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($n+$ t)-Color Partitions-A Survey

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ABSTRACT: A detailed account of the work done on $(n+t)-$color partitions with their origin is provided with an introduction to some basic concepts of partitions. Combinatorial counterparts of some q-series and mock theta functions are discussed.

Keywords: Partitions, $(\mathrm{n}+\mathrm{t})-$Color Partitions, Combinatorial Interpretations.

I. INTRODUCTION

"A partition of a positive integer ν is a non-increasing sequence of positive integers whose sum is ν". For example, 5 can be partitioned in following seven ways:
5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.

This simplicity can attract anyone having some interest in numbers. However, as the number increases it requires more and more precision in writing partitions since the number of partitions increases rapidly. The following table for some values $\boldsymbol{p(\nu)}$ verifies this fact:

$\boldsymbol{\nu}$	1	2	3	4	5	10	20	50	100	200
$\boldsymbol{p(\nu)}$	1	2	3	5	7	42	627	204226	190569292	3972999029388

The concept of partitions first arose in 1669 when Leibnitz asked Bernoulli about the number of ways of writing a positive integer as the sum of other positive integers. However, significant contributions to this theory were made in the eighteenth century by Euler. Euler proved many theorems of astonishing simplicity. Euler also gave the generating function for the number of partitions as:

$$\sum_{\nu=0}^{\infty} p(\nu)q^{\nu}=\frac{1}{(q;q)_{\infty}}.$$

Where $(a;q)_n$ is the rising q-factorial defined by $(a;q)_0=1$ and

$(a;q)_n=(1-a)(1-aq)\ldots\ldots(1-aq^{n-1})$ for $n>0$,

$(a;q)_{\infty}=\lim_{n\to\infty}(a;q)_n$.

Since then many great mathematicians have made significant contributions to the theory of partitions. Partitions have also played a pivotal role in understanding the combinatorics of q-series. The following theorem known as Euler's pentagonal number theorem is perhaps the first instance where a q-series is interpreted combinatorially:

$$\sum_{n\to-\infty}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}}=(q;q)_{\infty}.$$

And given below is the combinatorial version of this theorem:

Theorem 1.1: Let $D_e(\nu)$ denote the number of partitions of ν into an even number of distinct parts and $D_o(\nu)$ denote the number of partitions of ν into an odd number of distinct parts. Then

$$D_e(\nu)-D_o(\nu)=\begin{cases}(-1)^j & \text{if } \nu=j(3j\pm1)/2,\\ 0 & \text{otherwise}.\end{cases}$$

Partitions can also be represented graphically by using Ferrers graph. Ferrers graph of a partition $\lambda=\lambda_1+\lambda_2+\cdots+\lambda_k$ is a set of rows of left aligned and equally spaced dots, where i^{th} row has λ_i dots. The Ferrers graph of partition 9+7+5+5+2+1 is

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By deleting the main diagonal of Ferrers graph of partition and enumerating dots to the right in rows and below the diagonal in columns, two strictly decreasing sequences of non-negative integers are obtained. These sequences, say, $(c_1,c_2,\ldots,c_r)$ and $(d_1,d_2,\ldots,d_r)$ can be represented by the following $2\times r$ array:

$$\begin{pmatrix}c_1 & c_2 & \ldots c_r\\ d_1 & d_2 & \ldots d_r\end{pmatrix},$$

where $\nu=r+\sum_{i=1}^{r}c_i+\sum_{i=1}^{r}d_i$. This representation of a partition is known as Frobenius representation. The Frobenius representation of the partition 9+7+5+5+2+1 is

$$\begin{pmatrix}8 & 5 & 2 & 1\\ 5 & 3 & 1 & 0\end{pmatrix}.$$

II. FROBENIUS REPRESENTATION TO $(\mathrm{n}+\mathrm{t})-$COLOR PARTITIONS

In [11], Andrews *et al.,* proved a partition identity involving hook differences. For a partition π, hook difference at $(i,j)^{th}$ node, a node which lies in the i^{th} row and j^{th} column of Ferrers graph, is defined as the number of nodes in the i^{th} row minus the number of nodes in the j^{th} column. In [7], Agarwal and Andrews rephrased conditions on hook differences, in a special case of this identity, in terms of Frobenius symbol and mapped these to a new class of partitions known as $(\boldsymbol{n}+\boldsymbol{t})-$color partitions. However, $(\boldsymbol{n}+\boldsymbol{t})-$color partitions were first explicitly defined in their manuscript but they noted that these have been used by other authors while studying plane partitions. Before proceeding further, we must define $(\boldsymbol{n}+\boldsymbol{t})-$color partitions.

Definition 2.1. An $(\boldsymbol{n}+\boldsymbol{t})-$color partition, $\boldsymbol{t} \geq \mathbf{0}$ (also called a partition with $(\boldsymbol{n}+\boldsymbol{t})$ copies of $\boldsymbol{n}$) is a partition in which a part of size $\boldsymbol{n}$, $\boldsymbol{n} \geq \mathbf{0}$, can occur in $(\boldsymbol{n}+\boldsymbol{t})$ different colors denoted by $\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots, \boldsymbol{n}_{n+t}$. Note that for $\boldsymbol{t} > 0$ zeros can occur but cannot repeat.

Example 2.1. The partitions of $\mathbf{2}$ with "$\boldsymbol{n}+\mathbf{1}$ copies of $\boldsymbol{n}$" are given by

$$\begin{array}{llll} \mathbf{2}_3 & \mathbf{2}_3+\mathbf{0}_1 & \mathbf{1}_2+\mathbf{1}_2 & \mathbf{1}_2+\mathbf{1}_2+\mathbf{0}_1 \\ \mathbf{2}_2 & \mathbf{2}_2+\mathbf{0}_1 & \mathbf{1}_2+\mathbf{1}_1 & \mathbf{1}_2+\mathbf{1}_1+\mathbf{0}_1 \\ \mathbf{2}_1 & \mathbf{2}_1+\mathbf{0}_1 & \mathbf{1}_1+\mathbf{1}_1 & \mathbf{1}_1+\mathbf{1}_1+\mathbf{0}_1 \end{array}$$

Remark 2.1. The parts of $\boldsymbol{n}-$color partitions follow the lexicographic order as given below:

$\mathbf{1}_1 < \mathbf{2}_1 < \mathbf{2}_2 < \mathbf{3}_1 < \mathbf{3}_2 < \mathbf{3}_3 < \mathbf{4}_1 < \mathbf{4}_2 < \mathbf{4}_3 < \mathbf{4}_4$.

If $\boldsymbol{P}_N(\boldsymbol{\nu})$ denotes the number of $\boldsymbol{n}-$color partitions of $\boldsymbol{\nu}$ then it is immediate using standard techniques of partition theory that

$$\mathbf{1}+\sum_{\nu=1}^{\infty} \boldsymbol{P}_N(\boldsymbol{\nu})\boldsymbol{q}^{\boldsymbol{\nu}} = \prod_{\nu=1}^{\infty} \frac{\mathbf{1}}{(\mathbf{1}-\boldsymbol{q}^{\boldsymbol{\nu}})^{\boldsymbol{\nu}}}.$$

It is a very interesting fact that generating function for $\boldsymbol{n}-$color partitions is also a generating function for plane partitions. Since generating function for $\boldsymbol{n}-$color partitions and plane partitions is same, it is natural to ask whether there is a one to one correspondence between set of plane partitions and set of $\boldsymbol{n}-$color partitions of a positive integer $\boldsymbol{\nu}$. Agarwal [3] has found a bijection between these two sets of partitions via Bender-Knuth matrices. However, finding a direct bijection between these sets of partitions is still an open problem. Using this bijection, Agarwal [4] studied two restricted $\boldsymbol{n}-$color partition functions to obtain two analogues of Gaussian polynomials. In the same manuscript, Agarwal studied generating functions for various other restricted $\boldsymbol{n}-$color partition functions and obtained certain combinatorial identities for these restricted $\boldsymbol{n}-$color partition functions.

III. COMBINATORICS OF q-SERIES USING $(\boldsymbol{n}+\boldsymbol{t})-$COLOR PARTITIONS

The two famous q-series identities known as Rogers-Ramanujan identities are given by

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=0}^{\infty} (1-q^{5n-1})^{-1}(1-q^{5n-4})^{-1},$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=0}^{\infty} (1-q^{5n-2})^{-1}(1-q^{5n-3})^{-1}.$$

The combinatorial interpretations of these identities in terms of partitions were given by Major MacMahon [13]. Considering the form of these identities these are called sum-product identities. Identities similar in form to these identities are known as Rogers-Ramanujan type identities. A number of authors have found many Rogers-Ramanujan type identities. In the two manuscripts [12, 14], one can find lists of 200 and 130 identities respectively of Rogers-Ramanujan type.

In [2], Agarwal proved the following general theorem involving $n-$color partitions:

Theorem 3.1. For $k \geq -3$, let $A_k(\nu)$ denote the number of $n-$color partitions of ν such that each pair of summands m_i, n_j satisfies $|m-n| > i+j+k$. Then

$$\sum_{\nu=0}^{\infty} A_k(\nu)q^{\nu} = \sum_{\nu=0}^{\infty} \frac{q^{\nu[1+\frac{(k+3)(\nu-1)}{2}]}}{(q;q)_{\nu}(q;q^2)_{\nu}}.$$

To prove this result, Agarwal used the technique of splitting the set of partitions under consideration into certain classes thereby obtaining a recurrence relation which provides the required results. Using this method, Agarwal [1] provided combinatorial interpretations of seven identities from Slater's compendium in terms of $(n+t)-$color partitions. Agarwal and Bressoud [8] also linked restricted $(n+t)-$color partitions to certain weighted lattice paths introduced by them. In 2009, Agarwal and Rana [10] provided new combinatorial counterparts of Göllnitz-Gordon identities using $(n+t)-$color partitions.

In [5, 6], Agarwal used this idea of interpreting q-series to obtain interpretations of four mock theta functions in terms of $(n+t)-$color partitions and lattice paths respectively. In continuation, Agarwal and Rana [9] interpreted another mock theta function of fifth order in terms of $(n+t)-$color partitions and lattice paths. But there is a long list of mock theta functions which is still open to these combinatorial treatments. Also, there are many identities in Slater's and Chu-Zhang's compendium which have not been interpreted with these combinatorial tools till now.

REFERENCES

[1]. Agarwal, A.K. (1988). Rogers-Ramanujan identities for *n*-color partitions. *Journal of Number Theory,* **28**(3): 299-305.

[2]. Agarwal, A. K. (1986). Partitions with "N copies of N". Proceedings of the Colloque de Combinatoire Énumérative, University of Québec at Montréal, Lecture Notes in Mathematics (1234): 1-4.

[3]. Agarwal, A. K. (2000). Rogers-Ramanujan identities for plane partitions. *Journal of the Indian Mathematical Society*, **67**: 75-85.

[4]. Agarwal, A.K. (2001). *n*-color analogues of Gaussian polynomials, *ARS combinatoria*, **61**: 97-117 .

[5]. Agarwal, A.K. (2004). *n*-color partition theoretic interpretations of some mock theta functions. *Electronic Journal of combinatorics*, **11**(3):#N14.

[6]. Agarwal, A. K. (2005). Lattice paths and mock theta functions, Proceedings of the *6th Int. Conf., SSFA*, **6**: 95-102.

[7]. Agarwal, A.K. and Andrews, G.E. (1987). Rogers-Ramanujan identities for partitions with "N copies of N. *Journal of Combinatorial Theory Series A*, **45**(1): 40-49.

[8]. Agarwal, A.K. and Bressoud, D.M. (1989). Lattice paths and multiple basic hypergeometric series. *Pacific Journal of Mathematics*, **136**(2):209-228, 1989.

[9]. Agarwal, A.K. and Rana, M. (2008). Two new combinatorial interpretations of a fifth order mock theta Function. *Journal of Indian Mathematical Society, Special Centenary 1907-2007*: 11-24.

[10]. Agarwal, A. K. and Rana, M. (2009). New combinatorial versions of Göllnitz-Gordon identities, Utilitas *Mathematica*, **79**: 145-155.

[11]. Andrews, G.E., Baxter, R.J., Bressoud, D.M., Burge, W.H., Forrester, P.J. and Viennot, G. (1987). Partitions with prescribed hook differences, European Journal of Combinatorics, **8**(4): 341-350.

[12]. Chu, W. and Zhang, W. (2009). Bilateral Bailey lemma and Rogers-Ramanujan identities, *Advances in Applied Mathematics*, **42**(3): 358-391.

[13]. MacMahon, P.A. (2001). Combinatory analysis, volume **137**, American Mathematical Society.

[14.] Slater, L.J. (1952). Further identities of the Rogers-Ramanujan type*, Proceedings of The London Mathematical Society,* **2**(1): 147-167.