A Fixed Point Theorem In Complete Fuzzy 3-Metric Space Through Rational Expression

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ABSTRACT: Fuzzy metric space have introduced in many ways. We find some fixed point theorem in complete fuzzy 3-metric space through rational expression. Our paper is generalization form of Binayak Choudhary and Krishnapada Das [1] for Fuzzy 3-metric space motivated by Sushil Sharma [10].

I. INTRODUCTION

Fuzzy metric space have been introduced in many ways amongst specially to mention, fuzzy metric spaces were introduced by Kramosil and Michalek [7]. In this paper we use the concept of fuzzy metric space introduced by Kramosil and Michalek [7] and modified by George and Veeramani [5] to obtain Hausdorff topology for this kind of fuzzy metric space. Recently, Gregori and Sepena (2002) [6] extended Banach fixed point theorem to Fuzzy contraction mappings on complete fuzzy metric space in the sense of George and Veeramani [5].

In order to introduced a Hausdroff topology on the fuzzy metric space, in (Kramosil and Michalek 1975) the following definition was introduced.

Definition 2.3 : (George and Veeramani 1994) The 3-tuple (X, M, *) is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on X^2 \times [0,\infty) satisfying the following conditions:

(i) M(x, y, 0) = 0
(ii) M(x, y, t) = 1 for all t > 0 iff x = y,
(iii) M(x, y, t) = M(y, x, t),
(iv) M(x, y, t) * M(y, z, s) \leq M(x, z, t + s),
(v) M(x, y) : [0,\infty[ \rightarrow [0,1] is left-continuous, where x, y, z \in X and t, s > 0.

In order to introduced a Hausdroff topology on the fuzzy metric space, in (Kramosil and Michalek 1975) the following definition was introduced.

Definition 2.4 : (George and Veeramani 1994) In a metric space (X,d) the 3-tuple (X, Md, *) where Md(x, y, t) = t / (t + d(x, y)) and a*b = ab is a fuzzy metric space. This Md is called the standard fuzzy metric space induced by d.

Definition 2.5 : A binary operation * : [0,1] x [0,1] x [0,1]→[0,1] is called a continuous t-norm if (0,1], *) is an abelian topological monoid with unit 1 such that a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2 whenever a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2 for all a_1, a_2, b_1, b_2 and c_1, c_2 are in [0,1].
Definition 2.6: The 3-tuple $(X, M, \ast)$ is called a fuzzy 2-metric space if $X$ is an arbitrary set, $\ast$ is a continuous $t$-norm and $M$ is a fuzzy set in $X^3 \times [0,1)$ satisfying the following conditions for all $x, y, z, u \in X$ and $t_1, t_2, t_3 > 0$.

(FM' -1) $M(x, y, z, 0) = 0$, 
(FM' -2) $M(x, y, z, t) = 1$, $t > 0$ and when at least two of the three points are equal,
(FM' -3) $M(x, y, z, t) = M(x, z, y, t) = M(y, x, z, t)$,
(Symmetry about three variables)
(FM' -4) $M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, u, t_1) \ast M(x, u, z, t_2) \ast M(u, y, z, t_3)$
(This corresponds to tetrahedron inequality in 2-metric space)

The function value $M(x, y, z, t)$ may be interpreted as the probability that the area of triangle is less than $t$.

(FM' -5) $M(x, y, z, \cdot) : [0,1) \rightarrow [0,1]$ is left continuous.

Definition 2.7: Let $(X, M, \ast)$ be a fuzzy 2-metric space:

1. A sequence $\{x_n\}$ in fuzzy 2-metric space $X$ is said to be convergent to a point $x \in X$, if
   \[
   \lim_{n \to \infty} M(x_n, x, a, t) = 1
   \]
   for all $a \in X$ and $t > 0$.
2. A sequence $\{x_n\}$ in fuzzy 2-metric space $X$ is called a Cauchy sequence, if
   \[
   \lim_{n \to \infty} M(x_n + p, x_m, a, t) = 1
   \]
   for all $a \in X$ and $t > 0$, $p > 0$.
3. A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.8: A function $M$ is continuous in fuzzy 2-metric space iff whenever $x_n \to x$, $y_n \to y$, then
\[
\lim_{n \to \infty} M(x_n, y_n, a, t) = M(x, y, a, t)
\]
for all $a \in X$ and $t > 0$.

Definition 2.9: Two mappings $A$ and $S$ on fuzzy 2-metric space $X$ are weakly commuting iff
\[
M(ASu, SAu, a, t) \geq M(Au, Su, a, t)
\]
for all $u, a \in X$ and $t > 0$.

Definition 2.10: A binary operation $\ast : [0,1]^4 \to [0,1]$ is called a continuous $t$-norm if $([0,1], \ast)$ is an abelian topological monoid with unit 1 such that $a_1 \ast b_1 \ast c_1 \ast d_1 \leq a_2 \ast b_2 \ast c_2 \ast d_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$ and $d_1 \leq d_2$ for all $a_1, a_2, b_1, b_2, c_1, c_2$ and $d_1, d_2$ are in $[0,1]$.

Definition 2.11: The 3-tuple $(X, M, \ast)$ is called a fuzzy 3-metric space if $X$ is an arbitrary set, $\ast$ is a continuous $t$-norm and $M$ is a fuzzy set in $X^3 \times [0,1)$ satisfying the following conditions: for all $x, y, z, w, u \in X$ and $t_1, t_2, t_3, t_4 > 0$.

(FM'' -1) $M(x, y, z, w, 0) = 0$,
(FM'' -2) $M(x, y, z, w, t) = 1$ for all $t > 0$,
(FM'' -3) $M(x, y, z, w, t) = M(w, x, y, z, t)$,
(FM'' -4) $M(x, y, z, w, t_1 + t_2 + t_3 + t_4) \geq M(x, y, z, w, t_1) \ast M(x, y, u, t_2) \ast M(u, y, z, t_3) \ast M(u, z, w, t_4)$
(FM'' -5) $M(x, y, z, w, \cdot) : [0,1) \rightarrow [0,1]$ is left continuous.

Definition 2.12: Let $(X, M, \ast)$ be a fuzzy 3-metric space:

1. A sequence $\{x_n\}$ in fuzzy 3-metric space $X$ is said to be convergent to a point $x \in X$, if
   \[
   \lim_{n \to \infty} M(x_n, x, a, b, t) = 1
   \]
   for all $a, b \in X$ and $t > 0$.
2. A sequence $\{x_n\}$ in fuzzy 3-metric space $X$ is called a Cauchy sequence, if
   \[
   \lim_{n \to \infty} M(x_n + p, x_m, a, b, t) = 1
   \]
   for all $a, b \in X$ and $t > 0$, $p > 0$.
3. A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.13: A function $M$ is continuous in fuzzy 3-metric space iff whenever $x_n \to x$, $y_n \to y$
\[
\lim_{n \to \infty} M(x_n, y_n, a, b, t) = M(x, y, a, b, t)
\]
for all $a, b \in X$ and $t > 0$.

Definition 2.14: Two mappings $A$ and $S$ on fuzzy 3-metric space $X$ are weakly commuting iff
\[
M(ASu, SAu, a, b, t) \geq M(Au, Su, a, b, t)
\]
for all $u, a, b \in X$ and $t > 0$. 
Remark: Definitions and prepositions from Gregori and Sepena 2002 [6], Kumar and Chugh 2001 [8] are also used to prove our theorem.

III. MAIN RESULT

Theorem: Let $(X,M,\ast)$ be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy and $T$, $R$, and $S$ be mappings from $(X,M,\ast)$ into itself satisfying the following conditions:

$$T(X) \subseteq R(X) \text{ and } T(X) \subseteq S(X)$$

$$\frac{1}{M(T(x),T(y),a,b,c)} - 1 \leq k \left( \frac{1}{L(x,y,a,b,c)} - 1 \right)$$

with $0 < k < 1$ and

$$L(x, y, a, b, t) = \min \left\{ \frac{M(Rx,Sy,a,b,t)}{M(Rx,Tx,a,b,t)}, \frac{M(Sx, Ry,a,b,t)}{M(Sx,Tx,a,b,t)}, \frac{M(Rx,Ty,a,b,t)}{M(Ry,Ty,a,b,t)}, \frac{M(Sx,Tx,a,b,t)}{M(Sy,Tx,a,b,t)} \right\}$$

The pairs $T$, $S$ and $T$, $R$ are compatible. $R$, $T$, and $S$ are $\omega$-continuous. Then $R$, $T$, and $S$ have a unique common fixed point.

Proof: Let $x_0 \in X$ be an arbitrary point of $X$. Since $T(X) \subseteq R(X)$ and $T(X) \subseteq S(X)$, we can construct a sequence $\{x_n\}$ in $X$ such that

$$T(x_{n-1}) = R x_n = S x_n$$

Now,

$$L(x_n, x_{n+1}, a, b, t) = \min \left\{ \frac{M(Rx_{n+1},Sx_n,a,b,t)}{M(Rx_{n+1},Tx_n,a,b,t)}, \frac{M(Sx_{n+1},Rx_n,a,b,t)}{M(Sx_{n+1},Tx_n,a,b,t)}, \frac{M(Rx_{n+1},Tx_n,a,b,t)}{M(Rx_{n+1},Tx_{n+1},a,b,t)}, \frac{M(Sx_{n+1},Tx_n,a,b,t)}{M(Sx_{n+1},Tx_{n+1},a,b,t)} \right\}$$

$$= \min \left\{ \frac{M(Tx_{n-1},Tx_n,a,b,t)}{M(Tx_{n-1},Tx_{n+1},a,b,t)}, \frac{M(Tx_{n-1},Tx_n,a,b,t)}{M(Tx_{n-1},Tx_{n+1},a,b,t)}, \frac{M(Tx_{n-1},Tx_n,a,b,t)}{M(Tx_{n-1},Tx_{n+1},a,b,t)}, \frac{M(Tx_{n-1},Tx_n,a,b,t)}{M(Tx_{n-1},Tx_{n+1},a,b,t)} \right\}$$

We now claim that

$$L(x_n, x_{n+1}, a, b, t) = \frac{1}{M(Tx_{n-1},Tx_n,a,b,t)} - 1 \leq k \left( \frac{1}{M(Tx_{n-1},Tx_n,a,b,t)} - 1 \right)$$

which is a contradiction.

Hence,

$$\{T_n\} \text{ is a fuzzy contractive sequence in } (X,M,\ast). \text{ So } \{T_n\} \text{ is a Cauchy sequence in } (X,M,\ast).$$

As $X$ is a complete fuzzy metric space, $\{T_{n}\}$ is convergent. So, $\{T_{n}\}$ converges to some point $z$ in $X$.

$\therefore \{T_{n}\}, \{R_x\}, \{S_x\}$ converges to $z$. By $\omega$-continuity of $R$, $S$, and $T$, there exists a point $u$ in $X$ such that $x_n \to u$ as $n \to \infty$ and so $\lim R x_n = \lim S x_n = \lim T x_{n-1} = z$ implies

$$Ru = Su = Tu = z$$

Also by compatibility of pairs $T$, $S$ and $T$, $R$, and $Tu = Ru = Su = z$ implies

$$Tz = TRu = RTu = Rz \text{ and } Tz = TSu = STu = Sz$$
Therefore, \( Tz = Rz = Sz \).

We now claim that \( Tz = z \).

If not

\[
\frac{1}{M(Tx_{Tz}Tx_{Rz}a,b,t)} - 1 < \frac{1}{M(Tx_{Rz}a,b,t)} - 1
\]

\[
L(z,u,a,b,t) = \min \left\{ \frac{1}{M(Rz, Tu, a, b, t)} - 1, \frac{1}{M(Sz, Tu, a, b, t)} - 1, \frac{1}{M(Rz, Tu, a, b, t)} - 1, \frac{1}{M(Sz, Tu, a, b, t)} - 1 \right\}
\]

\[
= \min \left\{ \frac{1}{M(Tz, a, b, t)} - 1, \frac{1}{M(Tz, a, b, t)} - 1, \frac{1}{M(Tz, a, b, t)} - 1, \frac{1}{M(Tz, a, b, t)} - 1 \right\}
\]

\[
= \frac{1}{M(Tx_{Tz}Tx_{Rz}a,b,t)} - 1 < \frac{1}{M(Tx_{Rz}a,b,t)} - 1
\]

which is a contradiction.

Hence \( Tz = z \).

So \( z \) is a common fixed point of \( R, T \) and \( S \).

Now suppose \( v \neq z \) be another fixed point of \( R, T \) and \( S \).

\[
\frac{1}{M(Tx_{Tv}Tx_{Tv}a,b,t)} - 1 < \frac{1}{M(Tx_{Tv}a,b,t)} - 1
\]

\[
L(v,u,a,b,t) = \min \left\{ \frac{1}{M(Rv, Su, a, b, t)} - 1, \frac{1}{M(Sv, Su, a, b, t)} - 1, \frac{1}{M(Rv, Tv, a, b, t)} - 1, \frac{1}{M(Sv, Tv, a, b, t)} - 1 \right\}
\]

\[
= \min \left\{ \frac{1}{M(Tv, a, b, t)} - 1, \frac{1}{M(Sv, a, b, t)} - 1, \frac{1}{M(Tv, a, b, t)} - 1, \frac{1}{M(Sv, a, b, t)} - 1 \right\}
\]

\[
= \frac{1}{M(Tx_{Tv}Tx_{Tv}a,b,t)} - 1 < \frac{1}{M(Tx_{Tv}a,b,t)} - 1
\]

which is a contradiction. Hence \( v = z \).

Thus \( R, T \) and \( S \) have a unique common fixed point.

This completes the proof.
REFERENCES


