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Fixed Point and Common Fixed Point Theorems in Compact Metric Spaces and Pseudo Compact Tichnov Spaces for Self Mappings

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ABSTRACT : In the present paper we establish some fixed point theorems in compact spaces and pseudo compact tichnov spaces. Our result are more general than Bhardwaj et.al. and Edelstein.

Keywords : Fixed point, Compact spaces, Pseudo Compact Tichnov Spaces, self mappings

I. INTRODUCTION AND PRELIMINARY

There are several genralizations of classical contraction mapping theroem of Banach [1]. In 1961 Edelstein [4] established the existence of a unique fixed point of a self map T of a compact metric space satisfying the inequality $d(T(x), T(y)) < d(x, y)$ which is genralization of Banach. In the past few years a number of authers such as Fisher [5], Soni [11,12] have stablished a number of intresting results on compact metric spaces. More recently Fisher and Namdeo [6], Popa and Telci [10], Sahu [13] described some valuble results in compact metric spaces.

Jain and Dixit [7], Pathak [9], Khan, S. and Sharma [8] worked on pseudo-compact Tichonov spaces. Recently Bhardwaj *et.al.* [2,3] also worked for thease spaces.

On motivated by Bhardwaj et.al. [2, 3] and Edelstein [4] we diveded this paper in two sections.

1. Some fixed point theorems in compact metric spaces for self mappings and
2. Some fixed point and common fixed point theorems in pseudo compact tichonov spaces for self mappings.

Before starting main result we write some definitions.

Definition 1.1: A class $\{Gi\}$ of open subset of X is said to be an open cover of X, if each point in X belongs to one Gi that is $Ui\ Gi = X$.*A* subclass of an open cover which is at least an open cover is called a sub cover. A compact space is that space in which every open cover has finite sub cover

Definition 1.2: (Pseudo-compact Tichnov spaces) A Topological space X is said to be Pseudo compact space, if every real valued continuous function on X is bounded. It may be noted that every compact space is pseudo compact but converse is not necessarily true. However, in a metric space notation 'compact' and 'pseudo compact' coincide. By Tichonov space we mean a completely regular Hausdroff space.

II. MAIN RESULTS

Theorem 2.1: Let T be a continuous mapping of a compact metric space X into itself satisfying the condition

$$d(Tx,Ty) < a.\left[\frac{d(x,Tx)d(y,Ty)d(x,Ty)+d(x,y)d(y,Tx)d(y,Ty)}{[d(x,y)]^2+d(x,Ty)d(y,Ty)}\right] + \beta[d(y,Tx)+d(y,Ty)]+\delta[d(x,y)] \quad ...(2.1.1)$$

For all $x, y \, \varepsilon \, X, x \neq y$ where a , β, δ are non negative real's such that $a + \beta + \delta < 1$ then T has a unique fixed point.

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Proof: First we define a function F on X such that

$$F(x) = d(x, Tx), \text{ for all } x \, \varepsilon \, X$$

Since d and T are continuous on X therefore F is also. From the compactness of X, there exist a point $P \, \varepsilon \, X$ such that

$$F(p) = \inf\{F(x) : x \in X\}$$

If $F(p) \neq 0$, it follows that $T(p) \neq p$ then

$$F(Tp) = d(Tp, TTp)$$

$$< a.\left[\frac{d(p,Tp)d(Tp,TTp)d(p,TTp)+d(p,Tp)d(Tp,Tp)d(p,TTp)}{d(p,Tp)^2+d(p,TTp)d(Tp,TTp)}\right] + \beta[d(Tp,Tp)+d(Tp,TTp)]+\delta[d(p,Tp)]$$

$$< a\left[\frac{d(p,Tp)d(Tp,TTp)d(p,TTp)}{d(p,Tp)^2+d(p,TTp)d(Tp,TTp)}\right] +\beta\left[d(Tp,TTp)\right]+\delta\left[d(p,Tp)\right]$$

Since $\dfrac{d(p,Tp)d(Tp,TTp)d(p,TTp)}{d(p,Tp)^2+d(p,TTp)d(Tp,TTp)} < \dfrac{d(p,Tp)d(Tp,TTp)d(p,TTp)}{d(p,TTp)d(Tp,TTp)}$ we have

$$< a.\frac{d(p,Tp)d(Tp,TTp)d(p,TTp)}{d(p,TTp)d(Tp,TTp)}+\beta\left[d(Tp,TTp)\right]+\delta\left[d(p,Tp)\right] < a.d(p,Tp)+\beta\left[d(Tp,TTp)\right]+\delta\left[d(p,Tp)\right]$$

$\Rightarrow$ $$(1-\beta)\left[d(Tp,TTp)\right] < (a.+\delta)\left[d(p,Tp)\right]$$

$\Rightarrow$ $$d(Tp,TTp) < \frac{(a.+\delta)}{(1-\beta)}d(p,Tp)$$

Since $\alpha + \beta + \delta < 1$

$\Rightarrow$ $d(Tp, TTp) < d(p, Tp)$

$\Rightarrow$ $F(Tp) < F(p)$

Which is contradiction to (b), so $F(Tp) = F(p)$ Therefore p is fixed point of T.

Uniqueness: If possible suppose $q \neq p$ is another fixed point of T then

$$d(p,q) = d(Tp,Tq)$$

$$< a\frac{d(p,Tp)d(q,Tq)d(p,Tq)+d(p,q)d(q,Tp)d(q,Tq)}{\left[d(p,q)\right]^2+d(p,Tq)d(q,Tq)} +\beta\left[d(q,Tp)+d(q,Tq)\right]+\delta.d(p,q)$$

$$< a\frac{d(p,p)d(q,q)d(p,q)+d(p,q)d(q,p)d(q,q)}{\left[d(p,q)\right]^2+d(p,q)d(q,q)} +\beta\left[d(q,p)+d(q,q)\right]+\delta.d(p,q)$$

$$< \beta\left[d(q,p)\right]+\delta.d(p,q) < (\beta+\delta.)d(p,q)$$

i.e., $d(p,q) < (\beta+\delta.)d(p,q)$

Which is contradiction, since $a + \beta + \delta < 1$ implies $\beta + \delta < 1$. Thus p is unique fixed point of T. Now we shall prove the following theorem.

Corollary 2.2: Let T be a continuous mapping of a compact metric space X into itself satisfying the condition

$$d(Tx,Ty) < \alpha\left[\frac{d(x,Tx)d(y,Ty)d(x,Ty)+d(x,y)d(y,Tx)d(y,Ty)}{\left[d(x,y)\right]^2+d(x,Ty)d(y,Ty)}\right] +\beta\left[d(y,Tx)+d(y,Ty)\right]+\delta\left[d(x,y)\right]$$
$$+\eta\left[d(x,Tx)+d(x,Ty)\right]+\mu\frac{d(x,Tx)d(y,Ty)}{d(x,y)} \tag{2.2.1}$$

For all x, y ε X, $x \neq$ y where α , β, γ, η, μ, are non negative real's such that $\alpha + \beta + 3\eta + \mu < 1$ then T has a unique fixed point.

Proof: First we define a function F on X such that

$F(x) = d(x, Tx)$, for all x ε X

Since d and T are continuous on X therefore F is also. From the compactness of X, there exist a point $P \in X$ such that

$F(p) = \inf\{F(x) : x \in X)$

If $F(p) \neq 0$, it follows that $T(p) \neq$ p then

$$F(Tp) = d(Tp,TTp)$$

$$< a.\left[\frac{d(p,Tp)d(Tp,TTp)d(p,TTp)+d(p,Tp)d(Tp,Tp)d(p,TTp)}{d(p,Tp)^2+d(p,TTp)d(Tp,TTp)}\right] \begin{array}{l}+\beta\left[d(Tp,Tp)+d(Tp,TTp)\right]+\delta\left[d(p,Tp)\right]\\ +\eta\left[d(p,Tp)+d(p,TTp)\right]\end{array}$$

$$\mu\frac{d(p,Tp)d(Tp,TTp)}{d(p,Tp)} < a\left[\frac{d(p,Tp)d(Tp,TTp)d(p,TTp)}{d(p,Tp)^2+d(p,TTp)d(Tp,TTp)}\right] \begin{array}{l}+\beta\left[d(Tp,TTp)\right]+\delta\left[d(p,Tp)\right]\\ +\eta\left[d(p,Tp)+d(p,TTp)\right]+\mu d(Tp,TTp)\end{array}$$

Since $\dfrac{d(p,Tp)d(Tp,TTp)d(p,TTp)}{d(p,Tp)^2+d(p,TTp)d(Tp,TTp)} < \dfrac{d(p,Tp)d(Tp,TTp)d(p,TTp)}{d(p,TTp)d(Tp,TTp)}$ we have

$$< a.\frac{d(p,Tp)d(Tp,TTp)d(p,TTp)}{d(p,TTp)d(Tp,TTp)} + \beta\left[d(Tp,TTp)\right] + \delta\left[d(p,Tp)\right]$$

$$+\eta\left[d(p,Tp) + d(p,Tp) + d(Tp,TTp)\right] + \mu d(Tp,TTp)$$

$$< a.d(p,Tp) + \beta\left[d(Tp,TTp)\right] + \delta\left[d(p,Tp)\right]$$

$$+\eta\left[d(p,Tp) + d(p,Tp) + d(Tp,TTp)\right] + \mu d(Tp,TTp)$$

$$\Rightarrow \quad \left[d(Tp,TTp)\right] < (a. + \delta + 2\eta)\left[d(p,Tp)\right] + (\eta + \beta + \mu)d(Tp,TTp)$$

$$\Rightarrow \quad (1 - (\eta + \beta + \mu)\left[d(Tp,TTp)\right] < (\alpha + \delta + 2\eta)\left[d(p,Tp)\right]$$

$$\Rightarrow \quad \left[d(Tp,TTp)\right] < \frac{(a. + \delta + e + e)}{1 - e - \beta - f}\left[d(p,Tp)\right]$$

Since $\alpha + \beta + \delta + 3\eta + \mu < 1$

$$\Rightarrow \quad d(Tp,TTp) < d(p,Tp)$$

$$\Rightarrow \quad F(Tp) < F(p)$$

Which is contradiction to (b) so $F(Tp) = F(p)$. Therefore p is fixed point of T.

Uniqueness: If possible suppose $q \neq p$ is another fixed point of T then $d(p, q) = d(Tp, Tq)$

$$< \alpha\frac{d(p,Tp)d(q,Tq)d(p,Tq) + d(p,q)d(q,Tp)d(q,Tq)}{\left[d(p,Tq)\right]^2 + d(p,q)d(q,Tq)}$$

$$+\beta[d(q, Tp) + d(q, Tq)] + \delta.d(p, q)$$

$$+e\left[d(p,Tp) + d(p,Tq)\right] + f\,\frac{d(p,Tp)d(q,Tq)}{d(p,q)}$$

$$< \alpha\frac{d(p,p)d(q,q)d(p,q) + d(p,q)d(q,p)d(q,q)}{\left[d(p,q)\right]^2 + d(p,q)d(q,q)}$$

$$+\beta\left[d(q,p) + d(q,q)\right] + \delta.d(p,q)$$

$$+\eta\left[d(p,p) + d(p,q)\right] + \mu\frac{d(p,p)d(q,q)}{d(p,q)}$$

$$< \beta\left[d(q,p)\right] + \delta.d(p,q) + \eta d(p,q)$$

$$< (\beta + \delta + \eta)d(p,q)$$

i.e. $\quad d(p,q) < (\beta + \delta + e)d(p,q)$

Which is contradiction, since $\alpha + \beta + \delta + 3\eta + \mu < 1$ implies $\beta + \delta + \eta < 1$. Thus p is unique fixed point of T.

III. MAIN RESULTS

Theorem 3.1. Let p be pseudo compact Tichnov space and d be a non negative real valued continuous function over $p \times p$ ($p \times p$ is Tichnov space but not pseudo compact) satisfying

$$d(x, x) = 0, \text{ for all } x \in p \tag{3.1.1}$$

and $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in p$

Let $T : p \to p$ be continuous map satisfying

$$d(Tx,Ty) \prec \alpha\left[\frac{d(x,Tx)d(y,Ty)d(x,Ty) + d(x,y)d(y,Tx)d(y,Ty)}{\left[d(x,y)\right]^2 + d(x,Ty)d(y,Ty)}\right] + \beta\left[d(y,Tx) + d(y,Ty)\right] + \delta d(x,y) \tag{3.1.2}$$

For all distinct x, y in p with $2\alpha + 3\beta + \delta \prec 1$, then T has a fixed point in p, which is unique.

Proof: we define $\varphi : P \to R$ by

$$\varphi(p) = d(Tp, p), \text{ for all } p \in P$$

Where R is set of real numbers, clearly φ is continuous, being the composite of two functions T and d. Since p is pseudo compact Tichnov space, every real valued continuous function over p is bounded and attains its bounds. Thus there exist a point say $v \in P$ such that $\varphi(v) = \inf\ \{\varphi(p) : p \in P\}$, where ''inf'' denotes the infimum or the greatest lower bound in *R*. It may be noted that $\varphi(p) \subset R$. We now affirm that *v* is a fixed point for *T*. If not, let us suppose that $Tv \neq v$, then using, we have

$$\varphi(Tv) = d(T^2 v, Tv)$$
$$= d(TTv, Tv)$$
$$< \alpha\left[\frac{d(Tv,TTv)d(v,Tv)d(Tv,Tv)+d(Tv,v)d(v,TTv)d(v,Tv)}{[d(Tv,v)]^2+d(Tv,Tv)d(v,Tv)}\right]$$
$$+\beta[d(v,TTv)+d(v,Tv)]+\delta d(Tv,v)$$
$$< \alpha\left[\frac{d(Tv,v)d(v,TTv)d(v,Tv)}{d(Tv,v)d(Tv,v)}\right]$$
$$+\beta[d(v,Tv)+d(Tv,TTv)+d(v,Tv)]+\delta d(Tv,v)$$
$$< \alpha[d(v,TTv)]+\beta[d(v,Tv)+d(Tv,TTv)+d(v,Tv)]+\delta d(Tv,v)$$
$$< \alpha[d(v,Tv)+d(Tv,TTv)]+\beta[d(v,Tv)+d(Tv,TTv)+d(v,Tv)]+\delta d(Tv,v)$$
$$< (\alpha+2\beta+\delta)d(v,Tv)+(\alpha+\beta)d(Tv,TTv)$$
$$\Rightarrow \quad (1-\alpha-\beta)d(Tv,TTv) < (\alpha+2\beta+\delta)d(v,Tv)$$
$$\Rightarrow \quad d(Tv,TTv) < \frac{(\alpha+2\beta+\delta)}{(1-\alpha-\beta)}d(v,Tv)$$

Since $2\alpha + 3\beta + \delta \prec 1$

$\Rightarrow \quad d(Tv, TTv) < d(v, Tv)$

$\Rightarrow \quad \varphi(Tv) < \varphi(v)$

Which is contradiction and therefore $Tv = v$, *i.e.* *v* in *p* is fixed point for *T*.

To prove uniqueness of *v*, if possible, let $w \in P$ is another fixed point for *T*. *i.e.* $Tw = w$ and $w \neq v$, using (3.1.2) we have

$$\varphi(v,w) = d(Tv,Tw)$$
$$< \alpha\left[\frac{d(v,Tv)d(w,Tw)d(v,Tw)+d(v,w)d(w,Tv)d(w,Tw)}{[d(v,w)]^2+d(v,Tw)d(w,Tw)}\right] +\beta[d(w,Tv)+d(w,Tw)]+\delta d(v,w)$$
$$< \alpha\left[\frac{d(v,v)d(w,w)d(v,w)+d(v,w)d(w,v)d(w,w)}{[d(v,w)]^2+d(v,w)d(w,w)}\right] +\beta[d(w,v)+d(w,w)]+\delta d(v,w) < (\beta+\delta)d(v,w)$$

i.e., $d(v,w) < (\beta+\delta)d(v,w)$

$\Rightarrow d(v,w) < d(v,w)$, again leading to a contradiction, Hence $v \in P$ is unique for *T* in *P*. This completes the proof of theorem (7.6).

Theorem 3.2: Let *p* be pseudo compact Tichnov-space and d be a non negative real valued continuous function over $p \times p$ ($p \times p$ is Tichnov space but not pseudo-compact) satisfying.

If *S* and *T* are two continuous self mappings of *p* satisfying (3.2.1)

$$ST = TS$$

$$d(STx,Sy) \prec \alpha\left[\frac{d(Tx,STx)d(y,Sy)d(Tx,Sy)+d(Tx,y)d(y,STx)d(y,Sy)}{[d(Tx,y)]^2+d(Tx,Sy)d(y,Sy)}\right] +\beta[d(y,STx)+d(y,Sy)]+\delta d(Tx,y) \quad (3.2.2)$$

For all distinct *x*, *y* in *p* with $2\alpha + 3\beta + \delta \prec 1$, then *S* and *T* have a unique common fixed point.

Proof: We define $\varphi : P \to R$ by

$\varphi(p) = d(STp, Tp)$, for all $p \in P$

Where R is set of real numbers clearly φ is continuous, being the composite of two functions S, T and d. since p is pseudo compact Tichnov space, every real valued continuous function over p is bounded and attains its bounds. Thus there exist a point say $v \in P$ such that

$$\varphi(v) = \inf\{ \varphi(p) : p \in P\}$$

Where "inf" denotes the infimum or the greatest lower bound in R. It may be noted that $\varphi(p) \subset R$. We now affirm that v is a fixed point for S. If not, let us suppose that $Sv \neq v$, then we have

$$\varphi(Sv) = d(STSv, TSv) = d(STSv, STv)$$

$$< \alpha\left[\frac{d(TSv, STSv)d(Tv, ST)d(TSv, STv) + d(TSv, Tv)d(Tv, STSv)d(Tv, STv)}{\left[d(TSv, Tv)\right]^2 + d(TSv, STv)d(Tv, STv)}\right]$$

$$+\beta\left[d(Tv, STSv) + d(Tv, STv)\right] + \delta d(TSv, Tv)$$

$$< \alpha\left[\frac{d(TSv, Tv)d(Tv, STSv)d(Tv, STv)}{\left[d(TSv, Tv)\right]\left[d(TSv, Tv)\right]}\right]$$

$$+\beta\left[d(Tv, STSv) + d(Tv, STv)\right] + \delta d(TSv, Tv)$$

$$< \alpha\left[d(Tv, STSv)\right]$$

$$+\beta\left[d(Tv, STv) + d(STv, STSv) + d(Tv, STv)\right] + \delta d(TSv, Tv)$$

$$< \alpha\left[d(Tv, STv) + d(STv, STSv)\right]$$

$$+\beta.d(Tv, STv) + \beta.d(STv, STSv) + \beta.d(Tv, STv) + \delta.d(TSv, Tv)$$

$$\Rightarrow \quad (1 - \alpha - \beta)d(STSv, STv) < (\alpha + 2\beta + \delta)d(Tv, STv)$$

$$\Rightarrow \quad d(STSv, STv) < \frac{(\alpha + 2\beta + \delta)}{(1 - \alpha - \beta)}d(Tv, STv)$$

$$\Rightarrow \quad d(STSv, STv) < d(Tv, STv)$$

$$\Rightarrow \quad \varphi(Sv) < \varphi(v)$$

Which is contradiction because $d\,(STSv, STv) \geq 0$. Hence $v \in p$ is fixed point for S.

ie. $S(v) = v$

$ST(v) = TS\,(v) = T(v)$

Now we shall prove that $T(v) = v$. If possible let $Tv \neq v$, and then we have

$$d(Tv, v) = d(STv, Sv)$$

$$< \alpha\left[\frac{d(Tv, STv)d(v, Sv)d(Tv, Sv) + d(Tv, v)d(v, STv)d(v, Sv)}{\left[d(Tv, v)\right]^2 + d(Tv, Sv)d(v, Sv)}\right]$$

$$+\beta\left[d(v, STv) + d(v, Sv)\right] + \delta.d(Tv, v)$$

$$< \alpha\left[\frac{d(Tv, Tv)d(v, v)d(Tv, v) + d(Tv, v)d(v, Tv)d(v, v)}{\left[d(Tv, v)\right]^2 + d(Tv, v)d(v, v)}\right]$$

$$+\beta\left[d(v, Tv) + d(v, v)\right] + \delta.d(Tv, v)$$

$$< (\beta + \delta)d(v, Tv)$$

$$\Rightarrow \quad d(v, Tv) < (\beta + \delta)d(v, Tv)$$

$$\Rightarrow \quad d(v, Tv) < d(v, Tv)$$

Which is contradiction. Hence $v \in P$ is unique for T, *i.e.* $Tv = v$.

To prove the uniqueness of v, If possible let w be another fixed point of S and T

I.e. $T(v) = S(v) = v$ and $T(w) = S(w) = w$ and $w \neq v$ then we have

$$
\begin{aligned}
d(v,w) &= d(STv,Sw) \\
&< \alpha\left[\frac{d(Tv,STv)d(w,Sw)d(Tv,Sw)+d(Tv,w)d(w,STv)d(w,Sw)}{[d(Tv,w)]^2+d(Tv,Sw)d(w,Sw)}\right] \\
&\quad +\beta[d(w,STv)+d(w,Sw)]+\delta.d(Tv,w) \\
&< \alpha\left[\frac{d(Tv,Sv)d(w,w)d(Tv,Sw)+d(v,w)d(w,Sv)d(w,w)}{[d(v,w)]^2+d(v,w)d(w,w)}\right] \\
&\quad +\beta[d(w,Sv)+d(w,w)]+\delta.d(v,w) \\
&< (\beta+\delta)d(w,v)
\end{aligned}
$$

$\Rightarrow \quad d(w,v) < (\beta+\delta)d(w,v)$

$\Rightarrow \quad d(w,v) < d(w,v)$

$\Rightarrow$ Which is contradiction. which proves that is unique. This completes the proof of theorem.

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