Application of Meijer Transform in typical integral involving $\tilde{H}$ - Function

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ABSTRACT: In this paper a theorem on Meijer transform given by Agrawal has been used to evaluate a typical integral involving $\tilde{H}$ - Function. Our result is quite general in nature and number of known and new formulae can be obtained as applications section.

Keywords: $\tilde{H}$ - Function, Meijer transform.

Mathematical subject classification. 2011, 33C45

I. INTRODUCTION

The $\tilde{H}$- Function, a generalization of fox’s H- Function introduced by Inayat [5, P4107-17] and studied by Bushman and Shrivastava [2, P 4707-10] and other is defined and represented in the following manner.

$$\tilde{H}^{M,N}_{P,Q}
\left[\begin{array}{c}
\left(f_{1}, F_{1}ight)_{1,M}, \left(f_{2}, F_{2}ight)_{1,M}, \ldots, \left(f_{N}, F_{N}ight)_{1,M}, \left(f_{1}, F_{1}ight)_{P,1}, \ldots, \left(f_{P}, F_{P}ight)_{M,1}
\end{array}\right]
= \frac{1}{2\pi \omega} \int_{L} \theta(\xi)(pt)^{\xi} \, d\xi,$$

$$= \frac{1}{2\pi \omega} \int_{L} \left[ \prod_{j=1}^{M} \Gamma(f_{j} - F_{j}\xi) \prod_{j=1}^{N} \Gamma(1-e_{j} + E_{j}\xi)^{\epsilon_{j}} \prod_{j=1}^{P} \Gamma(1-e_{j} - E_{j}\xi)^{\epsilon_{j}} \right] (pt)^{\xi} \, d\xi,$$

(1.1)

And the contour L is the line form $c-\omega \to c + \omega \to \cdots$ to keep the pole of $\Gamma(f_{j} - F_{j}\xi), j = 1,2,\ldots,M$ to the right of the path and the singularities of $\Gamma(1-e_{j} + E_{j}\xi^{E_{j}}), j = 1,2,\ldots,N$ to the left of the path. The other details about $H$ – Function can be seen in the paper cited earlier evidently, if we take $E_{j}$ ($j = 1,2,\ldots,N$) and $J_{j}$ ($j = M=1,\ldots,Q$) equal to unity. The $H$–Function reduced to well known Fox’s $H$ – Function [3].


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The following sufficient condition for absolute convergence of the integral defined in equation (1.1) have been recently given by Gupta Jain, Agarwal [4,167-172].

\[
\begin{align*}
(i) \quad |\text{arg}(pt)| &< \frac{\pi}{2} \quad \text{and} \quad \tau > 0, \\
(ii) \quad |\text{arg}(pt)| &< \frac{\pi}{2} \quad \text{and} \quad \tau \geq 0,
\end{align*}
\]

\[
\begin{align*}
&\quad \text{And} \quad (a) \quad \tau \neq 0, \quad \text{and} \quad \text{contour L is so chosen that} \\
&\quad \text{and} \quad (b) \quad \tau = 0, \quad \text{and} \quad \tau + 1 = 0,
\end{align*}
\]

Where

\[
\begin{align*}
\Omega &= \sum_{j=1}^{M} F_j + \sum_{j=1}^{N} E_j - \sum_{j=M+1}^{O} F_j \tau_j - \sum_{j=N+1}^{P} E_j, \\
\tau &= \sum_{j=1}^{N} E_j + \sum_{j=N+1}^{P} E_j - \sum_{j=1}^{M} F_j - \sum_{j=M+1}^{O} F_j \tau_j, \\
\rho &= \left\{ \sum_{j=1}^{M} f_j + \sum_{j=M+1}^{O} F_j \tau_j, -\sum_{j=1}^{N} e_j - \sum_{j=N+1}^{P} e_j \right\} \\
&+ \frac{1}{2} \left\{ \sum_{j=1}^{N} E_j - \sum_{j=M+1}^{O} \tau_j + P - M - N \right\},
\end{align*}
\]

Meijer transform of the function \(f(t)\) denoted by \(\varphi(p)\) defined as

\[
\Phi(p) = \frac{k}{\tau} \int_{0}^{\infty} \frac{(pt)^{\tau}}{k_{\tau}(pt) \ f(t) \ dt}.
\]

Which is reduced to well known Laplace transform when \(\nu = \pm 1/2,\)

\[\text{II. THEOREM}\]

Theorem given by Agrawal [1.p77] on Meijer transform as, If

\[
\begin{align*}
\end{align*}
\]
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\[
\Phi(p) = \frac{R}{\lambda + \mu} h(t),
\]
And
\[
\Psi(\lambda, p, s) = \frac{k}{\lambda} k(st) h(t),
\]

Then
\[
\int_0^\infty \left[ t^{\frac{1}{2}(\lambda-\mu-\frac{1}{2})} (\frac{\alpha + \beta t}{\alpha t + \beta})^{\frac{1}{2}(\lambda+\mu+\frac{1}{2})} \right. \times (\alpha + \beta t)^{-\frac{1}{2}} \left\{ \sqrt{\frac{(\alpha + \beta t)(\alpha t + \beta t)}{t}} \right\} \left. \right] dt
\]
\[
= 2 \alpha^{-\frac{1}{2}} \psi(\lambda, \mu, \alpha, \beta),
\]

Provided: \( \Re(\alpha) > 0, \Re(\beta) > 0 \),

Integral involved are absolutely convergent and \( h(t) \) is independent of \( s \).

We shall make the use of this theorem in evaluating integral involving \( \tilde{H} \)-Function, we evaluate,

\[
\Phi(p) = \frac{1}{\lambda + \mu} \tilde{H}_{M, N}^{P, Q} (pt) \left[ \left[ (e_j; E_j, \epsilon_j)_{1.N} \right], \left[ (e_j; E_j)_{N+1.P} \right] \right]
\]

\[
\Phi(p) = \int_0^\infty \left\{ (pt)^2 k_{\lambda+\mu} (pt) \tilde{H}_{M, N}^{P, Q} (pt) \left[ \left[ (e_j; E_j, \epsilon_j)_{1.N} \right], \left[ (e_j; E_j)_{N+1.P} \right] \right] \right\} dt
\]

Using the definition of \( \tilde{H} \)-Function and changing the order of integration,

We get,

\[
\Phi(p) = \frac{1}{2 \pi \omega} \int L \theta(p \xi) p^2 \rho^\xi \left\{ \int_0^\infty (t^{\xi+\frac{1}{2}})^{-1} k_{\lambda+\mu} (pt) dt \right\} d\xi
\]

Using the result [2.pp 331],
\[ \Phi(p) = \frac{1}{2\pi\omega} \int_{-L}^{L} Q(r, \xi) \{ p^{\frac{1}{2}} \rho \xi \rho^{-(\xi + \frac{1}{2})} \}^{2} \times \]
\[ \Gamma \left( \frac{\xi}{2} + \frac{3}{4} - \frac{\lambda}{2} - \frac{-\mu}{2} \right) \Gamma \left( \frac{1}{2} \xi + \frac{3}{4} + \frac{1}{2} \lambda + \frac{1}{2} - \mu \right) \} d\xi. \]

Using the definition of \( \tilde{H} \)-Function and adjusting, we get,
\[ \Phi(p) = 2^{\frac{1}{2}} p^{-1} \tilde{H}_{p+2,Q}^{M+N+2} \left[ 2p^{\frac{1}{2}} \left( \frac{1}{4} + \frac{1}{2} \lambda + \frac{1}{2} \mu, \frac{1}{2}, 1 \right) \right] \left( \frac{1}{4} - \frac{1}{2} \lambda - \frac{1}{2} \mu, \frac{1}{2}, 1 \right) \]
\[ \left[ (e_{j};E_{j},\epsilon_{j})_{1,N},[(e_{j};E_{j})_{N+1,P}] \right] \]
\[ \left[ (f_{j},F_{j})_{1,M},[(f_{j},F_{j},\tau_{j})_{M+1,Q}] \right] \]

Now we shall evaluate,
\[ \psi(\lambda, \mu, \rho, s) = \frac{k}{\lambda} k_{\mu}(st) \times \tilde{H}_{P,Q}^{M,N} \left[ (\rho t) \right] \left[ (e_{j};E_{j},\epsilon_{j})_{1,N},[(e_{j};E_{j})_{N+1,P}] \right] \]
\[ \left[ (f_{j},F_{j})_{1,M},[(f_{j},F_{j},\tau_{j})_{M+1,Q}] \right] \]

Using the definition of \( \tilde{H} \)-Function and changing the order of integration,

We get,
\[ \psi(\lambda, \mu, \rho, s) = \frac{1}{2\pi\omega} \int_{0}^{\infty} \left\{ (pt)^{\frac{1}{2}} k_{\lambda}(pt) k_{\mu}(st) \tilde{H}_{P,Q}^{M,N} \left[ (\rho t) \right] \left[ (e_{j};E_{j},\epsilon_{j})_{1,N},[(e_{j};E_{j})_{N+1,P}] \right] \right\} dt \]

Using the result [2. pp 334], we get,
\[ \psi(\lambda, \mu, \rho, s) = \frac{1}{2\pi\omega} \int_{0}^{\infty} \theta(\xi) p^{\frac{1}{2}} \rho^{\xi} \int_{0}^{\xi} \left( t^{\frac{\xi}{2}} - 1 \right) k_{\mu}(st) k_{\lambda}(\rho t) dt \right\} d\xi \]

Using the result [2. pp 334], we get,
Expanding $2F_1$, adjusting the term and then changing the order of integration and summation, using the definition of $\tilde{H}$-Function, we get,

$$\psi(\lambda, \mu, \rho, s) = \sum_{k=0}^{\infty} \frac{2^k}{2} \frac{p^2 s^\mu}{k}(1 - \frac{s^2}{p^2})^k \cdot M,$$  

(2.5)

Where,

$$M \equiv \tilde{H} \cdot N + 6 \cdot P + 6 \cdot Q + 3 \left[ 2 \rho \ p \mid U \right]$$  

(2.6)

$U$ and $V$ are sets of parameters.

$$V : (\frac{1}{4} - \frac{1}{2} \lambda - \frac{1}{2} \mu, \frac{1}{2}, 1), (\frac{1}{4} + \frac{1}{2} \lambda - \mu, \frac{1}{2}, 1), (\frac{1}{4} - \frac{1}{2} \lambda + \mu, \frac{1}{2}, 1),$$

$$\left(\frac{1}{4} + \frac{1}{2} \lambda + \mu, \frac{1}{2}, 1\right)(\frac{1}{4} - \frac{1}{2} \lambda - \mu - k, \frac{1}{2}, 1), (\frac{1}{4} + \frac{1}{2} \lambda - \mu - k, \frac{1}{2}, 1)$$

$$\left[(e_j;E_j,e_i)_{1,N},[(e_j;E_i)_{N+1,P}]\right]$$

$$V : [(f_j,F_j)_{1,M},[(f_j,F_j,\tau_j)_{M+1,Q}],\left[\left(\frac{1}{4} - \frac{1}{2} \lambda - \mu, \frac{1}{2}, 1\right)(\frac{1}{4} + \frac{1}{2} \lambda - \mu, \frac{1}{2}, 1)(-\frac{1}{2} - k, 1)\right]$$

Now substituting the value of $\Phi\left(\sqrt{(\alpha+\beta t)(\alpha+\beta)} / t\right)$ and $\Psi(\lambda, \alpha, \beta)$ in (2.3), we get,
\[
\int_0^\infty t^{(\lambda - \mu - \frac{3}{2})} \left(\frac{\alpha + \beta t}{\alpha t + \beta}\right)^{\frac{1}{2}} \left(\frac{\lambda + \mu + 1}{2}\right)^{\frac{1}{2}} \left(\frac{\alpha + \beta t}{\alpha t + \beta}\right)^{-\frac{1}{2}} \sqrt{\frac{1}{(\alpha + \beta t)(\alpha t + \beta)}}
\]

\[
\int_0^\infty t^{(\lambda - \mu - \frac{3}{2})} \left(\frac{\alpha + \beta t}{\alpha t + \beta}\right)^{\frac{1}{2}} \left(\frac{\lambda + \mu + 1}{2}\right)^{\frac{1}{2}} \left(\frac{\alpha + \beta t}{\alpha t + \beta}\right)^{-\frac{1}{2}} \sqrt{\frac{1}{(\alpha + \beta t)(\alpha t + \beta)}}
\]

\[
H_{M,N+2}^{P+2,Q} \left[(2\rho) \left(\frac{1}{4} + \frac{1}{2} \left(\frac{\lambda + \mu + 1}{2}\right)^{\frac{1}{2}} \left(\frac{\alpha + \beta t}{\alpha t + \beta}\right)^{-\frac{1}{2}} \sqrt{\frac{1}{(\alpha + \beta t)(\alpha t + \beta)}} \right) \right] dt
\]

\[
= \sum_{k=0}^\infty \frac{1}{k} \left(1 - \frac{\beta^2}{\alpha^2}\right)^k H_{M,N+6}^{P+6,Q+3} \left[2 \rho \alpha \left|u \right|^v\right],
\]

Where \(u\) and \(v\) are mentioned in (2.6).

Provided \(\text{Re}(\alpha) > 0\), \(\text{Re}(\beta) > 0\), integrals involved are absolutely convergent. Where \(t\) is a non zero complex variable, \(L\) is Mellin Barnes type contour integral. The integral converges if,

\[
|\text{arg.}(pt)| < \frac{1}{2}, \pi, p \neq 0, p \text{ are mentioned in (1.4) and (1.6)}.
\]

### III. APPLICATIONS

(i) When \(\epsilon_j = \tau_j = 1\) (unity) reduce to Fox’s H-Function, [3].

\[
\int_0^\infty t^{(\lambda - \mu - \frac{3}{2})} \left(\frac{\alpha + \beta t}{\alpha t + \beta}\right)^{\frac{1}{2}} \left(\frac{\lambda + \mu + 1}{2}\right)^{\frac{1}{2}} \left(\frac{\alpha + \beta t}{\alpha t + \beta}\right)^{-\frac{1}{2}} \sqrt{\frac{1}{(\alpha + \beta t)(\alpha t + \beta)}}
\]

\[
H_{M,N+2}^{P+2,Q} \left[2 \rho \sqrt{\frac{1}{(\alpha + \beta t)(\alpha t + \beta)}} \right] dt
\]

\[
= \sum_{k=0}^\infty \frac{1}{k} \left(1 - \frac{\beta^2}{\alpha^2}\right)^k H_{M,N+6}^{P+6,Q+3} \left[2 \rho \alpha \left|u \right|^v\right],
\]

\[
2 \rho \alpha \left|u \right|^v \left((\frac{1}{4} - \frac{\lambda}{2} - \frac{\mu}{2} - \frac{k}{2} - \frac{1}{2}, \frac{1}{4} + \frac{\lambda}{2} - \frac{\mu}{2} - \frac{k}{2} - \frac{1}{2}), \frac{1}{4} + \frac{\lambda}{2} - \frac{\mu}{2} - \frac{k}{2} - \frac{1}{2}, \frac{1}{4} + \frac{\lambda}{2} - \frac{\mu}{2} - \frac{k}{2} - \frac{1}{2}\right]
\]

\[
(3.1)
\]
(iii) \( E_j = F_j = 1 \) and \( c_j = \tau_j = 1 \) for all value of \( j \) then (2.6) reduces to \( G \) – function,

\[
\int_0^{\infty} t^\left(\frac{1}{2}(\lambda-\mu-\frac{1}{2})\right) \left(\frac{\alpha + \beta t}{\alpha t + \beta}\right)^\frac{1}{2} (\alpha + \beta t)^{-\frac{1}{2}} \sqrt{\frac{1}{(\alpha + \beta t)(\alpha t + \beta)}}
\]

\[G_{M,N+2}^{P+2,Q} \left[ 2^{\rho \lambda/2} \sqrt{\frac{1}{(\alpha + \beta t)(\alpha t + \beta)}} \left( \frac{1}{4} + \frac{1}{2} \lambda + \frac{1}{2} \mu, \frac{1}{2}, \frac{1}{4} - \frac{\lambda}{2} - \frac{\mu}{2}, \frac{1}{2} \right) [e_j] [(f_j)] \right] dt
\]

\[= \sum_{k=0}^{\infty} \frac{1}{k} \frac{\alpha^2 \beta^2}{\alpha^2} \left( 1 - \frac{\beta^2}{\alpha^2} \right) G_{M,N+6}^{P+6,Q+3} \left[ 2^{\rho \lambda} \frac{u}{[(e_j)]_{1,p}}, [v] [f_j] \right] , \quad (3.2)
\]

Where \( U \) and \( V \) mentioned in (3.1)

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