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Strong Transitivity in Oriented Graphs

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ABSTRACT : An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops. The score of a vertex v_i in an oriented graph D is avi (or simply a_i) = n – 1 + d^+_{vi} – d^-_{vi}, where d^+_{vi} and d^-_{vi} are the outdegree and indegree, respectively, of v_i and n is the no. of vertices in D. In this paper we characterize strongly transitive oriented graphs. We also characterize irreducible score sequences with strongly transitivity.

Keywords : Oriented graph, tournament, score sequence, triples, reducible and irreducible, strongly transitive.

I. INTRODUCTION

Definition 1. A tournament is a directed graph (digraph) obtained by assigning a direction for each edge in an undirected complete graph. That is, it is a directed graph in which every pair of vertices is connected by a single directed edge.

Definition 2. The score (out degree) of vertex i is the number S_i of vertices that i dominates. Let vertices of T_n be labelled in such a way that $s_1 \leq s_2 \leq s_3 \leq \ldots \leq s_n$. The sequence $(s_1, s_2, \ldots, s_n)$ is called the score sequence of T_n. The score set S_t of a tournament T (an irreflexive, asymmetric, complete digraph) is a set of scores (out degrees) of the vertices of T.

The following result of Landau [1] gives a necessary and sufficient conditions for a score sequence to belong to some tournament.

Theorem 1. A sequence $(s_1, s_2, \ldots, s_n)$ of integers satisfying $0 \leq s_1 \leq s_2 \leq \ldots. \leq s_n$ is the score sequence of some n-tournament if and only if

$$\sum_i^i S_i \geq \binom{k}{2}, i \leq k \leq n$$

With equality when $k = n$.

Definition 3. An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops. Define $a_{vi} = n - 1 + d_{vi}^+ - d_{vi}^-$, the score of a vertex v_i in an oriented graph D, where d_{vi}^+ and d_{vi}^- are the outdegree and indegree, respectively, of v_i and n is the number of vertices in D. The score sequence of an oriented graph is formed by listing the vertex scores in non-decreasing order.

For any two vertices u and v in an oriented graph D, we have one of the following possibilities.

I. An arc directed from u to v, denoted by $u(1 - 0)v$.

II. An arc directed from v to u, denoted by $u(0 - 1)v$.

III. There is no arc from u to v and there is no arc from v to u, and is denoted by $u(0 - 0)v$.

Avery [2] has characterised the score sequence of oriented graphs.

If d_v^* is the number of those vertices u in D which have $v(0 - 0)v$, then $d_v^+ + d_v^- + d_v^* = n + 1$. Therefore, $a_v = 2d_v^+ + d_v^*$. This implies each vertex u with $v(1 - 0)u$ contributes two to the score of v and each vertex u with $v(0 - 0)u$ contributes one to the score of v. Since the number of arcs and non arcs in an oriented graph of order n is $\binom{n}{2}$ and each $v(0 - 0)u$ contributes two (one each at u and v) to scores. Therefore the sum total of all the scores is $2\binom{n}{2}$.

Definition 4. A triple in an oriented graph is an induced oriented subgraph with three vertices. For any three vertices u, v and w, the triples of the form $u(1 - 0)v(1 - 0)w(1 - 0)u$, or $u(1 - 0)v(1 - 0)w(0 - 0)u$ are said to be intransitive, while as the triples of the form $u(1 - 0)v(1 - 0)w(0 - 1)u$, or $u(1 - 0)v(0 - 1)w(0 - 0)u$, or $u(1 - 0)v(0 - 0)w(0 - 1)u$, or $u(1 - 0)v(0 - 0)w(0 - 0)u$ are said to be transitive.

An oriented graph is said to be transitive if all its triples are transitive. Also the triples of the form $u(1 - 0)v(1 - 0)w(0 - 1)u$, or $u(1 - 0)v(0 - 1)w(0 - 0)u$, or $u(1 - 0)v(0 - 0)w(0 - 1)u$ are said to be strongly transitive.

An oriented graph is strongly transitive if all its triples are strongly transitive.

The converse D' of an oriented graph D is obtained by reversing each arc of D.

Definition 5. An oriented graph D is reducible if it is possible to partition its vertices into two non-empty sets v_1 and v_2 in such a way that there is an arc from every vertex of v_2 to each vertex of v_1.

Let D_1 and D_2 be induced oriented graphs having vertex sets v_1 and v_2 respectively. Then D consists of D_1 and D_2 and arcs from every vertex of D_2 to each vertex of D_1 and we write $[D_1, D_2]$. If this is not possible, then the oriented graph D is irreducible.

If $D_1, D_2, \ldots, D_k$ are irreducible oriented graphs with disjoint vertex sets, then $D = [D_1, D_2, \ldots, D_k]$ denotes the oriented graph having all arcs D_i, $1 \leq i \leq k$, and arcs from every vertex of D_j to each vertex of D_i, $1 \leq i \leq j \leq k$. In this case, $D_1, D_2, \ldots, D_k$ are the irreducible components of D and such a decomposition is called the irreducible component decomposition of D.

Lemma 1.1: The score sequence A of an oriented graph is strongly transitive iff every irreducible component of A is strongly transitive.

Main results

Theorem 1. If D is an oriented graph having score sequence $A = [a_i]_1^n$, then D is irreducible iff for $k = 1, 2, \ldots, n-1$.

$$\sum_{i=1}^{k} a_i > k(k-1) \qquad \ldots (1)$$

$$\text{And } \sum_{i=1}^{2r} a_i = n(n-1) \qquad \ldots (2)$$

Proof: Suppose D is an irreducible oriented graph having score sequence $A = [a_i]_1^n$. Then we have to prove (1) and (2) conditions hold. Now, condition (2) holds since (theorem 1.1) has already established it for any oriented graph.

To verify inequalities (1) we observe that for any integer $k < n$, the sub-digraph induced by any set of k vertices has a sum of scores $k(k-1)$.

Since D is irreducible, these must be an arc from at least one of these vertices to one of the other $n - k$ vertices.

Thus, for $1 \leq k \leq n - 1$.

$$\sum_{i=1}^{k} S_i > k(k-1)$$

For the converse suppose conditions (1) and (2) hold, then we have to show that D is irreducible. We know by (theorem 1.1) that there exists an oriented graph D with these scores.

Assume that such an oriented graph D is reducible. Let $D = [D_1, D_2, \ldots, D_k,]$ be the irreducible component decomposition of D. If 'm' is the no. of vertices in D_1, then $m < n$ and the equation.

$$\sum_{i=1}^{m} S_i = m(m-1) \text{ holds.}$$

Which is a contradiction.

This proves the converse part.

Theorem 2. Let A be an irreducible score sequence. Then A is strongly transitive iff it is one of [0], [1, 1].

Proof: Let A be an irreducible score sequence and let D be an oriented graph having score sequence A.

Assume D has $n \geq 3$ vertices. Since A is irreducible, therefore there exists vertices u, v, w such that D has a triple of the form

$u(1-0)v(1-0)w(1-0)u$; or

$u(1-0)v(1-0)w(0-0)u$; or

$u(1-0)v(0-0)w(0-0)u$; or

$u(0-0)v(0-0)w(0-0)u$

Since we know that for any three vertices u, v, w the triples of the form

$u(1-0)v(1-0)w(0-1)u$;or

$u(1-0)v(0-1)w(0-0)u$; or

$u(1-0)v(0-0)w(0-1)u$

are said to be strongly transitive. As from above there is no such triple present. Thus, in this case A is not strongly transitive.

Assume D has two vertices, then $A = [1, 1]$ is the only irreducible score sequence and can be considered to be strongly transitive. As there are only two vertices there can either be edge from u to v or v to u. There can also be no edge between u and v.

If D has exactly one vertex, then $A = [0]$ which again is strongly transitive.

Theorem 3. The score sequence A is strongly transitive iff every irreducible component of A is one of [0], [1, 1].

Proof: Suppose A is strongly transitive, then we have to prove every irreducible component of A is one of [0], [1, 1].

Now for any three vertices u, v, w, the triples can be of the form

$u(1-0)v(1-0)w(0-1)u$; or

$u(1-0)v(0-1)w(0-0)u$; or

$u(1-0)v(0-0)w(0-1)u$.

Let $A = [A_1, A_2, \ldots, A_n]$ be the irreducible components of A.

Using (Lemma 1.1) every irreducible component of A is strongly transitive.

For $n \geq 3$, there exists vertices u, v, w, such that D has a triple of the form

$u(1-0)v(1-0)w(1-0)u$; or

$u(1-0)v(1-0)w(0-0)u$; or

$u(1-0)v(0-0)w(0-0)u$; or

$u(0-0)v(0-0)w(0-0)u$

which does not satisfy the conditions of strong transitivity. Thus, for $n \geq 3$ irreducible component of A is not strongly transitive. For $n = 2$, $A = [1, 1]$ is the only irreducible score sequence and can be considered to be strongly transitive.

If $n = 1$, then $A = [0]$. Which again is strongly transitive.

$\therefore$ It follows that every irreducible component of A is one of [0], [1, 1].

Conversely, suppose that every irreducible component of A is one of [0], [1, 1]. We have to prove A is strongly transitive.

When $A = [0]$, there is only one vertex which can trivially satisfy the conditions of strong transitivity.

When $A = [1, 1]$, then we have

$$u(1 - 0)v(0 - 0)u; \text{ or}$$

$$u(0 - 1)v(0 - 0)u$$

which is strongly transitive.

Thus, score sequence A is strongly transitive.

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