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Coupled Jacobsthal Sequence

Suman Jain*, Amitava Saraswati** and Kiran Sisodia***

**Government College, Barnagar, Vikram University, Ujjain, (MP) India.*

***Department of Mathematics, St. Paul's Indore, (MP) India.*

****Department of Mathematics, School of Studies, Vikram University, Ujjain, (MP) India*

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ABSTRACT : In this paper we have introduced interlinked coupled recurrence relation of Jacobsthal second order sequences and deduced some of its properties.

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I. INTRODUCTION

Atanassov [9] introduced the interlinked second order recurrence relation by constructing two sequences $\{\alpha\}_{i=0}^{\infty}$ and $\{\beta\}_{i=0}^{\infty}$ naming them as 2 – F sequences.

According to the scheme, $\alpha_{n+2} = \beta_{n+1} + \beta_n$, $n \geq 0$
$\beta_{n+2} = \alpha_{n+1} + \alpha_n$, $n \geq 0$

Taking, $\alpha_0 = a$, $\beta_0 = b$, $\alpha_1 = c$, $\beta_1 = d$, where a, b, c, d are integers, he extended his research in the same direction which can be seen in [10], [11] and [12]. Hirschhorn in [14] and [15] present explicit solutions to the longstanding problems on the second and third order recurrence relations posed by Atanassov [9]. Recently Singh, Sikhwal and Jain deduced coupled recurrence relations of order five [4]. Carlitz, *et. el*, [13] had also given a representation for a special sequence.

II. COUPLED JACOBSTHAL SEQUENCE

Here is an attempt to get similar relations using Jacobsthal sequence [7] defining it as

$J_{n+2} = J_{n+1} + 2J_n$ where, $J_o – 0$, $J_1 = 1$ and
$j_{n+2} = j_{n+1} + 2j_n$ where $j_o = 2$ and $j_1 = 1$, $n \geq 0$.

Applications to these two sequences to the curves are given in [3]. Moreover in [5] Horadam discussed the properties and has given the associated sequence with Jacobsthal numbers [6] and [8]. Recently Koken and Bozkurth in [1] and [2] have given some matrix properties of Jacobsthal numbers and Jacobsthal – Lucas numbers. Consequently Yilmaz and Bozkurt defined K – Jacobsthal numbers and described Binet's formula for the same [16].

We have introduced coupled order recurrence relations for Jacobsthal and Lucas – Lucas Jacobsthal numbers and called them as 2–J sequences.

Scheme # 2.1

$J_{n+2} = j_{n+1} + 2j_n \quad n \geq 0$
$j_{n+2} = J_{n+1} + 2J_n \quad n \geq 0$
$J_o = a, J_1 = b, j_o = c, j_1 = d$

According to our scheme if we set $a = b$ and $c = d$ then the sequence $\{J_i\}_{i=0}^{\infty}$ and $\{j_i\}_{i=0}^{\infty}$ shall coincide with each other and the sequence $\{J_i\}_{i=0}^{\infty}$ shall become a generalized Jacobsthal sequence where,

$J_0(a, c) = a$, $J_1(a, c) = c$
$J_{n+2}(a, c) = j_{n+1}(a, c) + j_n(a, c)$
$J_n = a, b, d + 2c, b + 2a + 2d,$
$j_n = c, d, b + 2a, d + 2c + 2b,$

By examining the above terms we obtain the following properties :

Theorem 1:

For every integer $m \geq 0$

(*a*) $J_{4m} \mid j_0 = j_{4m} \mid J_0$
(*b*) $J_{4m+1} + j_1 = j_{4m+1} + J_1$
(*c*) $J_{4m+3} + j_0 + j_1 = j_{4m+3} + J_0 + J_1$

Proofs :

For (*c*) the statement is obviously true for $n = 0$.

Assuming that the statement is true for some integer, $n \geq 1$, by the given scheme (1)

$$\begin{aligned} J_{4m+3} + j_0 + j_1 &= j_{4m+2} + 2j_{4m+1} + j_0 + j_1 \\ &= J_{4m+1} + 2J_{4m} + 2j_{4m+1} + j_0 + j_1 \\ &= J_{4m+1} + j_{4m+2} + j_{4m+1} + J_1 + J_0 \\ &\qquad \text{(by inductive hypothesis)} \\ &= J_{4m+1} + J_{4m+2} + j_{4m+1} + j_1 + j_0 \\ &= j_{4m+2} + j_1 + j_0 \end{aligned}$$

Hence the statement is true for all integers $n \geq 0$

Similar proofs can be given for parts (*a*) and (*b*). Adding the first n terms of $\{J_i\}_{i=0}^{\infty}$ and $\{J_i\}_{i=0}^{\infty}$ yield the following results.

Theorem 2:

For all integers $k \geq 0$

$(a)\ j_{3k+5} = \Sigma_{i=1}^{3k} J_{3k+i} + \Sigma_{i=-1}^{k+1} j_{3k+i} + \Sigma_{i=1}^{2k} j_{3k+i} + j_{3k-i}$

$(b)\ J_{3k+5} = \Sigma_{i=1}^{3k} j_{3k+i} + \Sigma_{i=-1}^{k+1} J_{3k+i} + \Sigma_{i=1}^{2k} j_{3k+i} + j_{3k-i}$

Proof (a) :

$j_{3k+5} = J_{3k+4} + 2J_{3k+3}$

$= j_{3k+3} + 2j_{3k+2} + 2J_{3k+3}$ by (1)

$= J_{3k+2} + 2J_{3k+1} + 2j_{3k+2} + 2j_{3k+2} + 4j_{3k+1}$

$= J_{3k+2} + 2J_{3k+1} + 2j_{3k+2} + 2J_{3k+3}$

$= \Sigma_{i=1}^{3k} J_{3k+i} + J_{3k+1} + J_{3k+3} + + 2j_{3k+2}$

$= \Sigma_{i=1}^{3k} J_{3k+i} + J_{3k+1} + 2J_{3k+2} + + 2j_{3k+1} + j_{3k+2}$ by (1)

$= \Sigma_{i=1}^{3k} J_{3k+i} + j_{3k} + 2j_{3k-1} + 2j_{3k+2} + 2j_{3k+1} + j_{3k+2}$

$= \Sigma_{i=1}^{3k} J_{3k+i} + \Sigma_{i=-1}^{k+1} j_{3k+i} + j_{3k-i} + j_{3k+1} + j_{3k+2}$

$= \Sigma_{i=1}^{3k} J_{3k+i} + \Sigma_{i=-1}^{k+1} j_{3k+i}\ \Sigma_{i=1}^{2k} J_{3k+i} + j_{3k-i}$

The proof of (b) is similar to the proof of (a), hence omitted for the sake of brevity. Adding the first n terms with even or odd subscripts for each sequence $\{J_i\}_{i=0}^{\infty}$ and $\{j_i\}_{i=0}^{\infty}$ we obtain more results which are similar to those obtained when one adds the first n terms of the Fibonacci sequence with even or odd subscripts. That is,

(i) $\Sigma_{i=0}^{k} j_i + j_1 = J_{2k}$

(ii) $\Sigma_{i=0}^{2k} j_i + j_{2k} = j_0 + J_{2k+1}$

(iii) $\Sigma_{i=0}^{3k} j_i + j_{3k} + j_{3k-2} = J_{3k-1} + J_{3k+2}$

(iv) $\Sigma_{i=0}^{4k} j_i + j_{4k} + j_{4k-2} = j_0 + J_{4k-1} + J_{4k+1}$

(v) $\Sigma_{i=0}^{5k} j_i + j_{5k} + j_{5k-2} + j_{5k-3} = j_0 + J_{5k-2} + J_{5k-3} + J_{5k+1}$

Proofs are omitted for the sake of brevity

After deriving relations between interlinked coupled recurrence relations using Jacobsthal and Lucas - Jacobsthal sequences we now derive some more relations between arbitrary coupled Integer sequences of the Jacobsthal progeny.

III. ARBITRARY INFINITE SEQUENCES

We consider two second order arbitrary infinite sequences $\{a_i\}_{i=0}^{\infty}$ and $\{b_i\}_{i=0}^{\infty}$ with the initial values a, c and b, $d \in R$

Out of the many schemes that emerge we study two of them

Scheme # 3.1

$a_{n+2} = b_{n+1} + 2a_n : b_{n+2} = a_{n+1} + 2b_n,\ n \geq 0$

$a_0 = a,\ b_0 = b,\ a_1 = c,\ b_1 = d$

Setting, $a - b$, $c - d$, the sequence $\{a_i\}$ and $\{b_i\}$ coincides and forms a generalized Jacobsthal sequence J_i.

Consider,

n	a_n	b_n
0	a	b
1	c	d
2	$d + 2a$	$c + 2b$
3	$3c + 2b$	$3d + 2a$

Theorem 3:

$a_n - b_n = (-1)^{n-1}(a_1 - b_1)J_n + (-1)^n.2.\ (a_0 - b_0)J_{n-1}$

Proof :

Using the principle of mathematical induction we get,

for $n = 2$ $a_2 - b_2 = (d + 2a) - (c + 2b)$

$= -(c - d) + 2(a - b)$

$= (-1)^{2-1}(c - d).\ 1 + (-1)^2,\ 2.(a - b).1$

$= (-1)^{2-1}.(a_1 - b_1).J_2 + (-1)^2 \cdot 2(a_0 - b_0).\ J_{2-1}$

If the statement is true for $n = k$

that is, $a_k - b_k = (-1)^{k-1}(a_1 - b_1)J_k + (-1)^k.\ 2.\ (a_0 - b_0)$-$J_{k-1}$

Hence for $n = k + 1$, we get

$(1)^{k+1-1}(a_1\ b_1)J_{k+1} \mid (1)^{k+1}2.\ (a_0\ b_0)J_{k+1-1}$

$= (-1)^k(a_1 - b_1)J_k + (-1)^{k+1}.2.(a_0 - b_0)J_k$

$= (-1)^k(a_1 - b_1)(J_k + 2J_{k-1}) + (-1)^{k+1}(a_0 - b_0)(2J_{k-1} + 2J_{k-2})$

$= (-1)^k(a_1 - b_1)(J_k) + (-1)^k(a_1 - b_1)(2J_{k-1}) + (-1)^{k+1}(a_0 - b_0)(2J_{k-1}) + (-1)^{k+1}(a_0 - b_0)(4J_{k-2})$

$= -[(-1)^{k-1}(a_1 - b_1)(J_k) + (-1)^k(a_0 - b_0)(2J_{k-1})] + (-1)^2[(-1)^{k-2}.$

$(a_1 - b_1)(J_{k-1}) + (-1)^{k-1}(a_0 - b_0)(2J_{k-2})]$

$= -(a_k - b_k) + 2[a_{k-1} - b_{k-1}]$

$-a_{k+1} - b_{k+1}$

Scheme 3.2

$a_{n+2} = a_{n+1} + 2a_n : b_{n+2} = b_{n+1} + 2b_n,\ n \geq 0$

:

Consider ,

n	a_n	b_n
0	a	b
1	c	d
2	$c + 2a$	$d + 2b$
3	$3c + 2a$	$3d + 2b$

Theorem 4

$a_n - b_n = J_n(a_1 - b_1) + 2J_{n-1}(a_0 - b_0)$

Using the principal of mathematical induction we get,

for $n = 2$ $a_2 - b_2 = (c - d) + 2(a - b)$

$= J_2(a_1 - b_1) + 2J_1(a_0 - b_0)$

Now, supposing that the statement is true for $n = k$

$a_k - b_k = J_k(a_1 - b_1) + 2J_{k-1}(a_0 - b_0)$

Thus, for , $n = k + 1$, we get

$$J_{k+1}(a_1 - b_1) + 2\,J_{k+1-1}(a_0 - b_0)$$
$$= [J_k + 2J_{k-1}](a_1 - b_1) + 2\,.\,[J_{k-1} + 2J_{k-2}](a_0 - b_0)$$
$$= J_k(a_1 - b_1) + 2J_{k-1}(a_1 - b_1) + 2\,.\,J_{k-1}(a_0 - b_0) + 4\,.\,J_{k-2}(a_0 - b_0)$$
$$= J_k(a_1 - b_1) + 2J_{k-1}(a_0 - b_0) + 2[J_{k-1}(a_1 - b_1) + 2J_{k-2}(a_0 - b_0)]$$
$$= (a_k - b_k) + 2[a_{k-1} - b_{k-1}]$$
$$= [a_k + 2a_{k-1}] - [b_k + 2b_{k-1}]$$
$$= a_{k+1} - b_{k+1}.$$

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