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Random Fixed Point Theorems in Polish Spaces

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ABSTRACT : The present paper deals with existence of fixed point and common fixed point theorems in polish space taking random operator. We take generalize contraction by taking rational expression.

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I. INTRODUCTION

Probabilistic functional analysis has emerged as one of the important mathematical disciplines in view of its role in analyzing probabilistic models in the applied sciences. The study of fixed points of random operators forms a central topic in this area. The Prague school of probabilistic initiated its study in the 1950. In recent years, the study of random fixed point have attracted much attention.

II. PRELIMINARIES

Definition 2.1 Let (X,d) be a Polish space, i.e. a separable complete metric space, and let (Ω, A) be a measurable space. Let 2^X be a family of all subsets of X and $CB(X)$ denote the family of all non-empty bounded closed subsets of X. A mapping $T : \Omega \to 2^X$ is called measurable if for all open subsets C of X,

$T^{-1}(C) = \{\omega \in \Omega : T(\omega) \cap C \neq \phi\} \in A.A$ A mapping $\xi : \Omega \to X$ is said to be measurable selector of a Measurable mapping $T : \Omega \to 2^X$ if ξ is measurable and $x(\omega) \in T(\omega)$ for all $\omega \in \Omega$. A mapping $f : \Omega \times X \to X$ is called a random operator if for all $x \in X$, $f(., x)$ is measurable.

A mapping $T : \Omega \times X \to CB(X)$ is called a random multi-valued operator if for each $x \in X$, $f(., x)$ is measurable.

A measurable mapping $T : \Omega \times X \to X$ is called a random fixed point of a random multi-valued operator $T : \Omega \times X \to CB(X)$ if for each.

$$\omega \in \Omega., \xi(\omega) \in T(\omega, \xi(\omega)).$$

Theorem 2.1 [3], [4]: Let T be an orbitally continuous mapping of a bounded complete 2-metric space X into itself. If T satisfies the condition:

$\min\{\rho(Tx, Ty, a), \rho(x, Tx, a), \rho(y, Ty, a) - \xi\rho(x, Ty, a), \rho(y, Tx, a)\} \leq q\rho(x, y, a)$ for all $x, y, a \in X$ and for some q with $0 < q < 1$, then for each $x \in X$, the sequence $\{T^n x\}$ $(n = 1, 2, 3.....)$ converges to a fixed point of T.

On taking this view of Theorem (2.1), we generalize this theorem for ran dom operator on Polish space.

III. MAIN RESULTS

Theorem 3.1: Let X be a Polish space. Let $T : \Omega \times X \to CB(X)$ be continuous random operator. If there exists measurable mappings $\alpha_1, \alpha_2, \alpha_3, \beta : \Omega \to (0, 1)$ such that

$$\alpha_1(\omega)\, p(T(\omega,x),T(\omega,y)) + \alpha_2(\omega)\, p(x,T(\omega,x)) + \alpha_3(\omega)\, p(y,T(\omega,y)) \leq \beta(\omega)\, p(x,y)$$
$$+\min\left\{ p(x,T(\omega,y)),\, p(y,T(\omega,x)),\, \frac{[p(x,T(\omega,y)) + p(y,T(\omega,x))]}{1+p(x,T(\omega,y)).p(y,T(\omega,x))} \right\} \quad ...(3.1.1)$$

for all $x, y \in X, \omega \in \Omega$ with $\alpha_1(\omega)+\alpha_2(\omega)+\alpha_3(\omega) > \beta(\omega)$,

Then the sequence $\{T^n x\}$ converges to fixed point of T.

Proof: Let $\xi_0 : \Omega \to X$ be an arbitrary measurable mapping and choose a measurable mapping $\xi_1 : \Omega \to X$ such that $\xi_1(\omega) \in T(\omega, \xi_0(\omega))$ for each. $\omega \in \Omega$. Then

$$\alpha_1(\omega)\,p\big(T(\omega,\xi_0(\omega)),T(\omega,\xi_1(\omega))\big)+\alpha_2(\omega)\,p\big(\xi_0(\omega),T(\omega,\xi_0(\omega))\big)+\alpha_3(\omega)\,p\big(\xi_1(\omega),T(\omega,\xi_1(\omega))\big)$$
$$\leq \beta(\omega)\,p\big(\xi_0(\omega),\xi_1(\omega)\big)+\min\left\{\begin{array}{l} p\big(\xi_0(\omega),T(\omega,\xi_1(\omega))\big),\,p\big(\xi_1(\omega),T(\omega,\xi_0(\omega))\big),\\ \dfrac{\big[p\big(\xi_0(\omega),T(\omega,\xi_1(\omega))\big)+p\big(\xi_1(\omega),T(\omega,\xi_0(\omega))\big)\big]}{1+p\big(\xi_0(\omega),T(\omega,\xi_1(\omega))\big).p\big(\xi_1(\omega),T(\omega,\xi_0(\omega))\big)}\end{array}\right\}$$

Further, there exists a measurable mapping $\xi_2 : \Omega \to X$ such that for all $\omega \in \Omega$, $\xi_2(\omega) \in T(\omega, \xi_1(\omega))$ and

$$\alpha_1(\omega)\,p\big(\xi_1(\omega),\xi_2(\omega)\big)+\alpha_2(\omega)\,p\big(\xi_0(\omega),\xi_1(\omega)\big)+\alpha_3(\omega)\,p\big(\xi_1(\omega),\xi_2(\omega)\big)$$
$$\leq \beta(\omega)\,p\big(\xi_0(\omega),\xi_1(\omega)\big)+\min\left\{\begin{array}{l} p\big(\xi_0(\omega),\xi_2(\omega)\big),\,p\big(\xi_1(\omega),\xi_1(\omega)\big),\\ \dfrac{\big[p\big(\xi_0(\omega),\xi_2(\omega)\big)+p\big(\xi_1(\omega),\xi_1(\omega)\big)\big]}{1+p\big(\xi_0(\omega),\xi_2(\omega)\big).p\big(\xi_1(\omega),\xi_1(\omega)\big)}\end{array}\right\}$$

$$\alpha_1(\omega)\,p\big(\xi_1(\omega),\xi_2(\omega)\big)+\alpha_2(\omega)\,p\big(\xi_0(\omega),\xi_1(\omega)\big)+\alpha_3(\omega)\,p\big(\xi_1(\omega),\xi_2(\omega)\big)$$
$$\leq \beta(\omega)\,p\big(\xi_0(\omega),\xi_1(\omega)\big)+\min\big\{p\big(\xi_0(\omega),\xi_2(\omega)\big),0,\,p\big(\xi_0(\omega),\xi_2(\omega)\big)\big\}$$

Then we get

$$\big[\alpha_1(\omega)+\alpha_3(\omega)\big]\,p\big(\xi_1(\omega),\xi_2(\omega)\big)\leq\big[\beta(\omega)-\alpha_2(\omega)\big]\,p\big(\xi_0(\omega),\xi_1(\omega)\big)$$

Thus $\quad p\big(\xi_1(\omega),\xi_2(\omega)\big)\leq\dfrac{\big[\beta(\omega)-\alpha_2(\omega)\big]}{\big[\alpha_1(\omega)+\alpha_3(\omega)\big]}\,p\big(\xi_0(\omega),\xi_1(\omega)\big)$

$$p\big(\xi_1(\omega),\xi_2(\omega)\big)\leq k\;\,p\big(\xi_0(\omega),\xi_1(\omega)\big)$$

Where. $k=\dfrac{\big[\beta(\omega)-\alpha_2(\omega)\big]}{\big[\alpha_1(\omega)+\alpha_3(\omega)\big]}<1$

Further, there exists a measurable mapping $\xi_3 : \Omega \to X$ such that for all $\omega \in \Omega$, $\xi_3(\omega) \in T(\omega, \xi_2(\omega))$ and

$$\alpha_1(\omega)\,p\big(T(\omega,\xi_1(\omega)),T(\omega,\xi_2(\omega))\big)+\alpha_2(\omega)\,p\big(\xi_1(\omega),T(\omega,\xi_1(\omega))\big)+\alpha_3(\omega)\,p\big(\xi_2(\omega),T(\omega,\xi_2(\omega))\big)$$
$$\leq \beta(\omega)\,p\big(\xi_1(\omega),\xi_2(\omega)\big)+\min\left\{\begin{array}{l} p\big(\xi_1(\omega),T(\omega,\xi_2(\omega))\big),\,p\big(\xi_2(\omega),T(\omega,\xi_1(\omega))\big),\\ \dfrac{\big[p\big(\xi_1(\omega),T(\omega,\xi_2(\omega))\big)+p\big(\xi_2(\omega),T(\omega,\xi_1(\omega))\big)\big]}{1+p\big(\xi_1(\omega),T(\omega,\xi_2(\omega))\big).p\big(\xi_2(\omega),T(\omega,\xi_1(\omega))\big)}\end{array}\right\}$$

Hence

$$\alpha_1(\omega)\,p\big(\xi_2(\omega),\xi_3(\omega)\big)+\alpha_2(\omega)\,p\big(\xi_1(\omega),\xi_2(\omega)\big)+\alpha_3(\omega)\,p\big(\xi_2(\omega),\xi_3(\omega)\big)$$
$$\leq \beta(\omega)\,p\big(\xi_1(\omega),\xi_2(\omega)\big)+\min\left\{\begin{array}{l} p\big(\xi_1(\omega),\xi_3(\omega)\big),\,p\big(\xi_2(\omega),\xi_2(\omega)\big),\\ \dfrac{\big[p\big(\xi_1(\omega),\xi_3(\omega)\big)+p\big(\xi_2(\omega),\xi_2(\omega)\big)\big]}{1+p\big(\xi_1(\omega),\xi_3(\omega)\big).p\big(\xi_2(\omega),\xi_2(\omega)\big)}\end{array}\right\}$$

Thus $\quad p\big(\xi_2(\omega),\xi_3(\omega)\big)\leq\dfrac{\big[\beta(\omega)-\alpha_2(\omega)\big]}{\big[\alpha_1(\omega)+\alpha_3(\omega)\big]}\,p\big(\xi_1(\omega),\xi_2(\omega)\big)$

$$p\big(\xi_2(\omega),\xi_3(\omega)\big)\leq k^2\;\,p\big(\xi_0(\omega),\xi_1(\omega)\big)$$

Similarly, proceeding in the same manner, by induction, we get a sequence of measurable mappings $\xi_n : \Omega \to X$ for $n > 0$ and for any $\omega \in \Omega$.

$$p\big(\xi_n(\omega),\xi_{n+1}(\omega)\big)\leq k\,p\big(\xi_{n-1}(\omega),\xi_n(\omega)\big)\leq ....\leq k^n\;\,p\big(\xi_0(\omega),\xi_1(\omega)\big)$$

Further, for $m > n$,

$$p(\xi_n(\omega),\xi_m(\omega)) \le p(\xi_n(\omega),\xi_{n+1}(\omega)) + ... + p(\xi_{m-1}(\omega),\xi_m(\omega))$$
$$\le (k^n + k^{n+1} + ... + k^{m-1})\, p(\xi_0(\omega),\xi_1(\omega))$$
$$\le \frac{k^n}{1-k}\, p(\xi_0(\omega),\xi_1(\omega))$$

Which tends to zero as $n \to \infty$. It follows that $\{\xi_n(\omega)\}$ is a Cauchy sequence. Which must converges to some point $\xi(\omega)$. Now, Let a measurable mapping $\xi : \Omega \to X$ such $\xi_n(\omega) \to \xi(\omega)$ that for each $\omega \in \Omega$.

It implies thatand. $\xi_{2n+1}(\omega) \to \xi(\omega)$ and $\xi_{2n+2}(\omega) \to \xi(\omega)$ Thus from (3.1.1), we have

$$\alpha_1(\omega)\, p(T(\omega,\xi(\omega)),T(\omega,\xi_{2n+1}(\omega))) + \alpha_2(\omega)\, p(\xi(\omega),T(\omega,\xi(\omega))) + \alpha_3(\omega)\, p(\xi_{2n+1}(\omega),T(\omega,\xi_{2n+1}(\omega)))$$
$$\le \beta(\omega)\, p(\xi(\omega),\xi_{2n+1}(\omega)) + \min\left\{\begin{array}{l} p(\xi(\omega),T(\omega,\xi_{2n+1}(\omega))),\ p(\xi_{2n+1}(\omega),T(\omega,\xi(\omega))),\\ \dfrac{\left[p(\xi(\omega),T(\omega,\xi_{2n+1}(\omega))) + p(\xi_{2n+1}(\omega),T(\omega,\xi(\omega)))\right]}{1+p(\xi(\omega),T(\omega,\xi_{2n+1}(\omega))).p(\xi_{2n+1}(\omega),T(\omega,\xi(\omega)))}\end{array}\right\}$$

Taking, $n \to \infty$., we have

$$\alpha_1(\omega)\, p(T(\omega,\xi(\omega)),\xi(\omega)) + \alpha_2(\omega)\, p(\xi(\omega),T(\omega,\xi(\omega))) + \alpha_3(\omega)\, p(\xi(\omega),\xi(\omega))$$
$$\le \beta(\omega)\, p(\xi(\omega),\xi(\omega)) + \min\left\{\begin{array}{l} p(\xi(\omega),\xi(\omega)),\ p(\xi(\omega),T(\omega,\xi(\omega))),\\ \dfrac{\left[p(\xi(\omega),\xi(\omega)) + p(\xi(\omega),T(\omega,\xi(\omega)))\right]}{1+p(\xi(\omega),\xi(\omega)).p(\xi(\omega),T(\omega,\xi(\omega)))}\end{array}\right\}$$

$\left[\alpha_1(\omega)+\alpha_2(\omega)\right] p(\xi(\omega),T(\omega,\xi(\omega))) \le 0$. Hence $\xi(\omega) \in T(\omega,\xi(\omega))$ for all $\omega \in \Omega$.

Therefore $\xi(\omega)$ is a fixed point of T.

Theorem 3.2: Let X be a Polish space. Let $T_1, T_2 : \Omega \times X \to CB(X)$ be two continuous random multi-valued operators. If there exists measurable mappings $\alpha_1, \alpha_2, \alpha_3, \beta : \Omega \to (0, 1)$ such that

$$\alpha_1(\omega)\, p(T_1(\omega,x),T_2(\omega,y)) + \alpha_2(\omega)\, p(x,T_1(\omega,x)) + \alpha_3(\omega)\, p(y,T_2(\omega,y)) \le \beta(\omega)\, p(x,y)$$
$$+\min\left\{p(x,T_2(\omega,y)),\ p(y,T_1(\omega,x)),\ \frac{\left[p(x,T_2(\omega,y)) + p(y,T_1(\omega,x))\right]}{1+p(x,T_2(\omega,y)).p(y,T_1(\omega,x))}\right\} \quad ...(3.2.1)$$

for all $x, y \in X, \omega \in \Omega$ with $\alpha_1(\omega)+\alpha_2(\omega)+\alpha_3(\omega) > \beta(\omega)$, Then T_1 and T_2 have a common fixed point .

Proof: Let $\xi_0 : \Omega \to X$ be an arbitrary measurable mapping and define a sequence of measurable mapping $\xi_0 : \Omega \to X$ such that for n > 0 and for any and.

$$\omega \in \Omega,\ \xi_{2n+1}(\omega) = T_1(\omega,\xi_{2n}(\omega)) \text{ and } \xi_{2n+2}(\omega) = T_2(\omega,\xi_{2n+1}(\omega))$$

Then from (3.2.1), we have

$$\alpha_1(\omega)\, p(T_1(\omega,\xi_{2n}(\omega)),T_2(\omega,\xi_{2n+1}(\omega))) + \alpha_2(\omega)\, p(\xi_{2n}(\omega),T_1(\omega,\xi_{2n}(\omega)))$$
$$+\alpha_3(\omega)\, p(\xi_{2n+1}(\omega),T_2(\omega,\xi_{2n+1}(\omega))) \le \beta(\omega)\, p(\xi_{2n}(\omega),\xi_{2n+1}(\omega))$$
$$+\min\left\{\begin{array}{l} p(\xi_{2n}(\omega),T_2(\omega,\xi_{2n+1}(\omega))),\ p(\xi_{2n+1}(\omega),T_1(\omega,\xi_{2n}(\omega))),\\ \dfrac{\left[p(\xi_{2n}(\omega),T_2(\omega,\xi_{2n+1}(\omega))) + p(\xi_{2n+1}(\omega),T_1(\omega,\xi_{2n}(\omega)))\right]}{1+p(\xi_{2n}(\omega),T_2(\omega,\xi_{2n+1}(\omega))).p(\xi_{2n+1}(\omega),T_1(\omega,\xi_{2n}(\omega)))}\end{array}\right\}$$

$$\alpha_1(\omega)\, p(\xi_{2n+1}(\omega),\xi_{2n+2}(\omega)) + \alpha_2(\omega)\, p(\xi_{2n}(\omega),\xi_{2n+1}(\omega)) + \alpha_3(\omega)\, p(\xi_{2n+1}(\omega),\xi_{2n+2}(\omega))$$
$$\le \beta(\omega)\, p(\xi_{2n}(\omega),\xi_{2n+1}(\omega)) + \min\left\{\begin{array}{l} p(\xi_{2n}(\omega),\xi_{2n+2}(\omega)),\ p(\xi_{2n+1}(\omega),\xi_{2n+1}(\omega)),\\ \dfrac{\left[p(\xi_{2n}(\omega),\xi_{2n+2}(\omega)) + p(\xi_{2n+1}(\omega),\xi_{2n+1}(\omega))\right]}{1+p(\xi_{2n}(\omega),\xi_{2n+2}(\omega)).p(\xi_{2n+1}(\omega),\xi_{2n+1}(\omega))}\end{array}\right\}$$

$$\left[\alpha_1(\omega)+\alpha_3(\omega)\right]p\left(\xi_{2n+1}(\omega),\xi_{2n+2}(\omega)\right)\le\left[\beta(\omega)-\alpha_2(\omega)\right]p\left(\xi_{2n}(\omega),\xi_{2n+1}(\omega)\right)$$

Thus $p\left(\xi_{2n+1}(\omega),\xi_{2n+2}(\omega)\right)\le k\,p\left(\xi_{2n}(\omega),\xi_{2n+1}(\omega)\right)$

where $k=\dfrac{\left[\beta(\omega)-\alpha_2(\omega)\right]}{\left[\alpha_1(\omega)+\alpha_3(\omega)\right]}<1.$

Then by the routine calculation we find that $p\left(\xi_n(\omega),\xi_{n+1}(\omega)\right)\le k^n\,p\left(\xi_0(\omega),\xi_1(\omega)\right)$

Which tends to zero as $n\to\infty$. It follows that $\{x_n(\omega)\}$ is a Cauchy sequence and there exists a measurable mapping $\xi:\Omega\to X$ such that $\xi_n(\omega)\to\xi(\omega)$ for each $\omega\in\Omega$. It implies that and $\xi_{2n+1}(\omega)\to\xi(\omega)$ and $\xi_{2n+2}(\omega)\to\xi(\omega)$. Thus we have for any $\omega\in\Omega$,

$$p\left(\xi(\omega),T_1\left(\omega,\xi(\omega)\right)\right)\le p\left(\xi(\omega),\xi_{2n+2}(\omega)\right)+p\left(\xi_{2n+2}(\omega),T_1\left(\omega,\xi(\omega)\right)\right) \quad ...(3.2.2)$$

Now using (3.2.1) we have

$$p\left(\xi_{2n+2}(\omega),T_1\left(\omega,\xi(\omega)\right)\right)=\alpha_1(\omega)\,p\left(T_1\left(\omega,\xi(\omega)\right),T_2\left(\omega,\xi_{2n+1}(\omega)\right)\right)+\alpha_2(\omega)\,p\left(\xi(\omega),T_1\left(\omega,\xi(\omega)\right)\right)$$

$$+\alpha_3(\omega)\,p\left(\xi_{2n+1}(\omega),T_2\left(\omega,\xi_{2n+1}(\omega)\right)\right)\le\beta(\omega)\,p\left(\xi(\omega),\xi_{2n+1}(\omega)\right)$$

$$+\min\left\{\begin{array}{l}p\left(\xi(\omega),T_2\left(\omega,\xi_{2n+1}(\omega)\right)\right),\,p\left(\xi_{2n+1}(\omega),T_1\left(\omega,\xi(\omega)\right)\right),\\ \dfrac{\left[p\left(\xi(\omega),T_2\left(\omega,\xi_{2n+1}(\omega)\right)\right)+p\left(\xi_{2n+1}(\omega),T_1\left(\omega,\xi(\omega)\right)\right)\right]}{1+p\left(\xi(\omega),T_2\left(\omega,\xi_{2n+1}(\omega)\right)\right).p\left(\xi_{2n+1}(\omega),T_1\left(\omega,\xi(\omega)\right)\right)}\end{array}\right\}$$

$$p\left(\xi_{2n+2}(\omega),T_1\left(\omega,\xi(\omega)\right)\right)=\alpha_1(\omega)\,p\left(T_1\left(\omega,\xi(\omega)\right),\xi_{2n+2}(\omega)\right)+\alpha_2(\omega)\,p\left(\xi(\omega),T_1\left(\omega,\xi(\omega)\right)\right)$$

$$+\alpha_3(\omega)\,p\left(\xi_{2n+1}(\omega),\xi_{2n+2}(\omega)\right)\le\beta(\omega)\,p\left(\xi(\omega),\xi_{2n+1}(\omega)\right)$$

$$+\min\left\{\begin{array}{l}p\left(\xi(\omega),\xi_{2n+2}(\omega)\right),\,p\left(\xi_{2n+1}(\omega),T_1\left(\omega,\xi(\omega)\right)\right),\\ \dfrac{\left[p\left(\xi(\omega),\xi_{2n+2}(\omega)\right)+p\left(\xi_{2n+1}(\omega),T_1\left(\omega,\xi(\omega)\right)\right)\right]}{1+p\left(\xi(\omega),\xi_{2n+2}(\omega)\right).p\left(\xi_{2n+1}(\omega),T_1\left(\omega,\xi(\omega)\right)\right)}\end{array}\right\}$$

Putting the value of $p\left(\xi_{2n+2}(\omega),T_1\left(\omega,\xi(\omega)\right)\right)$ and taking $n\to\infty$. in (3.2.2), we have

$$p\left(\xi(\omega),T_1\left(\omega,\xi(\omega)\right)\right)\le\alpha_1(\omega)\,p\left(T_1\left(\omega,\xi(\omega)\right),\xi(\omega)\right)+\alpha_2(\omega)\,p\left(\xi(\omega),T_1\left(\omega,\xi(\omega)\right)\right)\le 0$$

or $\left[1-\alpha_1(\omega)+\alpha_2(\omega)\right]p\left(T_1\left(\omega,\xi(\omega)\right),\xi(\omega)\right)\le 0$

Hence $\xi(\omega)=T_1\left(\omega,\xi(\omega)\right)$ *for all* $\omega\in\Omega$. Similarly for any $\omega\in\Omega$, $\xi(\omega)=T_2\left(\omega,\xi(\omega)\right)$

Thus $\xi(\omega)$ is a common fixed point of T_1 and T_2.

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