2 + e_3\psi_3 \qquad ...(26a)$$

and can be decompose as

$$\psi = \begin{pmatrix}\psi_a \\ \psi_b\end{pmatrix} = \begin{pmatrix}\psi_0 \\ \psi_1 \\ \psi_2 \\ -\psi_3\end{pmatrix} \qquad ...(26b)$$

in terms of two and four components Dirac spinors associated with symplectic representation of quaternions as $\psi = \psi_a + e_2\psi_b$ in terms of complex and accordingly $\psi_a = \psi_0 + e_1\psi_1$ and $\psi_a = \psi_2 - e_1\psi_3$ over the field of real number representation. In other words, we can write one component quaternion valued Dirac spinor which is isomorphic to two component complex spinor or four component real representation. In describing the theory of Dirac equation, the Dirac spinor is taken as the four-component spinor with complex (or real) coefficients. Hence in equation (26) the spinor ψ may be described as bi-quaternion valued where all components

i.e. ψ_0, ψ_1, ψ_2, ψ_3 are complex ones and the complex quantity $i = \sqrt{-1}$ commutes with all the quaternion basis elements $e_0 = 1$, e_1, e_2, e_3. The equations (24), (25) and (26*a*) may then be written as

$$E\begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix}=\left(\begin{bmatrix}0 & ie_l P_l\\ ie_l P & 0\end{bmatrix}+\begin{bmatrix}m & 0\\ 0 & -m\end{bmatrix}\right)\begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix} \quad ...(27)$$

and accordingly we get two coupled equations in terms of quaternionic component

$$(E_- + m)\,\psi_a = -ie_l P_l \Psi_b \quad ...(28a)$$

$$(E_+ + m)\,\psi_b = ie_l P_l \Psi_a \quad ...(28b)$$

We have obtained two equations in terms of quaternion component ψ_a and ψ_b. Equation (28*b*) is described for positive energy and equation (28*a*) is for negative energy. If we break ψ_a and ψ_b into further two components, we get four-component equation. So by using quaternions we obtain the Dirac equation in compact form. Let us write the solution of free particle Dirac equation in terms of spinor components

(*i*) One component spinor solutions

Using equations (28) and the symplectic representation $\psi = \psi_a + e_2 \psi_b$, where ψ_a and ψ_b are defined by $\psi_a = \psi_0 + e_1 \psi_1$ & $\psi_b = \psi_2 - e_1 \psi_3$ we get the solutions in the following ways

(*a*) $$\psi^{I} = 1 + e_2 \frac{ie_l P_l}{(E_+ + m)}$$

(Energy = positive, spin = $\uparrow$), ...(29a)

(*b*) $$\psi^{II} = \left(1 + e_2 \frac{ie_l P_l}{(E_+ + m)}\right) e_1$$

(Energy = positive, spin = $\downarrow$), ...(29b)

(*c*) $$\psi^{III} = -\frac{ie_l P_l}{(E_+ + m)} + e_2$$

(Energy = negative, spin = $\uparrow$), ...(29c)

(*d*) $$\psi^{IV} = \left(-\frac{ip_l P_l}{(E_+ + m)} + e_2\right) e_1$$

(Energy = negative, spin = $\downarrow$). ...(29d)

(*ii*) For Two component spinor amplitude

Let us consider $E = +E = E_+$ and hence the wave function component is defined as $\psi_a = [x_\pm]$, where $x_+ = \begin{bmatrix}1\\0\end{bmatrix}$ corresponds to spin up state and $x_- = \begin{bmatrix}0\\1\end{bmatrix}$ corresponds to spin down state. Thus we obtain the following cases to obtain the solution of free particle Dirac equation.

(*a*) Positive energy and spin up ($\uparrow$) solution *i.e.* for $\psi_a = x_+ = \begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ ($\psi_0 = 1$, $\psi_1 = 0$), we get

and $$\Psi_b = \frac{ie_1 P_1}{(E_+ + m)}$$

$$\psi_a = \psi_0 + e_1 \psi_1 = 1 \;\&\; \psi_b = \frac{ie_l P_l}{(E_+ + m)} \quad ...(30a)$$

Hence we obtain the Dirac spinor for positive energy spin up as

$$\psi^{I}(E, \uparrow)\begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix} = \begin{pmatrix}1\\ \dfrac{ie_l P_l}{(E_+ + m)}\end{pmatrix} \quad ...(30b)$$

(*b*) Positive energy and spin down ($\downarrow$) solution $\psi_a =$

$$x_- = \begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \; i.e.\; \psi_a = e_1,\; \psi_b = \frac{ie_l P_l}{(E_+ + m)} e_1 \quad ...(31a)$$

and we get the solution for positive energy spin down *i.e.*

$$\psi^{II}(E, \downarrow) = \begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix} = \begin{pmatrix}1\\ \dfrac{ie_l P_l}{(E_+ + m)}\end{pmatrix} e_1 \quad ...(31b)$$

(*c*) Negative energy and spin up ($E = -E = E_-$) solution

$$\psi_b = x_+ = \begin{pmatrix}\psi_2\\ -\psi_3\end{pmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \; i.e.\; \psi_b = 1 \;\&\; \psi_a$$

$$= -\frac{ie_l P_l}{(E_+ + m)} \quad ...(32a)$$

Hence we get the solution for negative energy spin up *i.e.*

$$\psi^{III} = (-E, \uparrow) = \begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix} = \begin{pmatrix}-\dfrac{ie_l P_l}{(E_+ + m)}\\ 1\end{pmatrix} \quad ...(32b)$$

(*d*) Similarly we may obtain the solution for negative energy and spin down *i.e.*

$$\psi^{IV} = (-E, \downarrow) = \begin{pmatrix}-\dfrac{ie_l P_l}{(E_+ + m)}\\ 1\end{pmatrix} e_1 \quad ...(33)$$

(*iii*) Four-component spinor amplitudes

These are obtained by restricting the propagation along Z-axis *i.e.* $P_x = P_y = 0$ (direction of propagation) and on substituting

$$e_1 = -i\sigma_l \quad i.e.\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2$$

$$= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad ...(34)$$

along with the usual definition of spin up and spin down amplitudes of spin *i.e.*

$$(a)\ \psi^{I}(E_+ \uparrow) = \begin{pmatrix} 1 \\ 0 \\ \frac{|P|}{E_+ + m} \\ 0 \end{pmatrix} \text{ (Energy = positive, spin =}\uparrow\text{)}, \qquad ...(35a)$$

$$(b)\ \psi^{II}(E_+ \downarrow) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{|P|}{E_+ + m} \end{pmatrix} \text{(Energy = positive, spin =}\downarrow\text{)}, \qquad ...(35b)$$

$$(c)\ \psi^{III}(-E_- \uparrow) = \begin{pmatrix} -\frac{|P|}{E_+ + m} \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{(Energy = negative, spin=}\uparrow\text{)}, \qquad ...(35c)$$

$$(d)\ \psi^{IV}(-E_- \downarrow) = \begin{pmatrix} 0 \\ -\frac{|P|}{E_+ + m} \\ 0 \\ 1 \end{pmatrix} \text{(Energy = negative, spin=}\downarrow\text{)} \qquad ...(35d)$$

As such we have obtain the solution of quaternion Dirac equation in terms of one component quaternion, two-component complex and four-component real spinor amplitudes. Equation (35) is same as obtained in the case of usual Dirac equation. Thus we may interpret that the $N = 1$ quaternion spinor amplitude is isomorphic to $N = 2$ complex and $N = 4$ real spinor amplitude solution of Dirac equation. We can accordingly interpret the minimum dimensional representation for Dirac equation is $N = 1$ in quaternionic case, $N = 2$ in complex case and $N = 4$ for the case of real number field.

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