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Common Fixed Point Theorems in Fuzzy Metric Space using Occasionally Weakly Compatible Mappings

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ABSTRACT: In this paper, the concept of occasionally weakly compatible maps in fuzzy metric space has been introduced to prove common fixed point theorem in discontinuous maps.

Keywords: Common fixed points, fuzzy metric space, compatible maps, occasionally weakly compatible mappings, weak compatible mappings and semi-compatible mappings.
AMS Subject Classification: Primary 47H10, Secondary 54H25.

I. INTRODUCTION

Zadeh [19] introduced the concept of fuzzy set as a new way to represent vagueness in our everyday life. A fuzzy set A in X is a function with domain X and values in [0, 1]. Since then, many authors regarding the theory of fuzzy sets and its applications have developed a lot of literatures.

However, when the uncertainty is due to fuzziness rather than randomness, as sometimes in the measurement of an ordinary length, it seems that the concept of a fuzzy metric space is more suitable. We can divide them into following two groups: The first group involves those results in which a fuzzy metric on a set X is treated as a map where X represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distances are set up between fuzzy objects. On the other hand in second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy. In this paper we deal with the Fuzzy metric space defined by Kramosil and Michalek [11] and modified by George and Veeramani [5]. Recently, Grabiec [6] has proved fixed point results for Fuzzy metric space. In the sequel, Singh and Chauhan [15] introduced the concept of compatible mappings in Fuzzy metric space and proved the common fixed point theorem. Jungck et. al. [9] introduced the concept of compatible maps of type (A) in metric space and proved fixed point theorems. Cho [3, 4] introduced the concept of compatible maps of type (α) and compatible maps of type (β) in fuzzy metric space. In 2011, using the concept of compatible maps of type (A) and type (β), Singh *et. al.* [16, 17] proved fixed point theorems in a fuzzy metric space.

Recently in 2012, Jain *et. al.* [7, 8] and Sharma *et. al*. [13] proved various fixed point theorems using the concepts of semi-compatible mappings, property (E.A.) and absorbing mappings.

For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

II. PRELIMINARIES

Definition 2.1. [12] A binary operation $*$: $[0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for $a, b, c, d \in [0, 1]$.

Examples of t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2.2. [12] The 3-tuple $(X, M, *)$ is said to be a *Fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm and M is a Fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions : for all $x, y, z \in X$ and $s, t > 0$.

(FM-1) $M(x, y, 0) = 0$,

(FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(FM-5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous,

(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$.

Note that $M(x, y, t)$ can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a Fuzzy metric space.

Example 2.1. [12] Let (X, d) be a metric space. Define $a * b = \min \{a, b\}$ and $M(x,y,t) = \frac{t}{t + d(x,y)}$ for all $x, y \in X$ and all $t > 0$. Then (X, M, *) is a Fuzzy metric space. It is called the Fuzzy metric space induced by d.

Definition 2.3. [12] A sequence $\{x_n\}$ in a Fuzzy metric space (X, M, *) is said to be a Cauchy sequence if and only if for each $\varepsilon > 0$, $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

The sequence $\{x_n\}$ is said to converge to a point x in X if and only if for each $\varepsilon > 0$, $t > 0$ there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \varepsilon$ for all $n \geq n_0$.

A Fuzzy metric space (X, M, *) is said to be complete if every Cauchy sequence in it converges to a point in it.

Definition 2.4. [15] Self mappings A and S of a Fuzzy metric space (X, M, *) are said to be compatible if and only if $M(ASx_n, SAx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n, Ax_n \to p$ for some p in X as $n \to \infty$.

Definition 2.5. [16] Two self maps A and B of a fuzzy metric space (X, M, *) are said to be weak compatible if they commute at their coincidence points, i.e. Ax = Bx implies ABx = BAx.

The following concept [1] is a proper generalization of nontrivial weakly compatible maps which do have a coincidence point. The counterpart of the concept of occasionally weakly compatible maps in Fuzzy metric space is as follows:

Definition 2.6. Self maps A and S of a Fuzzy metric space (X, M, *) are said to be occasionally weakly compatible (owc) if and only if there is a point x in X which is coincidence point of A and S at which A and S commute.

Lemma 2.1. Let (X, M, *) be a Fuzzy metric space, A and B are occasionally weakly compatible self maps of X. If A and B have a unique point of coincidence, w = Ax = Bx, then w is the unique common fixed point of A and B.

Proof. Since A and B are occasionally weakly compatible, there exists a point $x \in X$ such that Ax = Bx = w and ABx = BAx. Thus, AAx = ABx = BAx, which says that Ax is also a point of coincidence of A and B. Since the point of coincidence w = Ax is unique by hypothesis, BAx = AAx = Ax, and w = Ax is a common fixed point of A and B. Moreover, if z is any common fixed point of A and B, then z = Az = Bz = w by the unique of the point of coincidence.

Lemma 2.2. [6] Let (X, M, *) be a fuzzy metric space. Then for all $x, y \in X$, M(x, y, .) is a non-decreasing function.

Lemma 2.3. [2] Let (X, M, *) be a fuzzy metric space. If there exists $k \in (0, 1)$ such that for all $x, y \in X$ $M(x, y, kt) \geq M(x, y, t)$ $\forall$ $t > 0$ then x = y.

Lemma 2.4.[10] The only t-norm * satisfying $r * r \geq r$ for all $r \in [0, 1]$ is the minimum t-norm, that is $a * b = \min \{a, b\}$ for all $a, b \in [0, 1]$.

Proposition 2.1. [17] In a fuzzy metric space (X, M, *) limit of a sequence is unique

III. MAIN RESULT

Now we prove the following results:

Theorem 3.1 Let (X, M, *) be a Fuzzy metric spacee. Further, let (L,A) and (P, S) are occasionally weakly compatible maps in X satisfying

(3.1.1) $\min\{M(Lx, Py, kt), M(Sy, Lx, kt)\} + cM(Sy, Py, kt) \geq [aM(Ax, Lx, t) + bM(Ax, Sy, t)]$

for all $x, y \in X$, $k \in (0,1)$ and $t > 0$, where $0 < a, b < 1$ and $0 \leq c < 1$, such that $a + b + c = 1$;

Then L, A, P and S have a unique common fixed point in X.

Proof. Since the pairs (L,A) and (P, S) are occasionally weakly compatible, there exist points u, v $\in$ X such that Lu = Au, LAu = ALu and Pv = Sv, PSv = SPv.

Now we show that Lu = Pv.

By putting x = u and y = v in inequality (3.1.1), then we get

$\min\{M(Lu, Pv, kt),M(Sv, Lu, kt)\} + cM(Sv, Pv, kt)$
$\geq [aM(Au, Lu, t) + bM(Au, Sv, t)]$,
$\min\{M(Lu, Pv, kt),M(Pv, Lu, kt)\} + cM(Pv, Pv, kt)$
$\geq [aM(Lu, Lu, t) + bM(Lu, Pv, t)]$,
$M(Lu, Pv, kt) + c \geq [a + bM(Lu, Pv, t)]$,
$M(Lu, Pv, kt) \geq [bM(Lu, Pv, t) + (a - c)]$,

$$M(w,z,kt) \geq \left(\frac{a-c}{1-b}\right)$$

$M(Lu, Pv, kt) \geq 1$.

Thus, we have Lu = Pv. Therefore, Lu = Au = Pv = Sv. Moreover, if there is another point z such that Lz = Az. Then using the inequality (3.1.1) it follows that Lz = Az = Pv = Sv, or Lu = Lz. Hence w = Lu = Au is the unique point ofcoincidence of L and A. By Lemma 2.1, w is the unique common fixed point of L and A. Similarly, there is a unique point $z \in X$ such that z = Pz = Sz. Suppose that $w \neq z$ and taking x = w, y = z in inequality (3.1.1), then we get

$\min\{M(Lw, Pz, kt),M(Sz, Lw, kt)\} + cM(Sz, Pz, kt)$
$\geq [aM(Aw, Lw, t) + bM(Aw, Sz, t)]$,
$\min\{M(w, z, kt), M(z, w, kt)\} + cM(z, z, kt)$
$\geq [aM(w, w, t) + bM(w, z, t)]$,
$M(w, z, kt) + c \geq [a + bM(w, z, t)]$,
$M(w, z, kt) \geq [bM(w, z, t)+ (a - c)]$,

$$M(w,z,kt) \geq \left(\frac{a-c}{1-b}\right)$$

$M(w,z,kt) \geq 1$.

Thus, we have w = z. That is w is the unique common fixed point of L, A, P and S in X.

On taking P = L and S = A in Theorem 3.1. then we get the following interesting result.

Corollary 3.1. Let (X, M, *) be a Fuzzy metric spacee. Further, let (L,A) are occasionally weakly compatible maps in X satisfying

min{M(Lx, Ly, kt),M(Ay, Lx, kt)} + cM(Ay, Ly, kt)
≥ [aM(Ax, Lx, t) + bM(Ax, Ay, t)]

for all x, y □ X, k□ (0,1) and t > 0, where 0 < a, b <
and 0 ≤ c < 1 such that a + b + c = 1;

Then L and A have a unique common fixed point in X.

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