International Journal of Theoretical & Applied Sciences **6**(1): 102-113(2014)

ISSN No. (Print): 0975-1718
ISSN No. (Online): 2249-3247

On Construction of Hadamard Codes Using Hadamard Rhotrices

P.L. Sharma, S. Kumar and M. Rehan
Department of Mathematics,
Himachal Pradesh University, Shimla, (HP), India

(Corresponding author P.L. Sharma)
(Received 26 April, 2014, Accepted 25 May, 2014)

ABSTRACT: Hadamard matrices have wide applications in image analysis, signal processing, coding theory, cryptology and combinatorial designs. The codes generated from Hadamard matrices are of much importance due to the large distance between them. These codes can correct large number of errors and are essential component of the study in communication channels. Rhotrix is a new paradigm of research study and has wide applications in cryptography and coding theory. In the present paper, we introduce Hadamard codes using Hadamard rhotrices.

AMS Classifications: 15A33, 14G50, 11T71.

Keywords: Hadamard Matrix; Hadamard Rhotrix; Optimal Codes; Hadamard Codes.

I. INTRODUCTION

Transformation of data over insecure channels is prevailing all over the world. Security of data from hackers is the need of present scenario. Cryptography is the science which provides confidentiality, authenticity and integrity of data travelling through insecure channels. In cryptography sequences of symbols are used for the pieces of information. This process of representation is called coding and the symbols are called code symbols. A sequence of code symbols is called a code word. In electronic transmission, it is necessary that the code symbols should be small. Hence, binary code is very applicable.

Code words are of constant length. A block code is one in which all code words have the same length n. Given any two code words, the Hamming distance between two code words is defined as the number of components in which the words disagree. A distance d code is one in which the minimum of all the Hamming distances between the words is at least d. An (n, M, d; q) code means a set of M code words of length n with q symbols and Hamming distance d [1]. An (n, M, d; q) code is optimal if M is as large as possible for given n, d and q. Plotkin [2] obtained the following bounds for binary codes:

$$M \le 2\left[\tfrac{d}{2d-n}\right] \quad \text{if } d \text{ is even and } d \le n < 2d, \qquad \ldots(1.1)$$

$$M \le 2n \quad \text{if } d \text{ is even and } n = 2d, \qquad \ldots(1.2)$$

$$M \le 2\left[\frac{d+1}{2d+1-n}\right] \quad \text{if } d \text{ is odd and } d \le n < 2d+1, \qquad \ldots(1.3)$$

$$M \le 2n+2 \quad \text{if } d \text{ is odd and } n = 2d+1. \qquad \ldots(1.4)$$

The famous matrix with orthogonal property was defined by Sylvester [3] in 1867 and further studied by Hadamard [4] in 1893 and now known as Hadamard matrix. Hadamard matrices have received much attention in the recent past, owing to their well-known and promising applications [5]. Sarukhanyan *et al.* [6] studied the Hadamard matrices and their applications in image analysis, signal processing, coding theory, cryptology and combinatorial designs [7-10]. The orthogonal codes from Hadamard matrices are useful in encoding and decoding a information in very noisy channels because these codes have large distance and can correct large number of errors. There are several methods to construct Hadamard matrices. Kimura and Ohmori [11] constructed Hadamard matrices of order 28.

Koukouvinos and Seberry [12] used orthogonal designs for the construction of Hadamard matrices. Singh *et al.* [13, 14] constructed Hadamard matrix using BIBD and Frobenius groups. Szollosi [15] studied the construction as well as classification of Hadamard matrices. Sajadieh *et al.* [16] used Vandermonde matrices for the construction of Finite Field Hadamard matrices.

Rhotrix is a new concept introduced in the literature of mathematics in 2003 [17]. It is a mathematical object which is, in some way between 2×2 - dimensional and 3×3 - dimensional matrices. A rhotrix of dimension 3 is defined as

$$R_3 = \left\langle \begin{matrix} & a_1 & \\ a_2 & a_3 & a_4 \\ & a_5 & \end{matrix} \right\rangle, \quad \ldots(1.5)$$

where $a_1, a_2, a_3, a_4, a_5 \in \mathrm{R}$. A rhotrix of higher order is defined in [18]. Algebra and analysis of rhotrices is discussed in the literature [17-28]. Hadamard rhotrix over finite field is defined in [29]. We give necessary and sufficient conditions for Hadamard rhotrices and its sub-rhotrices in Theorem 2.1 and Theorem 2.3.We introduce Hadamard codes by making the use of Hadamard rhotrices in Theorem 2.5.

II. MAINS RESULTS

Theorem 2.1 A rhotrix R_n is Hadamard rhotrix over GF(2) iff there exist two square matrices whose rows are orthogonal to each other.

Proof: Let R_n be a Hadamard rhotrix over GF(2) defined as

$$R_n = \left\langle \begin{matrix} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{t1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & a_{1t} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ & & a_{tt-2} & c_{t-1t-2} & a_{t-1t-1} & c_{t-2t-1} & a_{t-2t} & & \\ & & & a_{tt-1} & c_{t-1t-1} & a_{t-1t} & & & \\ & & & & a_{tt} & & & & \end{matrix} \right\rangle. \quad \ldots(2.1)$$

Then M_1 and M_2 are two coupled matrices in R_n defined as

$$M_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1t} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2t} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3t} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{t1} & a_{t2} & a_{t3} & \ldots & a_{tt} \end{pmatrix} \quad \ldots(2.2)$$

$$M_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1t-1} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2t-1} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3t-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{t1} & a_{t2} & a_{t3} & \ldots & a_{t-1t-1} \end{pmatrix}. \quad \ldots(2.3)$$

By the definition of Hadamard rhotrix,

$$(a_{11},a_{12},a_{13},.,.,.,a_{1t}).(a_{21},a_{22},a_{23},.,.,.,a_{2t})=0,$$
$$(a_{11},a_{12},a_{13},.,.,.,a_{1t-1}).(a_{21},a_{22},a_{23},.,.,.,a_{2t-1})=0,$$
$$(a_{21},a_{22},a_{23},.,.,.,a_{2t}).(a_{31},a_{32},a_{33},.,.,.,a_{3t})=0,$$
$$(a_{21},a_{22},a_{23},.,.,.,a_{2t-1}).(a_{31},a_{32},a_{33},.,.,.,a_{3t-1})=0,$$
$$(a_{31},a_{32},a_{33},.,.,.,a_{3t}).(a_{41},a_{42},a_{43},.,.,.,a_{4t})=0,$$
$$(a_{31},a_{32},a_{33},.,.,.,a_{3t-1}).(a_{41},a_{42},a_{43},.,.,.,a_{4t-1})=0,$$
$$(.,.,.,.,.,.,.,.,.,.).(.,.,.,.,.,.,.,.,.,.)=0,$$
$$(.,.,.,.,.,.,.,.,.,.).(.,.,.,.,.,.,.,.,.,.)=0,$$
$$(.,.,.,.,a_{t-1t-1},a_{t-1t}).(a_{t1},.,.,a_{tt-2},a_{tt-1},a_{tt})=0.$$

This gives, M_1 and M_2 are Hadamard matrices as rows of two matrices are orthogonal to each other.

Example 2.2 Let the rhotrix of order 9 be defined as

$$R_9=\left\langle\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & 1 & 0 & & & \\ & & 1 & 0 & 1 & 0 & 0 & & \\ & 0 & 1 & 0 & 1 & 1 & 1 & 1 & \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \\ & & 0 & 0 & 0 & 0 & 0 & & \\ & & & 1 & 1 & 0 & & & \\ & & & & 0 & & & & \end{array}\right\rangle. \qquad \ldots(2.4)$$

Two coupled matrices of (2.4) are

$$M_1=\begin{pmatrix}1&0&0&1&0\\0&1&1&0&1\\1&0&0&1&0\\0&1&1&0&0\\1&0&0&1&0\end{pmatrix},\ M_2=\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{pmatrix}.$$

The inner product of different rows in M_1 over GF(2) is

$$(1, 0, 0, 1, 0)\ (0, 1, 1, 0, 1) = 0,$$
$$(0, 1, 1, 0, 1)\ (1, 0, 0, 1, 0) = 0,$$
$$(1, 0, 0, 1, 0)\ (0, 1, 1, 0, 0) = 0,$$
$$(0, 1, 1, 0, 0)\ (1, 0, 0, 1, 0) = 0,$$

(1, 0, 0, 1, 0) (1, 0, 0, 1, 0) = 0,
(1, 0, 0, 1, 0) (0, 1, 1, 0, 0) = 0,
(1, 0, 0, 1, 0) (1, 0, 0, 1, 0) = 0,
(0, 1, 1, 0, 1) (1, 0, 0, 1, 0) = 0,
(1, 0, 0, 1, 0) (1, 0, 0, 1, 0) = 0,
(1, 0, 0, 1, 0) (0, 1, 1, 0, 1) = 0,
(0, 1, 1, 0, 0) (1, 0, 0, 1, 0) = 0,
(0, 1, 1, 0, 0) (0, 1, 1, 0, 1) = 0,
(0, 1, 1, 0, 0) (1, 0, 0, 1, 0) = 0,
(1, 0, 0, 1, 0) (0, 1, 1, 0, 1) = 0,
(1, 0, 0, 1, 0) (1, 0, 0, 1, 0) = 0,
(1, 0, 0, 1, 0) (1, 0, 0, 1, 0) = 0.

Since all the rows in M_1 are orthogonal to each other. Therefore, it is Hadamard matrix.
Similarly, the inner product of different rows in

$$M_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

over GF(2) is

(1, 0, 1, 0) (0, 1, 0, 1) = 0,
(1, 0, 1, 0) (1, 0, 1, 0) = 0,
(1, 0, 1, 0) (0, 1, 0, 1) = 0,
(0, 1, 0,1) (1, 0, 1, 0) = 0,
(0, 1 ,0, 1) (0, 1, 0, 1) = 0 etc.

Since all the rows in M_2 are orthogonal to each other. Therefore, it is also a Hadamard matrix. Both the coupled matrices M_1 and M_2 of the rhotrix R_9 are orthogonal. Therefore, R_9 is Hadamard rhotrix.

Theorem 2.3 A rhotrix R_n of odd order $n > 3,$ is Hadamard rhotrix iff the sub-rhotrices of R_n given by $R_{n-(2p+2)}$, $p = 1, 2, 3, \ldots$ are Hadamard over GF(2).

Proof: Let R_n be a Hadamard rhotrix defined

$$R_n = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{31} & a_{21} & a_{12} & & & \\ & & . & a_{41} & a_{32} & a_{22} & . & & \\ & a_{d-2,1} & . & . & . & . & . & a_{1,n-1} & \\ a_{d,1} & a_{d-1,1} & a_{d-2,2} & . & . & . & a_{3,n-1} & a_{2,n-1} & a_{1,n} \\ & a_{d,2} & a_{d-1,2} & . & . & . & a_{4,n-1} & a_{3,n} & . \\ & & . & . & a_{d-2,n-1} & . & a_{d-2,n} & & \\ & & & a_{d,n-1} & a_{d-1,n-1} & a_{d-2,n} & & & \\ & & & & a_{d,n} & & & & \end{array} \right\rangle. \qquad \ldots(2.5)$$

The sub-rhotrix of R_n for $p = 1$ is

$$R_{n-4} = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{31} & a_{21} & a_{12} & & & \\ & & . & a_{41} & a_{32} & a_{22} & . & & \\ & a_{d-2,1} & . & . & . & . & . & . & \\ a_{d,1} & a_{d-1,1} & a_{d-2,2} & . & . & . & . & . & a_{1,n-1} \\ & a_{d,2} & a_{d-1,2} & . & . & . & . & a_{3,n-1} & . \\ & & . & . & . & . & . & & \\ & & & . & . & a_{d-2,n-1} & & & \\ & & & & a_{d,n-1} & & & & \end{array} \right\rangle. \quad \ldots(2.6)$$

The sub-rhotrix of R_n for $p = 2$ is

$$R_{n-6} = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{31} & a_{21} & a_{12} & & \\ & a_{d-2,1} & . & a_{32} & a_{22} & . & \\ a_{d,1} & a_{d-1,1} & a_{d-2,2} & . & . & . & . \\ & a_{d,2} & a_{d-1,2} & . & . & . & \\ & & . & . & . & & \\ & & & a_{d,n-1} & & & \end{array} \right\rangle. \quad \ldots(2.7)$$

The sub-rhotrix of R_n for $p = 3$ is

$$R_{n-8} = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{31} & a_{21} & a_{12} & \\ a_{d,1} & . & a_{32} & a_{22} & . \\ & a_{d,2} & . & . & \\ & & . & & \end{array} \right\rangle. \quad \ldots(2.8)$$

By the definition of Hadamard rhotrix, R_{n-4}, R_{n-6}, R_{n-8} are Hadamard. Similarly, for all the values of p, all the sub-rhotrices of R_n are Hadamard. If all the sub-rhotrices of R_n are Hadamard, then all the rows of coupled matrices in the sub-rhotrices are orthogonal to each other. Therefore, R_n is Hadamard rhotrix.

Example 2.4 Let the rhotrix of order 9 be represented as

$$R_9 = \left\langle \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & 1 & 0 & & & \\ & & 1 & 0 & 1 & 0 & 0 & & \\ & 0 & 1 & 0 & 1 & 1 & 1 & 1 & \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \\ & & 0 & 0 & 0 & 0 & 0 & & \\ & & & 1 & 1 & 0 & & & \\ & & & & 0 & & & & \end{array} \right\rangle. \qquad \text{...(2.9)}$$

The coupled matrices in the rhotrix (2.9) are

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \; M_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Since the rows in both matrices are orthogonal to each other .Therefore R_9 is Hadamard rhotrix given in Example 2.2. The sub rhotrix of order R_{n-4} for $p = 1$ is

$$R_5 = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 1 & 0 & \\ 1 & 0 & 1 & 1 & 1 \\ & 0 & 0 & 0 & \\ & & 1 & & \end{array} \right\rangle. \qquad \text{...(2.10)}$$

The coupled matrices in R_5 are

$$M_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, M_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

In matrix M_1, the inner product of the rows over GF(2) is

$$(1, 0, 1)\ (0, 1, 0) = 0,$$

$$(0, 1, 0)\ (1, 0, 1) = 0,$$

$$(1, 0, 1))\ (1, 0, 1) = 0.$$

In M_1, all the rows are orthogonal to each other. Therefore, it is Hadamard matrix. In the matrix M_2, the inner product of the rows over GF(2) is

$$(1, 1)\ (0, 0) = 0\ .$$

The rows are orthogonal to each other in M_2. Therefore, M_2 is Hadamard matrix. Hence, R_5 is Hadamard rhotrix because all the rows in both the coupled matrices are orthogonal to each other. Now, the sub rhotrix of order R_{n-6} for $p = 2$ is

$$R_3 = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 1 & \end{array} \right\rangle. \qquad \text{...(2.11)}$$

The coupled matrices in R_3 are

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ M_2 = (0).$$

Therefore, R_3 is Hadamard rhotrix because rows are orthogonal to each other in both the coupled matrices. Hence all the sub rhotrices of R_9 are Hadamard.

Theorem 2.5 If Hadamard rhotrix has coupled matrix of order $4t$ and $4t-1$, then this implies the existence of the following optimal codes: $(4t, 8t, 2t; 2)$, $(4t-1, 4t, 2t; 2)$, $(4t-1, 8t, 2t-1; 2)$, $(4t-2, 2t, 2t; 2)$, $(4t-2, 4t, 2t-1; 2)$ and $(4t-3, 2t, 2t-1; 2)$.

Proof: Let R_n be Hadamard rhotrix having coupled matrices H_1 and H_2 of order $4t$ and $4t-1$ respectively. Consider J_1 and J_2 be two matrices of order $4t$ and $4t-1$ whose all entries are 1. The $8t$ rows of two matrices

$$W_{4t}^1 = \tfrac{1}{2}(J_1 + H_1)$$

and

$$W_{4t}^2 = \tfrac{1}{2}(J_1 - H_1)$$

form a $(4t, 8t, 2t; 2)$ code. In order to construct the remaining codes, we delete the first column of H_1 and obtain matrices

$$W_{4t}^3$$

and

$$W_{4t}^4 = \tfrac{1}{2}(J_1 - H_1),$$

then the $4t$ rows of

$$W_{4t}^3$$

and the $8t$ rows of

$$W_{4t}^3 \text{ and } W_{4t}^4$$

give second and third code. Now, we delete the first column of W_{4t}^3 and the rows which start with 1 to obtain the matrix W_{4t}^5. This matrix gives the fourth code. Further, we consider the matrix H_2. The $8t$ rows of two matrices

$$W_{4t-1}^1 = \tfrac{1}{2}(J_2 + H_2)$$

and

$$W_{4t-1}^2 = \tfrac{1}{2}(J_2 - H_2)$$

form the same code $(4t-1, 8t, 2t-1; 2)$ code. On deleting the first column of H_2, we get the matrix W_{4t-1}^3. This matrix gives $(4t-2, 4t, 2t-1; 2)$ code. Now, we delete the first column of

$$W_{4t-1}^3$$

and the rows which start with 1 to obtain the matrix W_{4t-1}^4. This matrix gives the last code.

Example 2.6 Let R_{23} be a Hadamard rhotrix defined as

$$R_{23} = \left\langle \begin{array}{ccccccccccccccccccccccc}
&&&&&&&&&&&1&&&&&&&&&&& \\
&&&&&&&&&&1&1&1&&&&&&&&&& \\
&&&&&&&&&1&1&0&1&1&&&&&&&&& \\
&&&&&&&&1&1&0&0&1&1&1&&&&&&&& \\
&&&&&&&1&1&1&0&0&1&0&1&1&&&&&&& \\
&&&&&&1&1&0&1&0&0&1&0&1&1&1&&&&&& \\
&&&&&1&1&0&0&1&0&0&1&0&1&1&1&1&&&&& \\
&&&&1&1&0&0&0&1&0&0&1&0&1&1&1&1&1&&&& \\
&&&1&1&1&0&0&0&1&0&0&1&0&1&1&1&0&1&1&&& \\
&&1&1&1&1&0&0&0&0&0&0&1&0&1&1&1&0&0&1&1&& \\
&1&1&1&1&1&0&0&0&1&0&0&1&0&1&1&1&0&0&0&1&1& \\
1&1&0&1&1&1&0&0&0&1&0&0&1&0&1&1&1&0&0&0&1&1&1 \\
&1&0&1&1&1&0&0&0&1&0&0&1&0&1&1&1&0&0&0&1&0& \\
&&0&1&1&1&0&0&0&1&0&0&1&0&1&1&1&0&0&0&1&& \\
&&&1&1&1&0&0&0&1&0&0&1&0&1&1&1&0&0&0&&& \\
&&&&1&1&0&0&0&1&0&0&1&0&1&1&1&0&0&&&& \\
&&&&&1&0&0&0&1&0&0&1&0&1&1&1&0&&&&& \\
&&&&&&0&0&0&1&0&0&1&0&1&1&1&&&&&& \\
&&&&&&&0&0&1&0&0&1&0&1&1&&&&&&& \\
&&&&&&&&0&1&0&0&1&0&1&&&&&&&& \\
&&&&&&&&&1&0&0&1&0&&&&&&&&& \\
&&&&&&&&&&0&0&1&&&&&&&&&& \\
&&&&&&&&&&&0&&&&&&&&&&&
\end{array} \right\rangle, \tag{2.12}$$

then the coupled matrices of the rhotrix R_{23} are of order 12 and 11 respectively defined as

$$M_1 = \begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1 \\
1&0&1&0&1&1&1&0&0&0&1&0 \\
1&0&0&1&0&1&1&1&0&0&0&1 \\
1&1&0&0&1&0&1&1&1&0&0&0 \\
1&0&1&0&0&1&0&1&1&1&0&0 \\
1&0&0&1&0&0&1&0&1&1&1&0 \\
1&0&0&0&1&0&0&1&0&1&1&1 \\
1&1&0&0&0&1&0&0&1&0&1&1 \\
1&1&1&0&0&0&1&0&0&1&0&1 \\
1&1&1&1&0&0&0&1&0&0&1&0 \\
1&0&1&1&1&0&0&0&1&0&0&1 \\
1&1&0&1&1&1&0&0&0&1&0&0
\end{pmatrix}$$

and

$$M_2 = \begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1 \\
1&0&1&0&1&1&1&0&0&0&1 \\
1&0&0&1&0&1&1&1&0&0&0 \\
1&1&0&0&1&0&1&1&1&0&0 \\
1&0&1&0&0&1&0&1&1&1&0 \\
1&0&0&1&0&0&1&0&1&1&1 \\
1&0&0&0&1&0&0&1&0&1&1 \\
1&1&0&0&0&1&0&0&1&0&1 \\
1&1&1&0&0&0&1&0&0&1&0 \\
1&1&1&1&0&0&0&1&0&0&1 \\
1&0&1&1&1&0&0&0&1&0&0
\end{pmatrix}.$$

From the coupled matrix M_1, we construct the following codes for (2.12):

The matrices

$$W_{12}^{1}=\begin{pmatrix}1&1&1&1&1&1&1&1&1&1&1&1\\1&0&1&0&1&1&1&0&0&0&1&0\\1&0&0&1&0&1&1&1&0&0&0&1\\1&1&0&0&1&0&1&1&1&0&0&0\\1&0&1&0&0&1&0&1&1&1&0&0\\1&0&0&1&0&0&1&0&1&1&1&0\\1&0&0&0&1&0&0&1&0&1&1&1\\1&1&0&0&0&1&0&0&1&0&1&1\\1&1&1&0&0&0&1&0&0&1&0&1\\1&1&1&1&0&0&0&1&0&0&1&0\\1&0&1&1&1&0&0&0&1&0&0&1\\1&1&0&1&1&1&0&0&0&1&0&0\end{pmatrix} \qquad \ldots(2.13)$$

and

$$W_{12}^{2}=\begin{pmatrix}0&0&0&0&0&0&0&0&0&0&0&0\\0&1&0&1&0&0&0&1&1&1&0&1\\0&1&1&0&1&0&0&0&1&1&1&0\\0&0&1&1&0&1&0&0&0&1&1&1\\0&1&0&1&1&0&1&0&0&0&1&1\\0&1&1&0&1&1&0&1&0&0&0&1\\0&1&1&1&0&1&1&0&1&0&0&0\\0&0&1&1&1&0&1&1&0&1&0&0\\0&0&0&1&1&1&0&1&1&0&1&0\\0&0&0&0&1&1&1&0&1&1&0&1\\0&1&0&0&0&1&1&1&0&1&1&0\\0&0&1&0&0&0&1&1&1&0&1&1\end{pmatrix}, \qquad \ldots(2.14)$$

give code (12, 24, 6; 2).
Now,

$$W_{12}^{3}=\begin{pmatrix}1&1&1&1&1&1&1&1&1&1&1\\0&1&0&1&1&1&0&0&0&1&0\\0&0&1&0&1&1&1&0&0&0&1\\1&0&0&1&0&1&1&1&0&0&0\\0&1&0&0&1&0&1&1&1&0&0\\0&0&1&0&0&1&0&1&1&1&0\\0&0&0&1&0&0&1&0&1&1&1\\1&0&0&0&1&0&0&1&0&1&1\\1&1&0&0&0&1&0&0&1&0&1\\1&1&1&0&0&0&1&0&0&1&0\\0&1&1&1&0&0&0&1&0&0&1\\1&0&1&1&1&0&0&0&1&0&0\end{pmatrix} \qquad \ldots(2.15)$$

And

$$W_{12}^{4}=\begin{pmatrix}0&0&0&0&0&0&0&0&0&0&0\\1&0&1&0&0&0&1&1&1&0&1\\1&1&0&1&0&0&0&1&1&1&0\\0&1&1&0&1&0&0&0&1&1&1\\1&0&1&1&0&1&0&0&0&1&1\\1&1&0&1&1&0&1&0&0&0&1\\1&1&1&0&1&1&0&1&0&0&0\\0&1&1&1&0&1&1&0&1&0&0\\0&0&1&1&1&0&1&1&0&1&0\\0&0&0&1&1&1&0&1&1&0&1\\1&0&0&0&1&1&1&0&1&1&0\\0&1&0&0&0&1&1&1&0&1&1\end{pmatrix} \qquad \ldots(2.16)$$

give code (11, 24, 5; 2) and the matrix (2.15) gives code (11,12,6;2).
Further,

$$W_{12}^{5}=\begin{pmatrix}1&0&1&1&1&0&0&0&1&0\\0&1&0&1&1&1&0&0&0&1\\1&0&0&1&0&1&1&1&0&0\\0&1&0&0&1&0&1&1&1&0\\0&0&1&0&0&1&0&1&1&1\\1&1&1&0&0&0&1&0&0&1\end{pmatrix}, \qquad \ldots(2.17)$$

which gives code (10,6,6;2).
Similarly,

$$W_{11}^{1}=\begin{pmatrix}1&1&1&1&1&1&1&1&1&1&1\\1&0&1&0&1&1&1&0&0&0&1\\1&0&0&1&0&1&1&1&0&0&0\\1&1&0&0&1&0&1&1&1&0&0\\1&0&1&0&0&1&0&1&1&1&0\\1&0&0&1&0&0&1&0&1&1&1\\1&0&0&0&1&0&0&1&0&1&1\\1&1&0&0&0&1&0&0&1&0&1\\1&1&1&0&0&0&1&0&0&1&0\\1&1&1&1&0&0&0&1&0&0&1\\1&0&1&1&1&0&0&0&1&0&0\end{pmatrix} \qquad \ldots(2.18)$$

and

$$W_{11}^{2}=\begin{pmatrix}0&0&0&0&0&0&0&0&0&0&0\\0&1&0&1&0&0&0&1&1&1&0\\0&1&1&0&1&0&0&0&1&1&1\\0&0&1&1&0&1&0&0&0&1&1\\0&1&0&1&1&0&1&0&0&0&1\\0&1&1&0&1&1&0&1&0&0&0\\0&1&1&1&0&1&1&0&1&0&0\\0&0&1&1&1&0&1&1&0&1&0\\0&0&0&1&1&1&0&1&1&0&1\\0&0&0&0&1&1&1&0&1&1&0\\0&1&0&0&0&1&1&1&0&1&1\end{pmatrix} \qquad \ldots(2.19)$$

give code (11, 24, 5;2).

Now,

$$W_{11}^{3}=\begin{pmatrix}1&1&1&1&1&1&1&1&1&1\\0&1&0&1&1&1&0&0&0&1\\0&0&1&0&1&1&1&0&0&0\\1&0&0&1&0&1&1&1&0&0\\0&1&0&0&1&0&1&1&1&0\\0&0&1&0&0&1&0&1&1&1\\0&0&0&1&0&0&1&0&1&1\\1&0&0&0&1&0&0&1&0&1\\1&1&0&0&0&1&0&0&1&0\\1&1&1&0&0&0&1&0&0&1\\0&1&1&1&0&0&0&1&0&0\end{pmatrix}. \qquad \ldots(2.20)$$

This matrix gives code (10, 12, 5; 2). Further,

$$W_{11}^{4}=\begin{pmatrix}1&0&1&1&1&0&0&0&1\\0&1&0&1&1&1&0&0&0\\1&0&0&1&0&1&1&1&0\\0&1&0&0&1&0&1&1&1\\0&0&1&0&0&1&0&1&1\\1&1&1&0&0&0&1&0&0\end{pmatrix}. \qquad \ldots(2.21)$$

This matrix gives code (9, 6, 5; 2).

III. CONCLUSION

In this paper, we have introduced Hadamard codes using Hadamard rhotrices over finite field $GF(2)$ which play inevitable role in coding theory and cryptography.

REFERENCES

[1]. Hedayat A. and Walis W. D., Hadamard matrices and their applications, *The Annals of Statistics*, **6**(6), 1184-1238, (1978).

[2]. Plotkin M., Binary codes with given of minimal distance, *Proceedings of Cybernetics*, **7**, 60-67, (1963).

[3]. Sylvester J.J., Thoughts on orthogonal matrices, simultaneous sign successions and tessellated pavements in two or more colors, with applications to Newton's rule, *Ornamental Tile-Work and the Theory of Numbers Phil. Mag.*, **34**, 461- 475, (1867).

[4]. Hadamard J., Resolution d'une question relative aux determinants. *Bull. Des Sciences Mathematiques*, **17**, 240-246, (1893).

[5]. Turyn R.J., Complex Hadamard matrices, structures and their applications, Gordon and Breach, New York, (1970).

[6]. Sarukhanyan H., Agaian S. and Astola K.E.J., Hadamard transforms. TICSP Series # 23, ISBN 952-15-1172-9, (2004).

[7]. Rao R.K. *et al.,* Hadamard matrix analysis and synthesis with applications to communications and signal/image processing, *Kluwer Academic Publishers*, Boston, London, 1997.

[8]. Levenshtein V.I., Application of Hadamard matrices on one coding problem, *Problems of Cybernetics*,(in Russian), **5**, 123-136, (1961).

[9]. Chang S. C. and Weldon E. J., Coding for T-user multiple-access channels, *IEEE Trans. on Information Theory*, **25**(6), 684-691,(1979).

[10]. Seberry J. and Yamada M., Hadamard matrices, sequences and block designs, Surveys in Contemporary Design Theory, *Wiley-Inter science Series in Discrete Mathematics*, John Wiley, New York, (1992).

[11]. Kimura H. and Ohmori H., Construction of Hadamard matrices of order 28, *Graphs and Combinatorics*, **2**, 247-257, (1986).

[12]. Koukouvinos C. and Seberry J., Constructing Hadamard matrices from orthogonal designs, *Australasian J. of Combinatorics*, **6**, 267-278, (1992).

[13]. Singh M.K., Sinha K.and Kageyama S., A construction of Hadamard matrices from BIBD ($2k^2$-2k+1,k,1) *Australasian J. of Combinatorics*, **26**, 93-97, (2002).

[14]. Singh M.K. and Manjhi P.K., Construction of Hadamard matrices from certain Frobenius groups, *Global J. of Computer Science and Technology,* **11**(10), 45-50, (2011).

[15]. Szollosi F., Construction, classification and parametrization of complex Hadamard matrices, Thesis, Central European University, Budapest, Hungary, (2011).

[16]. Sajadieh M., Dakhilalian M., Mala H. and Omoomi B., On construction of involutory MDS matrices from Vandermonde matrices, *Des. Codes and Crypt.*, **64**, 287-308, (2012).

[17]. Ajibade A. O., The concept of rhotrices in mathematical enrichment, *Int. J. Math. Educ. Sci. Tech.*, **34**(2), 175-179, (2003).

[18]. Sani B., The row-column multiplication of high dimensional rhotrices, *Int. J. Math. Educ. Sci. Tech.,* **38**(5), 657-662, (2007).

[19]. Sani B., An alternative method for multiplication of rhotrices, *Int. J. Math. Educ. Sci. Tech.*, **35**, 777-781, (2004).

[20]. Sani B., Conversion of a rhotrix to a coupled matrix, *Int. J. Math. Educ. Sci. Tech.*, **39**, 244-249, (2008).

[21]. Aminu A., Rhotrix vector spaces, *Int. J. Math. Educ. Sci. Tech*., **41**(4), 531-573, (2010).

[22]. Tudunkaya S.M. and Makanjuola S.O., Rhotrices and the construction of finite fields, *Bulletin of Pure and Applied Sciences*, **29E**(2), 225-229, (2010).

[23]. Sharma P.L. and Kanwar R. K., Adjoint of a rhotrix and its basic properties, *International J. Mathematical Sciences*, **11**(3-4), 337-343, (2012).

[24]. Sharma P.L. and Kanwar R.K., On inner product space and bilinear forms over rhotrices, *Bulletin of Pure and Applied Sciences*, **31E** (1), 109-118, (2012).

[25]. Sharma P.L. and Kanwar R.K., A note on relationship between invertible rhotrices and associated invertible matrices, *Bulletin of Pure and Applied Sciences*, **30 E** (Math & Stat.) (2), 333-339, (2011).

[26]. Sharma P.L. and Kanwar R.K., The Cayley-Hamilton theorem for rhotrices, *International J. Mathematics and Analysis*, **4**(1), 171-178, (2012).

[27]. Sharma P.L. and Kanwar R.K., On involutory and Pascal rhotrices, *International J. of Math. Sci. & Engg. Appls.* (IJMSEA), **7**(IV), 133-146, (2013).

[28]. Kanwar R. K., A study of some analogous properties of rhotrices and matrices, Ph. D. Thesis, Department of Mathematics and Statistics, Himachal Pradesh University, Shimla, India, (2013).

[29]. Sharma P.L., Kumar S. and Rehan M., On Hadamard rhotrix over finite field, *Bulletin of pure and Applied Sciences*, **32 E** (*Maths. & Stat*.) (2), 181-190, (2013).