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Quaternionic Formulation of Gravito-Dyonic Field Equations

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ABSTRACT: Defining the quaternion in terms of Pauli spin matrices we have reformulated the generalized wave function, generalized four-potential, generalized current, Lorentz force equation of motion associated with gravito-dyons in simple and compact quaternion notations. It has been shown that quaternionic form of generalized potential generalized current and Lorentz force equation of motion not only compact and simple but also remain invariant under quaternion transformations and as such the theory is manifestly covariant. Generalized Maxwell-Dirac equation, generalized field tensor and continuity equation have also been reformulated by means of consistent quaternion notation in simple and compact forms. It has been shown that the reformulation of quantum equations by means of quaternions of gravito-dyons reproduces the theory of gravitational mass in the absence of Heavisidian mass gravito-dyons.

Keywords: Generalized Maxwell-Dirac equation, Lorentz force equation of motion, reformulation of quantum equations

I. INTRODUCTION

The asymmetry between electricity and magnetism became very clear at the end of nineteenth century with the formulation of Maxwell's equations for electromagnetism. Quaternions were very first example of hyper-complex numbers having the significant impact of mathematics and physics. Because of their beautiful and unique properties quaternions, quaternions attracted many to study the laws of nature over the fields of these numbers. Recently, there has been a revival in the formulation of natural laws within the framework of general quaternion algebra and many of the basic physical equations have been reformulated by means of quaternions [2]. Maxwell's equations are reformulated in terms of quaternions [3-6].
In order to understand the theoretical existence of monopoles (dyons) and keeping in view of their recent potential importance, with the fact that formalism necessary to describe has been clumsy and not manifestly covariant the quaternionic form of generalized field of dyons have been developed in unique, simple, compact and consistent manner [7]. Rajput et al.[8] has developed the unique consistent quaternionic formulation for dyons, which reproduces to usual electrodynamics in absence of magnetic charge. Postulating the existence of the Heavisidian monopoles and taking the linear equation for gravito-Heavisian field, Rajput [9] has been demonstrated the structural symmetry between the generalized electromagnetic field associated with gravito-dyons.
Keeping in view the interest in linear gravity and the structural symmetry between generalized gravito-Heavisidian and electromagnetic field in this paper we have reformulated the quantum equations of gravito-dyons in terms of simple, compact and consistent representation of quaternions. Defining the quaternion in terms of Pauli spin matrices we have reformulated the generalized wave function, generalized four-potential, generalized current, Lorentz force equation of motion associated with gravito-dyons in simple and compact quaternion notations. It has been shown that quaternionic form of generalized potential generalized current and Lorentz force equation of motion not only compact and simple but also remain invariant under quaternion transformations and as such the theory is manifestly covariant. Generalized Maxwell-Dirac equation, generalized field tensor and continuity equation have also been reformulated by means of consistent quaternion notation in simple and compact forms. It has been shown that the reformulation of quantum equations by means of quaternions of gravito-dyons reproduces the theory of gravitational mass in the absence of Heavisidian mass gravito-dyons.

II. GRAVITO-DYONS AND THEIR FIELD EQUATIONS

Let us define the generalized charge associated with gravio-Heavisidian fields (gravito-dyons) as complex quantity by

$$q \;=\; m \;-\; ih \qquad \ldots(1)$$

where m and h are gravitational and Heavisidian charges (masses) respectively.
Then generalized fields of gravito-dyons can be expressed as

$$\vec{G} = -\frac{\partial \vec{a}}{\partial t} - \vec{\nabla}\Phi_g + \vec{\nabla}\times\vec{b} \quad \& \quad \vec{M} = -\frac{\partial \vec{b}}{\partial t} - \vec{\nabla}\Phi_M - \vec{\nabla}\times\vec{a} \qquad \ldots(2)$$

where $\vec{G}$ and $\vec{M}$ represents gravitational and gravi-magnetic fields respectively. We can also define the generalized vector field as $\vec{\psi} \;=\; \vec{G} - i\,\vec{M}$ …(3)
which satisfy the following form of equation

$$\vec{\nabla}.\vec{\psi} \;=\; J_0 \quad \& \quad \vec{\nabla}\times\vec{\psi} \;=\; -i\frac{\partial \vec{\Psi}}{\partial t} - i\vec{J} \qquad \ldots(4)$$

where J_0 and $\vec{J}$ are the temporal and vector components of generalized four-current associated with gravito-dyons *i.e.*

$$J_\mu^{GH} = j_\mu^{G} - i\, k_\mu^{H} \qquad \ldots(5)$$

The generalized four-potential associated with gravito-dyons may be written as

$$V_\mu = a_\mu - i\, b_\mu \qquad \ldots(6)$$

The generalized gravito-Heaviseadian field tensor for gravito-dyons is given as

$$G_{\mu\nu} = F_{\mu\nu} - i\, \tilde{F}_{\mu\nu} \qquad \ldots(7)$$

Which gives the following covariant form of linear field equation

$$G_{\mu\nu;\nu} = -J_\mu^{GH} \quad \& \quad \tilde{G}_{\mu\nu;\nu} = 0 \qquad \ldots(8)$$

and reduces to following form of generalized Maxwell's – Dirac equation in presence of gravito-dyons associated with generalized four-potential i.e.

$$\Box V_\mu = -J_\mu^{GH} \qquad \ldots(9)$$

where $\Box$ represents the D'Alembertian operator and is given as

$$\Box = \frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \qquad \ldots(9a)$$

The Lorentz force equation of motion for gravito-dyon may be written as

$$f_\mu = m_o \frac{d^2 x_\mu}{d\tau^2} = \mathrm{Re}\,(q^* G_{\mu\nu}\, v^\nu) \qquad \ldots(10)$$

where v^ν is the four velocity and M is the effective mass of gravito-dyon given by M = m + h.

III. QUATERNION ANALYSIS OF GRAVITO-DYONIC FIELDS

The complex vector field $\vec{\Psi}^{GH}$ may be written as

$$\vec{\Psi}^{GH} = \vec{G} - i\vec{H} = \left(\frac{\partial \vec{a}}{\partial t} + \vec{\nabla}\Phi_g + \vec{\nabla}\times\vec{b}\right) - i\left(\frac{\partial \vec{b}}{\partial t} + \vec{\nabla}\Phi_h - \vec{\nabla}\times\vec{a}\right)$$

$$= \frac{\partial}{\partial t}\left(\vec{a} + i\vec{b}\right) - \vec{\nabla}\left(\Phi_g - i\Phi_h\right) + i\vec{\nabla}\times\left(\vec{a} - i\vec{b}\right) = \frac{\partial \vec{V}}{\partial t} + \vec{\nabla}V_0 + i\vec{\nabla}\times\vec{V}$$

$$\Rightarrow \quad \vec{\Psi}^{GH} = \frac{\partial \vec{V}}{\partial t} + grad\ V_0 + i\ curl\ \vec{V} \qquad \ldots(11)$$

where $\vec{V}$ and V_0 are sptatial and temporal components of generalized four potentials associated to gravito–dyons. Using equation (4) and the property of vector triple product we have

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{\Psi}) = \vec{\nabla}\times\left(-i\frac{\partial \vec{\Psi}}{\partial t} + i\vec{J}\right) \Rightarrow -\vec{\nabla}J_0 - \nabla^2\vec{\Psi} = -\frac{\partial^2 \vec{\Psi}}{\partial t^2} + \frac{\partial \vec{J}}{\partial t} + i\vec{\nabla}\times\vec{J}$$

$$\Rightarrow \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{\Psi} = \frac{\partial \vec{J}}{\partial t} + \vec{\nabla}J_0 + i\vec{\nabla}\times\vec{J}$$

$$\Rightarrow \quad \vec{\Psi} = \frac{\partial \vec{J}}{\partial t} + grad\ J_0 + i\ curl\ \vec{J} = S \text{ (say)} \qquad \ldots(12)$$

Thus from equation (11) and (12) it is clear that the role of V in Ψ is same as the role of J in $\vec{\Psi}$. Therefore J must be related with V by the same operator which can be shown with the help of quaternionic formulation.

In order to write the field equation (11) and (12) in quaternionic form for gravito-dyons we define the quaternion as

$$q = q_0 + \sigma_1 q_1 + \sigma_2 q_2 + \sigma_3 q_3 \qquad \ldots(13)$$

where $\sigma_i = ie_i$, $\sigma_i^2 = 1$ & $\sigma_i \sigma_{j_i} = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$.

The four–potential $\{V_\mu\}$ and four–current $\{J_\mu\}$ of gravi-magnetic field may be written in the following form

$$V = V_0 + \sigma_1 V_1 + \sigma_2 V_2 + \sigma_3 V_3 \qquad \ldots(14)$$

and

$$J = J_0 + \sigma_1 J_1 + \sigma_2 J_2 + \sigma_3 J_3 \qquad \ldots(15)$$

The quaternion differential operator D can be written as

$$D = \partial_0 + \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3 \qquad \ldots(16)$$

And the quaternion conjugate of V, J and D are defined as

$$\bar{V} = V_0 - \sigma_1 V_1 - \sigma_2 V_2 - \sigma_3 V_3$$
$$\bar{J} = J_0 - \sigma_1 J_1 - \sigma_2 J_2 - \sigma_3 J_3 \qquad \ldots(17)$$
$$\bar{D} = \partial_0 - \sigma_1\partial_1 - \sigma_2\partial_2 - \sigma_3\partial_3$$

From these equations we have

$$\begin{aligned} D\,V &= (\partial_0 + \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3)(\partial_0 V_0 + \partial_1 V_1 + \partial_2 V_2 + \partial_3 V_3) \\ &= (\partial_0 V_0 + divV) + \sigma_1\{\partial_0 V_1 + \partial_1 V_0 + i(\nabla\times V)_1\} \\ &\quad + \sigma_2\{\partial_0 V_2 + \partial_2 V_0 + i(\nabla\times V)_2\} + \sigma_3\{\partial_0 V_3 + \partial_3 V_0 + i(\nabla\times V)_3)\} \\ &= \Psi_0 + \sigma_1\Psi_1 + \sigma_2\Psi_2 + \sigma_3\Psi_3 = \Psi \qquad \ldots(18) \end{aligned}$$

where $\Psi_0 = \partial_0 V_0 + div\ \vec{V} = 0$ (due to Lorentz gauge condition) and $\Psi_j = \partial_0 V_j + \partial_j V_0 + i(\vec{\nabla}\times\vec{V})_j$. Also

$$\begin{aligned} D\,J &= (\partial_0 + \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3)(\partial_0 J_0 + \partial_1 J_1 + \partial_2 J_2 + \partial_3 J_3) \\ &= (\partial_0 J_0 + divJ) + \sigma_1\{\partial_0 J_1 + \partial_1 J_0 + i(\nabla\times J)_1\} \\ &\quad + \sigma_2\{\partial_0 J_2 + \partial_2 J_0 + i(\nabla\times J)_2\} + \sigma_3\{\partial_0 J_3 + \partial_3 J_0 + i(\nabla\times J)_3)\} \\ &= S_0 + \sigma_1 S_1 + \sigma_2 S_2 + \sigma_3 S_3 = S \qquad \ldots(19) \end{aligned}$$

where $S_0 = \partial_0 J_0 + divJ = 0$ (due to continuity equation) and $S_j = \partial_0 J_j + \partial_j J_0 + i(\nabla\times J)_j$

Now operating Ψ by $\bar{D}$ we have

$$\bar{D}\,\Psi = (\partial_0 - \sigma_1\partial_1 - \sigma_2\partial_2 - \sigma_3\partial_3)\ (\sigma_1\Psi_1 + \sigma_2\Psi_2 + \sigma_3\Psi_3)$$

($\Psi_0 = 0$ due to Lorentz condition)

$$\begin{aligned} &= -\ \partial_1\Psi_1 - \partial_2\Psi_2 - \partial_3\Psi_3 + \sigma_1\{\partial_0\Psi_1 - i(\partial_2\Psi_3 - \partial_3\Psi_2)\} \\ &\quad + \sigma_2\{\partial_0\Psi_2 - i(\partial_3\Psi_1 - \partial_1\Psi_3)\} + \sigma_3\{\partial_0\Psi_3 - i(\partial_1\Psi_2 - \partial_2\Psi_1)\} \\ &= -\vec{\nabla}\cdot\vec{\Psi} + \sigma_1\{\partial_0\Psi_1 - i(\nabla\times\Psi)_1\} + \sigma_2\{\partial_0\Psi_2 - i(\nabla\times\Psi)_2\} + \sigma_3\{\partial_0\Psi_3 - i(\nabla\times\Psi)_3\} \\ &= J_0 + \sigma_1 J_1 + \sigma_2 J_2 + \sigma_3 J_3 = J \qquad \ldots.(20) \end{aligned}$$

Similarly

$$\begin{aligned} \overline{(DV)} &= (V_0 - \sigma_1 V_1 - \sigma_2 V_2 - \sigma_3 V_3)(\partial_0 - \sigma_1\partial_1 - \sigma_2\partial_2 - \sigma_3\partial_3) \\ &= (\partial_0 V_0 + divV) - \sigma_1\{\partial_0 V_1 + \partial_1 V_0 + i(\nabla\times V)_1\} \\ &\quad - \sigma_2\{\partial_0 V_2 + \partial_2 V_0 + i(\nabla\times V)_2\} - \sigma_3\{\partial_0 V_3 + \partial_3 V_0 + i(\nabla\times V)_3)\} \\ &= \Psi_0 - \sigma_1\Psi_1 - \sigma_2\Psi_2 - \sigma_3\Psi_3 = \bar{\Psi} \qquad (21) \end{aligned}$$

Likewise

$$\begin{aligned} \overline{(DJ)} &= (\partial_0 J_0 - \sigma_1 J_1 - \sigma_2 J_2 - \sigma_3 J_3)(\partial_0 - \sigma_1\partial_1 - \sigma_2\partial_2 - \sigma_3\partial_3) \\ &= \partial_0 J_0 + divJ - \sigma_1\{\partial_0 J_1 + \partial_1 J_0 + i(\nabla\times J)_1\} \\ &\quad - \sigma_2\{\partial_0 J_2 + \partial_2 J_0 + i(\nabla\times J)_2\} - \sigma_3\{\partial_0 J_3 + \partial_3 J_0 + i(\nabla\times J)_3)\} \\ &= S_0 - \sigma_1 S_1 - \sigma_2 S_2 - \sigma_3 S_3 = \bar{S} \qquad \ldots.(22) \end{aligned}$$

Now J and S can be expressed in terms of V and Ψ as follows:

The temporal component of J can be written as

$$\begin{aligned} J_0 &= -\vec{\nabla}\cdot\vec{\Psi} = \partial_1\Psi_1 - \partial_2\Psi_2 - \partial_3\Psi_3 \\ &= -\ \partial_0(\partial_1 V_1 + \partial_2 V_2 + \partial_3 V_3) - \partial_1^2 V_0 - \partial_2^2 V_0 - \partial_3^2 V_0 = -\ \partial_0(divV) - (\partial_1^2 + \partial_2^2 + \partial_3^2)V_0 \\ &= -\ \partial_0(\partial_0 V_0) - (\nabla^2)V_0 \qquad [\ \because \partial_0 V_0 + divV = 0\ ] \\ &= \partial_0^2 V_0 - \nabla^2 V_0 = (\partial_0^2 - \nabla^2)V_0 = V_0 \end{aligned}$$

Thus the temporal components of J is related with the temporal component of V.

The spatial component of J can be expressed in terms of Ψ and V as

$$J_1 = \partial_0 \Psi_1 - i(\nabla \times \Psi)_1 \quad = -\partial_0\{\partial_0 V_1 + \partial_1 V_0 + i(\nabla \times V)_1\} - i\{\partial_2 \Psi_3 - \partial_3 \Psi_2\}$$

$$= -\partial_0^2 V_1 + \partial_0 \partial_1 V_0 + i\partial_0(\partial_2 V_3 - \partial_3 V_2) - i\partial_2\{\partial_0 V_3 + \partial_2 V_0 + i(\partial_1 V_2 - \partial_3 V_1)\}$$

$$+ i\partial_3\{\partial_0 V_2 + \partial_2 V_0 + i(\partial_3 V_1 - \partial_1 V_3)\}$$

$$= \partial_0^2 V_1 - \partial_2^2 V_1 - \partial_3^2 V_1 - \partial_1(\partial_0 V_0 + \partial_2 V_2 + \partial_3 V_3)$$

$$= \partial_0^2 V_1 - \partial_1^2 V_1 - \partial_2^2 V_1 - \partial_3^2 V_1 - \partial_1(\partial_0 V_0 + \partial_1 V_1 + \partial_2 V_2 + \partial_3 V_3)$$

$$= \quad V_1 + \partial_1(\partial_0 V_0 + divV) \quad = \quad V_1 \qquad [\because \partial_0 V_0 + divV = 0]$$

Similarly $\quad J_2 = \quad V_2 \quad \& \quad J_3 = \quad V_3$

Thus the spatial components of J are related with the spatial components of V. Therefore we can write

$$V = J \qquad \text{…(23)}$$

which is similar to field equation (9). Similarly we can express S in terms of Ψ as follows.

The temporal component of S i.e. S_0 is zero due to continuity equation. The spatial components of S are expressed as

$$S_1 = \partial_0 J_1 + \partial_1 J_0 + i(\nabla \times J)_1$$

$$= \partial_0\{\partial_0 \Psi_1 - i(\nabla \times \Psi)_1\} + \partial_1(-\nabla \cdot \Psi)$$

$$+ i[\partial_2\{-\partial_0 \Psi_3 + i(\nabla \times \Psi)_3\} - \partial_3\{-\partial_0 \Psi_2 + i(\nabla \times \Psi)_2\}]$$

$$= \partial_0^2 \Psi_1 - i\partial_0 \partial_2 \Psi_3 + i\partial_0 \partial_3 \Psi_2 - \partial_1^2 \Psi_1 \quad - \partial_1 \partial_3 \Psi_3 - i\partial_2 \partial_0 \Psi_3 + \partial_2 \partial_1 \Psi_2 - \partial_2^2 \Psi_1$$

$$- i\partial_3 \partial_0 \Psi_2 - \partial_3^2 \Psi_1 + \partial_3 \partial_1 \Psi_3 - \partial_1 \partial_2 \Psi_2$$

$$= \partial_0^2 \Psi_1 - \partial_1^2 \Psi_1 - \partial_2^2 \Psi_1 - \partial_3^2 \Psi_1 = (\partial_0^2 - \nabla^2)\Psi_1 \quad = \quad \Psi_1$$

Similarly $\quad S_2 = \quad \Psi_2 \quad \& \quad S_3 = \quad \Psi_3$

Therefore the spatial components of S are related with the spatial components of Ψ and we can write

$$\Psi = S \qquad \text{…(24)}$$

The generalized gravito-Heavisidian field tensor may be expressed in quaternionic form as

$$G^{gh} = \sigma_0 G_0^{gh} + \sigma_1 G_1^{gh} + \sigma_2 G_2^{gh} + \sigma_3 G_3^{gh} \qquad \text{…(25)}$$

where

$$G_i = \sigma_0 G_{i0} + \sigma_1 G_{i1} + \sigma_2 G_{i2} + \sigma_3 G_{i3} \quad (i = 0,1,2,3) \qquad \text{…(25a)}$$

As $G_{\mu\nu}$ is anti-symmetric field tensor therefore

$$G_{00} = G_{11} = G_{22} = G_{33} = 0$$

Now operating G_0 by quaternion differential operator D and $\overline{D}$ by $\overline{G_0}$, we have

$$D\, G_0 = (\partial_0 + \sigma_1 \partial_1 + \sigma_2 \partial_2 + \sigma_3 \partial_3)(\sigma_1 G_{01} + \sigma_2 G_{02} + \sigma_3 G_{03})$$

$$\overline{G_0}\, \overline{D} = (-\sigma_1 G_{01} - \sigma_2 G_{02} - \sigma_3 G_{03})(\partial_0 - \sigma_1 \partial_1 - \sigma_2 \partial_2 - \sigma_3 \partial_3)$$

Since $G_{\mu\nu;\nu} = -J_\mu$ for gravito-dyons, therefore

$$D\, G_0 + \overline{G_0}\, \overline{D} = 2[\partial_1 G_{01} + \partial_2 G_{02} + \partial_3 G_{03}] = -2\, J_0$$

$$\Rightarrow \quad [\overline{D}, G_0] = -J_0 \qquad \text{…(26a)}$$

Similarly

$$[\overline{D}, G_1] = -J_1 \quad , \quad [\overline{D}, G_2] = -J_2 \quad \& \quad [\overline{D}, G_3] = -J_3 .$$

Also

$$D\, G + \overline{G}\, \overline{D} = 2[\partial_0 G_0 + \partial_1 G_1 + \partial_2 G_2 + \partial_3 G_3] = -2\, J \qquad \text{…(27)}$$

Thus we may write the equation $G_{\mu\nu,\nu} = -J_\mu$ in quaternion form as

$$[\overline{D}, G] = -J \qquad \text{…(28)}$$

Also

$$D\,J + \overline{J}\,\overline{D} \;=\; 2[\partial_0 J_0 + \partial_1 J_1 + \partial_2 J_2 + \partial_3 J_3]$$
$$= 2[\,\partial_0 J_0 + divJ\,] \;=\; 0 \text{ (due to continuity equation)}$$

Thus

$$[\,\overline{D}\,,\,J\,] \;=\; 0 \qquad \ldots(29)$$

which is continuity equation in quaternionic form. The quaternionic form of force equation is described as

$$f \;=\; q[\,v\,,G\,] \;=\; \tfrac{1}{2}\,q[\,\overline{v}\,G + \overline{G}\,v\,] \qquad \ldots(30)$$

IV. DISCUSSION

We have obtained the relation between the components of generalized four-current with the generalized field associated with gravito-dyons . The electromagnetic four potentials and four-currents have been expressed in quaternion form . We have also written the differential operator in quaternion form and operated the four potentials and four-currents by quaternion differential operator. After using Lorentz gauge condition and continuity equation we have obtained the equations in quaternionic form of generalized potential and generalized four-current of dyons respectively. As such these two equations are quantum equations respectively for generalized four-potential and generalized four-current of gravito-dyons. These equations are the short hand notation and written in compact and consistent way. These two equations remain invariant under quaternion transformations and as such the quaternionic formulation becomes manifestly covariant. We have obtained the relations between gravito-Heavisidian field and the components of generalized four-current of gravito-dyons and hence establishes the quaternionic form of generalized Maxwell-Dirac equation of gravito-dyons in simple, compact and consistent manner. On the other hand, we have describe the quaternion conjugate forms of generalized potential and current of gravito-dyons. The generalized field tensor, four-velocity and four force have been written is quaternionic form. The components of field tensor have also been written in quaternionic form. The Lorentz force equation has also been expressed in quaternionic form.

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