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A Fixed Point Theorem in Complete Metric Space

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ABSTRACT : The aim of the present paper is to prove a fixed point theorem in complete metric spaces.

Keywords : Fixed point, Complete metric space.

I. INTRODUCTION

In 1922, the polish mathematician, Banach, proved a theorem which established the existence and uniqueness of a fixed point. His result is known as Banach fixed point theorem or the Banach Contraction principle.This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. This theorem has been generalized and extended by many authors [1-4] in various ways.

In this paper we generalize some results of Fisher [5], Kannan [6] and Sessa [7].

Preliminaries:

Defination 1. A sequence $\{x_n\}$ in metric space (X, d) is said to be convergent to a point $x \in X$, denoted by $\lim_{n\to\infty} x_n = x,$ if $\lim_{n\to\infty} (x_n, x) = 0.$

Defination 2. A sequence $\{x_n\}$ in metric space (X, d) is said to be a Cauchy sequence if $\lim_{t\to\infty} d(x_n, x_m) = 0$ for all $n, m > t$.

Defination 3. A metric space (X, d) is said to be complete if every cauchy sequence in X is convergent.

Theorem : Let X be a complete metric space $x_0 \in X$ and for $r > 0$, T: $clsr(x_0, r)B \to X$ be a linear transformation. Suppose that for any $x, y \in clsrB\ (x_0, r)$. We have

$$d(Tx, Ty) \leq \alpha\{d(x, Tx)d(y, Ty)d(x, y)\}^{1/3} \qquad ... (1)$$

where $\alpha \in (0,1)$ satisfies the condition

$$d(x_0, Tx_0) < (1 - \alpha)\mathrm{r} \qquad ... (2)$$

Then T has a fixed point.

Proof :

For $x_0 \in B(x_0, r)$, we can choose $x_1 \in X$ such that $Tx_0 = x_1$

Then from (2), we have $d(x_0, x_1) = d(x_0, Tx_0) < (1 - \alpha)r$

Hence, $x_1 \in B(x_0, r)$

Again, if we choose $x_2 \in X$ such that $x_2 = Tx_1$

Then, $d(x_1, x_2) = d(x_1, Tx_1) = d(Tx_0, Tx_1)$

From (1) we have,

$\Rightarrow \quad d(Tx_0, Tx_1) \leq \alpha\{d(x_0, Tx_0)d(x_1, Tx_1)d(x_0, x_1)\}^{1/3} \leq \alpha\{d(x_0, x_1)d(x_1, x_2)d(x_0, x_1)\}^{1/3}$

$\Rightarrow \quad d(x_1, x_2) \leq \alpha\{[d(x_0, x_1)]^2 d(x_1, x_2)\}^{1/3}$

$\Rightarrow \quad \{d(x_1, x_2)\}^2 < \alpha^3\{d(x_0, x_1)\}^2$

$\Rightarrow \quad d(x_1, x_2) < \alpha^{3/2} d(x_0, x_1)$

$\Rightarrow \quad d(Tx_0, Tx_1) < \alpha^{3/2}(1 - \alpha)r$

$\Rightarrow \quad x_2 \in B(x_0, r)$

Now,

$d(x_0, x_2) \leq d(x_0, x_1) + d(x_1, x_2) < (1 - \alpha)r + \alpha^{3/2}(1 - \alpha)r < (1 - \alpha)r\{1 + \alpha^{3/2}\}$

$\Rightarrow \quad d(x_0, x_2) < r$

$\Rightarrow \quad x_2 \in B(x_0, r)$

Proceeding in the same manner,we get, $x_n \in B(x_0, r)$

$\Rightarrow \quad x_{n-1} = Tx_n$

Proceeding inductively, we obtain x_n, such that $x_n \in B(x_0, r)$

Also, $x_n = Tx_{n+1}$

Hence the sequence $\{x_n\}$ is Cauchy sequence. i.e. $\lim_{n\to\infty} x_n = x$

Claim : x is a fixed point of T

On the contrary if we assume that $x \neq Tx$

Then, $d(x, Tx) \leq d(x, x_n) + d(x_n, Tx) \leq d(x, x_n) + d(Tx_{n-1}, Tx)$

From the continuity of T and using (1), we get $d(x, Tx) < 0$ which is a contradiction.

Hence, x is the fixed point of T.

II. REFERENCES

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