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A Fixed Point Theorem on Three Metric Spaces

Rajesh Shrivastava*, Kiran Rathore*and K. Qureshi**

**Department of Mathematics, Government Science and Commerce College, Benazir, Bhopal, (M.P.)*
***Additional Director, Higher Education Department, Govt. of M.P., Bhopal, (M.P.) India*

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ABSTRACT : In this paper we prove fixed point theorem for three metric spaces. A fixed point theorem for set valued mappings on three complete metric spaces is obtained which generalizes a result of [4]. A new related fixed point theorem for three pairs of mappings on three complete metric spaces is obtained.

Keywords : Complete metric space, common fixed point.

I. INTRODUCTION AND PRELIMINARIES

Let (X, d) be a complete metric space and let $B(X)$ be the set of all non-empty subsets of X. The function $\delta(A, B)$ with A and B in $B(X)$ is defined by

$\delta(A, B) = \sup\{d(a, b) : a \in A,\ b \in B\}$

If A consists of a single point a, we write $\delta(A, B) = \delta(a, b)$ and if B also consists of a single point b, we write $\delta(A, B) = \delta(a, b) = d(a, b)$. It follows immediately from the definition that

$$\delta(A, B) = \delta(B, A) \geq 0,$$

$$\delta(A, B) \leq \delta(A, C) + \delta(C, B)$$

for all A, B, C in $B(X)$.

Related fixed point theorems on three complete metric spaces have been studied by Fisher and Rao [4,6,7] Nung [5], Jain and Rao [1- 3]. In this paper, we prove a related fixed point theorem for three mappings. Not of all which are necessary continuous on three metric spaces.

II. MAIN RESULTS

According to Fisher and Rao [4]

2.1. Theorem :

Let (X, d), (Y, ρ) and (Z, σ) be three metric spaces and $T : X \to Y$, $S : Y \to Z$, and $R : Z \to X$ mappings satisfying the inequalities :

$$d(RSy, RSTx) \leq \max\{d(x, RSy), d(x, RSTx), \rho(y, Tx), \rho(y, TRSy), \rho(Tx, TRSy)\}$$

for all x in X and y in Y with $y \neq Tx$,

$$\rho(TRz, TRSy) \leq \max\{\rho(y, TRz), \rho(y, TRSy), \sigma(z, Sy), \sigma(z, STRz), \sigma(Sy, STRz)\}$$

for all y in Y and z in Z with $z \neq Sy$, and

$$\sigma(STx, STRz) \leq \max\{\sigma(z, STx), \sigma(z, STRz), d(x,Rz), d(x, RSTx), d(Rz, RSTx)\}$$

for all x in X and z in Z with $x \neq Rz$. Further, assume one of the following conditions:

(a) (X, d) is compact and RST is continuous,

(b) (Y, ρ) is compact and TRS is continuous,

(c) (Z, σ) is compact and STR is continuous.

Then RST has a unique fixed point w in X, TRS has a unique fixed point u in Y and STR has a unique fixed point v in Z. Further $Su = v$, $Rv = w$ and $Tw = u$.

Our Result:

2.2. Theorem :

Let (X, d), (Y, δ) and (Z, μ) be three metric spaces and $A : X \to Y$, $B : Y \to Z$, and $C : Z \to X$ be mappings satisfying the inequalities:

$$d(ABz, ABCx) \leq \max\{d(x, ABz), d(x, ABCx), \delta(z, Cx), \delta(z, CBAz)\} \quad ...(1)$$

for all x in X and z in Z with $z \neq Cx$,

$$\delta(BCx, BCAy) \leq \max\{\delta(y, BCx), \delta(y, BCAy), \mu(x, By), \mu(x, ABCx)\} \quad ...(2)$$

for all y in Y and x in X with $x \neq Ay$, and

$$\mu(CAy, CABz) \leq \max\{\mu(z, CAy), \mu(z, CABy), d(y, Az)\ d(y, BCAy)\} \quad ...(3)$$

for all y in Y and z in Z with $y \neq Bz$. Further, assume one of the following conditions :

(a) (X, d) is compact and ABC is continuous,

(b) (Y, δ) is compact and BCA is continuous,

(c) (Z, μ) is compact and CAB is continuous.

Then ABC has a unique fixed point α in X, BCA has a unique fixed point β in Y and CAB has a unique fixed point γ in Z.

Proof : Suppose (a) holds. Define $\lambda(x) = d(x, ABCx)$ for $x \in X$. Then there exists 'a' in X such that

$$\lambda(a) = d(a, ABCa) = \inf\{\lambda(x) : x \in X\}$$

Suppose

$$ABCABCABCa \neq ABCABCa, \text{ Then}$$

$$BCABCABCa \neq BCABCa,$$

$$CABCABCa \neq CABCa,$$

$$ABCABCa \neq ABCa,$$

$$BCABCa \neq BCa,$$

$$CABCa \neq Ca, \qquad ABCa \neq a$$

From (1), with $z = CABCa$, $x = ABCABCa$, we have

$$d(ABCABCa, ABCABCABCa) \leq \max\{d(ABCABCa, ABCABCa), \delta(ABCABCa, ABCABCABCa), \delta(CABCa, CABCABCa), \delta(CABCa, CABCABCa), \delta(CABCABCa, CABCABCa)\} \quad ...(4)$$

so that

$$\lambda(ABCABCa) \leq \delta(CABCa, CABCABCa)$$

From (2), with $x = ABa$, $y = BCABa$, we have

$$\delta(BCABa, BCABCABa) \leq \max\{\delta(BCABa, BCABa), \delta(BCABa, BCABCABa), \mu(ABa, ABCABa), \mu(ABa, ABCABa), \mu(ABCABa, ABCABa)\} \quad ...(5)$$

so that $\delta(BCABa, BCABCABa) \leq \mu(ABa, ABCABa)$

From (3) with $y = a$, $z = CAa$, we have

$$\mu(CAa, CABCAa) \leq \max\{\mu(CAa, CAa), \mu(CAa, CABCAa), d(a, BCAa), d(a, BCAa), d(BCAa, BCAa)\} \quad ...(6)$$

so that $\mu(CAa, CABCAa) \leq \lambda(a)$

From (4), (5) and (6), we have $\lambda(ABCABCa) < \lambda(a)$, contradicting the existence of a.

Hence, $ABCABCABCa = ABCABCa$.

Putting $ABCABCa = w$ in X we have,

$$ABC\alpha = \alpha$$

Now let $CT\alpha = \beta$ in Y and $B\beta = \gamma$ in Z. Then $A\gamma = AB\beta = ABC\alpha = \alpha$, and it follows that

$$BCA\gamma = BC\alpha = B\beta = \gamma$$

and $$CAB\beta = CA\gamma = C\alpha = \beta$$

To prove uniqueness, suppose that ABC has a second distinct fixed point $\alpha 0$ in X then,

$$ABC\alpha \neq ABC\alpha 0, BC\alpha \neq BC\alpha 0; C\alpha \neq C\alpha 0:$$

Using (1), with $z = C\alpha$, $x = \alpha 0$, we get

$$dABC\alpha, ABC\alpha 0) \leq \max\{d(\alpha 0, ABC\alpha), d(\alpha 0, ABC\alpha 0), \delta(C\alpha, C\alpha 0), \delta(C\alpha, CABC\alpha), \delta(C\alpha 0, CABC\alpha)\} \quad ...(7)$$

so that $d(\alpha, \alpha 0) \leq \delta(C\alpha, C\alpha 0)$

Using (2) with $x = BC\alpha$, $y = C\alpha 0$, we get

$$\delta(CABC\alpha, CABC\alpha 0) \leq \max\{\delta(C\alpha 0, CABC\alpha), \delta(C\alpha 0, CABC\alpha 0), \mu(BC\alpha, BC\alpha 0), \mu(BC\alpha, BCABC\alpha), \mu(BC\alpha 0, BCABC\alpha)\} \quad ...(8)$$

so that $\delta(C\alpha, C\alpha 0) < \mu(BC\alpha, BC\alpha 0)$

Using (3) with $x = \alpha$, $z = CA\alpha 0$, we get

$$\mu(CA\alpha, CABCA\alpha 0) \leq \max\{\mu(CA\alpha 0, CA \leq \alpha), \mu(CA\alpha 0, CABCA\alpha 0), d(\alpha, BCA\alpha 0), d(\alpha, BCA\alpha), d(BCA\alpha 0, BCA\alpha)\} \quad ...(9)$$

so that $\mu = (CA\alpha, CA\alpha 0) \leq d(\alpha, \alpha 0)$

From (7), (8) and (9), we have $d(\alpha, \alpha 0) < d(\alpha, \alpha 0)$ so that $\alpha = \alpha 0$, proving the uniqueness of α.

Similarly, we can show that γ is the unique fixed point of CAB and β is the unique fixed point of CAB.

It follows similarly that the theorem holds if (*b*) or (*c*) holds instead of (*a*).

Now, we give the following example to illustrate our theorem.

Example. Let $X = [1, 2]$; $Y = [2, 3)$; $Z = (3, 4]$, and let $d = \delta = \mu$ be the usual metric for the real numbers. Define $A : X \to Y$, $B : Y \to Z$ and $C : Z \to X$ by :

$$Ax = \begin{cases} 1 & \text{if } x \,\varepsilon\, [1, 3/2) \\ 5/2 & \text{if } x \,\varepsilon\, (3/2, 2] \end{cases}$$

$$By = \{4 \; y \,\varepsilon\, Y,$$

$$Cz = \begin{cases} 7/4 & \text{if } z \,\varepsilon\, (3, 7/2] \\ 2 & \text{if } z \,\varepsilon\, (7/2, 4] \end{cases}$$

Here Y and Z are not compact spaces and A and B are not continuous. However, all the conditions of Theorem 2.1 are satisfied. Clearly,

$ABC(1) = 1$; $CAB(5/2) = 5/2$; $BCA(4) = 4$; $B(5/2) = 4$; $A\,4 = 2$ and $C\,1, 5/2$.

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