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Common Fixed Point Theorems for Occasionally Weakly Compatible Mappings in Cone Metric Space

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ABSTRACT: In this paper, we prove common fixed point theorems for occasionally weakly compatible mappings satisfying general contractive condition in cone metric spaces. Our result extend and generalize some results of Prudhvi [11], Chen and Chen [4].

Key words. Cone metric space, occasionally weakly compatible, coincidence point, fixed point.
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I. INTRODUCTION AND PRELIMINARIES

In 2007 Huang and Zhang [6] have generalized the concept of metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying various contractive conditions. Many authors study this subject and many fixed point theorems are proved. For example [1,7,12,5,9]. Jungck [8] gave a common fixed point theorem for commuting mappings, which generalizes the Banach's fixed point theorem and he also introduced the concept of compatible maps which is weaker than weakly commuting maps. Further this result was generalized by Pant [10], Amari and Moutawakil [3], Jungck [9] Al-Thagafi and Shahzad [2]. In [2] author defend the concept of occasionally weakly compatible which is more general than the concept of weakly compatible maps.

The following definitions are due to Huang and Zhang [6].

Definition 1.1. Let E be a real Banach space and P be a subset of E. P is called a cone if;

(a) P is closed, nonempty and $P \neq \{0\}$;

(b) a, b ∈ R, a b ≥ 0, $x, y \in P \Longrightarrow ax + by \in P$;

(c) $x \in P$ and $-x \in P \Longrightarrow x = 0$

Given a cone P ∈ E, we define a partial ordering "$\leq$ " with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to denote $x \leq y$ but $x \neq y$ and $x \ll y$ to denote $y - x \in P^o$, where P^o stands for the interior of P.

The cone P is called normal if there is a number K > 0 such that for all x,y ∈ E,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|$$

The least positive number satisfying above is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$

For some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

In the following we always suppose E is a Banach space, P is a cone in E with

P ≠ ∅ and ≤is partial ordering with respect to P.

Definition 1.2. A cone metric space is an ordered pair (X,d), where X is any set and d : X×X→ E is a mapping satisfying :

(a) 0< d (x, y) for all x, y ∈X and d (x, y) = 0 if and only if x = y,

(b) d (x, y) = d (y, x) for all x, y∈ X

(c) d (x, y) ≤ d (x, z) + d (z, y) for all x ,y ,z∈ X.

Definition 1.3. Let (X,d) be a cone metric space $\{x_n\}$ a sequence in X and $x \in X$. If for any c∈ E with $c \gg 0$, there is N such that for all n>N, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converge to x. i.e $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 1.4. Let (X,d) be a cone metric space $\{x_n\}$ a sequence in X, if for any c E with c>>0, there is N such that for all n, m >N, d $(x_m, x_n) << c$, then $\{x_n\}$ is called Cauchy sequence in X.

Lemma 1.1[13]. Let (X,d) be a cone metric space, P a normal cone with a normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $y_n \to y, x_n \to x$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

Lemma 1.2[13]. Let (X,d) be a cone metric space, P a normal cone with a normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converge to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

Lemma 1.3[13]. Let (X, d) be a cone metric space, P a normal cone with a normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.

Definition 1.5[11]. Let X be a set and let f,g be two self-mappings of X. A point x in X is called a coincidence point of f and g if and only if fx=gx, We shall call w=fx=gx a point of coincidence point.

Definition 1.6 [11]. Two self-maps f and g of a set X are occasionally weakly compatible (owc) if and only if there is a point x in X which is a coincidence point of f and g at which f and g commute.

Lemma 1.4 [11]. Let X be a set f, g owcself mappings of X. If f and g have a unique point of coincidence, w=fx=gx, then w is the unique common fixed point of f and g.

II. MAIN RESULT

In this section we extand and generalize some results of Prudhvi [11], Chen and Chen[4] and also we prove common fixed point theorems for occasionally weakly compatible mappings satisfying general contractive condition in cone metric spaces.

Theorem 2.1. Let (X,d) be a complete cone metric space and P be a normal cone. Let A, B, S, T: X → X be mappings such that

(i) A(X)T(X) and B(X)S(X)

(ii) The pairs (A,S) and (B,T) are occasionally weakly compatible.

and satisfying the following condition

$d(Ax,By) \leq a_1 d(Sx,Ty) + a_2 d(Ax,Sx) + a_3 d(By,Ty) + a_4 d(Sx,By) + a_5 d(Ax,Ty)$ … (1)

for all x,y ϵX and $\varphi : R_+ \to R_+$ continuous. Then A, B, S, and T have unique common fixed point.

Proof. Since the pairs (A,S) and (B,T) are occasionally weakly compatible, there exists a point u X such that

Au = Su and AAu = ASu = SAu = SSu …(2)

And also w X such that

Bw = Tw and BBw = TBw = BTw = TTw …(3)

Since A(X) T(X), there exist a point w X such that

Au = Tw …(4)

Also B(X) S(X), there exist a point u X such that

Bw = Su …(5)

Now we claim that Au is the unique common fixed point of A and S. First we assert that Au is a fixed point of A. If AAu ≠ Au, then by (6), we have

d(AAu,Au) = d(AAu,Su)

=d(AAu,Bw)

$\leq a_1 d(SAu,Tw) + a_2 d(AAu,SAu) + a_3 d(Bw,Tw) + a_4 d(SAu,Bw) + a_5 d(AAu,Tw)$

$= a_1 d(AAu,Bw) + 0 + 0 + a_4 d(AAu,Bw) + a_5 d(AAu,Tw)$

$= a_1 d(AAu,Au) + a_4 d(AAu,Au) + a_5 d(AAu,Au)$

$= (a_1+a_4+a_5)\, d(AAu,Au)$

which is a contradiction. Hence Au is a fixed point of A. By (2) Au is a common fixed point of A and S.

Now we claim that Bw is the unique common fixed point of B and T and we assert that Bw is a fixed point of B. If BBw ≠ Bw, then by (1), we have

d(BBw,Bw) < d(BBw,Bw)

which is a contradiction. Hence Bw is a fixed point of B. By (3) Bw is a common fixed point of B and T.

Now by (2) and (5) we have Au = Su = Bw.

Hence Au = Bw is a common fixed point A, B, S and T.

For the uniqueness, let v be another common fixed point of A, B, S and T.

Let Au = Su = u and Bv = Tv = v.

If u ≠ v, then from (6), we have

d(u,v) = d(Au,Bv)

$\leq a_1 d(Su,Tv) + a_2 d(Au,Su) + a_3 d(Bv,Tv) + a_4 d(Su,Bv) + a_5 d(Au,Tv)$

\’ $= a_1 d(u,v) + 0 + 0 + a_4 d(u,v) + a_5 d(u,v)$

$= (a_1+a_4+a_5) d(Au,Tv)$, a contradiction.

Therefore, u = v. Hence A, B, S, and T have unique common fixed point.

Theorem 2.2. Let (X,d) be a complete cone metric space and P be a normal cone. Let A, B, S, T: X → X be mappings such that

(i) A(X)T(X) and B(X)S(X)

(ii) The pairs (A,S) and (B,T) are occasionally weakly compatible.

and satisfying the following condition,

$d(Ax,By) \leq \varphi(g(x,y))$ …(6)

where $g(x,y)=d(Sx,Ty)+\Upsilon(d(Sx,Ax)+d(Ty,By))$, for all x,y ϵ X and $\varphi: R_+ \to R_+$ continuous.

Then A, B, S, and T have unique common fixed point.

Proof. Since the pairs (A,S) and (B,T) are occasionally weakly compatible, there exists a point u X such that

Au = Su and AAu=ASu=Sau=SSu …(7)

And also w X such that

Bw = Tw and BBw = TBw = BTw = TTw ...(8)

Since A(X) T(X), there exist a point w X such that

Au = Tw …(9)

Also B(X) □S(X), there exist a point u □ X such that

Bw = Su ...(10)

Now we claim that Au is the unique common fixed point of A and S. First we assert that Au is a fixed point of A. If AAu ≠Au, then by (6), we have

d(AAu,Au) = d(AAu,Su)

=d(AAu,Bw)

$\leq \varphi$ (g(Au,u))

$\leq \varphi$ (d(SAu,Tw)+ Υ[d(SAu,AAu)+d(Tw,Bw)])

$\leq \varphi$ (d(AAu,Au)+ Υ[0+0])

$\leq \varphi$ (d(AAu,Au))

<d(AAu,Au)

which is a contradiction. Hence Au is a fixed point of A. By (7) Au is a common fixed point of A and S.

Now we claim that Bw is the unique common fixed point of B and T and we assert that Bw is a fixed point of B. If BBu≠Bu, then by (6), we have

d(BBw,Bw)<d(BBw,Bw)

which is a contradiction.Hence Bw is a fixed point of B. By (8) Bw is a common fixed point of B and T.

Now by (7) and (10) we have Au = Su = Bw.

Hence Au = Bw is a common fixed point A, B, S and T.

For the uniqueness, let v be another common fixed point of A, B, S and T.

Let Au = Su = u and Bv = Tv = v.

If u ≠ v, then from (6), we have

d(u,v) = d(Au,Bv)

$\leq \varphi$ (g(u,v))

\' = φ (d(Su,Tv)+ Υ[d(Su,Au)+d(Tv,Bv)])

= φ (d(Au,Bv)+ Υ[0+0])

= φ (d(u,v)) < φ (d(u,v)), a contradiction.

Therefore, u = v. Hence A, B, S, and T have unique common fixed point.

Corollary 2.1. Let (X,d) be a complete cone metric space and P be a normal cone. Let A, B, S, T: X→X be mappings such that

(i) A(X)□T(X) and B(X)□S(X)

(ii) The pairs (A,S) and (B,T) are weakly compatible.

and satisfying the following condition

$d(Ax,By) \leq a_1d(Sx,Ty) + a_2d(Ax,Sx) + a_3d(By,Ty) + a_4d(Sx,By) + a_5d(Ax,Ty)$ (11)

for all x,y ϵX and φ :$R_+ \to R_+$ continuous. Then A, B, S, and T have unique common fixed point.

Proof. Since weakly compatible maps are occasionally weakly compatible. Therefore the result follows from theorem 2.1.

Theorem 2.2. Let (X,d) be a complete cone metric space and P be a normal cone. Let A, B, S, T: X→X be mappings such that

(i) A(X)□T(X) and B(X)□S(X)

(ii) The pairs (A,S) and (B,T) are compatible.

and satisfying the following condition,

d(Ax,By)≤ φ(g(x,y)) ...(12)

where g(x,y)=d(Sx,Ty)+Υ(d(Sx,Ax)+d(Ty,By)), for all x,y ϵ X and φ:$R_+ \to R_+$continuous. Then A, B, S, and T have unique common fixed point.

Proof. Since weakly compatible maps are occasionally weakly compatible. Therefore the result follows from theorem 2.2.

Remark. If we put S = T = g and A = B = f in theorem 2.1 and theorem 2.2 we get theorem 2.1 and theorem 2.2 of Prudhvi [11].

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