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Numerical Solution of Burger's Equation using efficient computational Technique using Cubic B-splines

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ABSTRACT: A numerical technique is presented to solve nonlinear parabolic Burgers' equation arising in unsteady flow of the generalized Newtonian fluid. Cubic B-spline functions are used for discretization along with the zeros of Chebyshev polynomial as collocation points. Using the operational matrix of derivative, the problem reduces to a set of differential algebraic equations, which is solved by MATLAB ode15s solver. The practical example is included to demonstrate the validity and applicability of technique.

Keywords: Cubic B-spline; Burgers' equation; Orthogonal collocation; Chebyshev polynomial.

I. INTRODUCTION

Consider the one-dimensional Burgers' equation

$$U_t + UU_x = \lambda U_{xx},\ \ a \le x \le b,\ t \ge 0, \ldots(1)$$

with the initial condition

$$U(x,0) = f(x),\ a \le x \le b, \ldots(2)$$

and the boundary condition

$$U(a,t) = \beta_1,\ \ U(b,t) = \beta_2, \ldots(3)$$

where $\lambda > 0$ is the coefficient of kinematics viscosity, and β_1, β_2 and $f(x)$ will be chosen in a later section. This is a nonlinear parabolic equation and describes in a simple manner a balance between nonlinear convection and linear diffusion or dissipation. Burger's equation is well-known to show shock formation. It is the 1D Navier-Stokes equation without the pressure term and the volume forces. Due to its similarity to the Navier–Stokes equation, Burgers' equation often arises in the mathematical modelling used to solve problems in fluid dynamics involving turbulence. Bateman [1] has first introduced Burgers' equation as worthy of study and gave its steady solutions. After that Burgers [2] simplified the Navier-Stokes equation by just dropping the pressure term. It was later treated as a mathematical model for turbulence and such an equation is widely referred to as Burgers' equation. Since then the equation has found applications in field as diverse as number theory, gas dynamics, heat conduction, elasticity, etc. However, Hopf [3] and Cole [4] have shown that the homogeneous Burgers' equation lacks the most important property attributed to turbulence. The exact solutions of the one dimensional Burgers' equation have been surveyed by Benton and Platzman [5].

In many cases these solutions involve infinite series which may converge very slowly for small values of the viscosity coefficient λ which correspond to steep wave fronts in the propagation of the dynamic wave-forms [6]. Many studies have been done on the numerical solutions of Burgers' equation to deal with solutions for the small values of λ. A finite element method has been given by Caldwell *et. al.* [7], to solve Burgers' equation by altering the size of the element at each stage. Moreover Caldwell and Smith [8] have discussed the comparison of a number of numerical approaches to the equation. Nguyen and Reynen [9] also suggested a space-time finite element method based on a least square weak formulation using piecewise linear shape functions. A kind of finite element method based on weighted residual formulation is given by Varo glu and Liam Finn [10] and demonstrated the high accuracy and the stability. Rubin and Graves [11] have used the spline function technique and quasi-linearization for the numerical solution of the Burgers' equation in one space variable. A cubic spline collocation procedure is developed for the Burgers' equation in the papers [12, 13]. The implicit-finite difference scheme, together with cubic splines interpolating space derivatives in the Burgers' equation, has been proposed in the papers [14–17].

In the present work, the model equations are discretized using the technique of cubic B-spline collocation method (CSCM). The cubic B-spline is used for the interpolation function, with zeros of shifted Chebyshev polynomials as collocation points. Solution and its principal derivatives over the subinterval are approximated by the combination of the cubic B-splines and unknown element parameters.

Placing nodal values and its derivatives in the Burgers' equation are resulting in system consisting of $n+1$ equations for $n+3$ parameters. The resulting system can be solved with MATLAB after the boundary conditions are applied.

II. NUMERICAL SCHEME

A. Mesh Selection

Consider the uniform partition $\pi \equiv \{a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b\}$, over $\bar{\Omega}$ with $h = x_m - x_{m-1}$, $m = 1, \ldots, n$. Then cubic *B*-splines functions $B_m(x)$ are used as trial function [18], which can be written in terms of nodes as follows:

$$B_m(x) = \frac{1}{6h^3}\begin{cases} (x - x_{m-2})^3 & ,[x_{m-2}, x_{m-1}] \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3 & ,[x_{m-1}, x_m] \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3 & ,[x_m, x_{m+1}] \\ (x_{m+2} - x)^3 & ,[x_{m+1}, x_{m+2}] \\ 0 & ,otherwise \end{cases} . \qquad (4)$$

where $B_m(x)$ are the *B*-spline basis functions (four control points or blending functions). The blending functions sum to one and are positive on $[a,b]$ and not everywhere. The cubic *B*-spline functions are defined as:

$$B_1(x) = \tfrac{1}{6}(1 - 3x + 3x^2 - x^3), \qquad B_2(x) = \tfrac{1}{6}(4 - 6x^2 + 3x^3),$$

$$B_3(x) = \tfrac{1}{6}(1 + 3x + 3x^2 - 3x^3), \qquad B_4(x) = \tfrac{1}{6}(x^3).$$

B. Selection of Collocation Points

The roots of shifted Chebyshev polynomials are used as collocation points because these roots have the tendency to keep the error down to a minimum at the corners. Usually, Chebyshev polynomials can be written in the following form:

$$T_r(x) = \cos r\theta, \qquad \cos\theta = x, \qquad -1 \le x \le 1,$$

where $T_r(x) = 1$ at $x = \pm 1$, the value of x is $+1$ for even r and -1 for odd r. The turning points of $T_r(x)$ occur at the zeros of $\frac{\sin r\theta}{\sin\theta}$, *i.e.*, at the $r-1$ points as:

$$\theta_i = \frac{i\pi}{r}, \; x_i = \cos\left(\frac{i\pi}{r}\right), \quad i = 1, 2, \ldots, r-1.$$

Both turning points and zeros are symmetrical about the origin $x = 0$. The finite range $a \le \xi \le b$, can be transformed into any range by using $\xi = 0.5(b-a)x + 0.5(b+a)$. Particularly, for required range $0 \le \xi \le 1$, it will be $\xi = 0.5(x+1)$, *i.e.*, $x = 2\xi - 1$. Accordingly, for this range $T_r(x)$ can be written with a special notation as follows:

$$T_r^*(\xi) = T_r(2\xi - 1), \quad 0 \le \xi \le 1.$$

The properties of $T_r^*(\xi)$ can be deduced from those of $T_r(2\xi - 1)$ and in particular, all values of $T_r^*(\xi)$ can be generated from the recurrence system:

$$T_{r+1}^*(\xi) = 2(2\xi - 1)T_r^*(\xi) - T_{r-1}^*(\xi), \qquad \text{with } T_0^*(\xi) = 1, \; T_1^*(\xi) = 2\xi - 1.$$

C. Choice of trial function and discretization process

The approximate solution $\tilde{U}$ of exact solution U can be written in the following form:

$$\tilde{U}_{\bar{\Omega}}(x,t) = \tilde{U}(x,t) = \sum_{i=0}^{n+1} \delta_i(t) B_{i,3}(x), \qquad \ldots(5)$$

where $B_{i,3}(x)$ is a cubic B-splines.

From Eq. (5) the approximate solution $\tilde{U}(x,t)$ is substituted for $U(x,t)$ in the differential equation (1). Thereby obtained discretized form can be written as follows:

$$\frac{\partial U(x,t)}{\partial x} = \frac{1}{h_i}\frac{\partial \tilde{U}(x,t)}{\partial x} = \frac{1}{h_i}\sum_{i=0}^{n+1} \delta_i(t)\frac{dB_{i,3}(x)}{dx}, \qquad \ldots(6)$$

$$\frac{\partial^2 U(x,t)}{\partial x^2} = \frac{1}{h_i^2}\frac{\partial^2 \tilde{U}(x,t)}{\partial x^2} = \frac{1}{h_i^2}\sum_{i=0}^{n+1} \delta_i(t)\frac{d^2 B_{i,3}(x)}{dx^2}, \qquad \ldots(7)$$

$$\frac{\partial U(x,t)}{\partial t} = \frac{\partial \tilde{U}(x,t)}{\partial t} = \sum_{i=0}^{n+1} \frac{d\delta_i(t)}{dt} B_{i,3}(x). \qquad \ldots(8)$$

After discretization, the collocation equations and the boundary conditions form a system of $n+3$ differential algebraic equation (DAEs) in $n+3$ unknowns. The system is solved with MATLAB ode15s differential equation solver using initial approximation from Eq. (3).

III. APPLICATION OF NUMERICAL APPROACH

Consider the Burger's equation (1) with the initial condition:

$$U(x,0) = \sin \pi x, \quad 0 \le x \le 1 \qquad \ldots(9)$$

and the homogeneous boundary conditions:

$$U(0,t) = U(1,t) = 0, \ t > 0 \qquad \ldots(10)$$

The exact solution of Burger's equation (1) with conditions (9) and (10) was given by Cole [4] as:

$$U(x,t) = 2\pi\lambda \frac{\sum_{n=1}^{\infty} b_n \exp(-n^2\pi^2\lambda t) n \sin n\pi x}{b_0 + \sum_{n=1}^{\infty} b_n \exp(-n^2\pi^2\lambda t) n \cos n\pi x} \qquad \ldots(11)$$

where $b_0 = \int_0^1 \exp[-(2\pi\lambda)^{-1}(1-\cos\pi x)]\,dx$ and $b_n = 2\int_0^1 \exp[-(2\pi\lambda)^{-1}(1-\cos\pi x)]\cos n\pi x\,dx, \ n \ge 1$ are Fourier coefficients.

The theoretical solution of this problem was expressed as an infinite series by Cole [4].

The discretized form of Eq. (1) along with initial and boundary conditions (9) and (10) respectively, using CSCM is given below:

$$\sum_{i=0}^{n+1} \frac{d\delta_i(t)}{dt} B_{i,3}(x) = \frac{\lambda}{h_i^2}\sum_{i=0}^{n+1} \delta_i(t)\frac{d^2B_{i,3}(x)}{dx^2} - \sum_{i=0}^{n+1} \delta_i(t) B_{i,3}(x)\left(\frac{1}{h_i}\sum_{i=0}^{n+1} \delta_i(t)\frac{dB_{i,3}(x)}{dx}\right), \qquad \ldots(12)$$

with initial condition

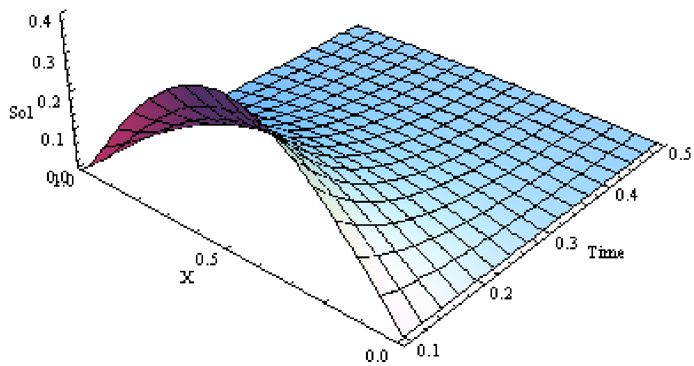
$$\delta_1(t) + 4\delta_2(t) + \delta_3(t) = \sin(\pi x), \qquad \ldots(13)$$

and boundary conditions

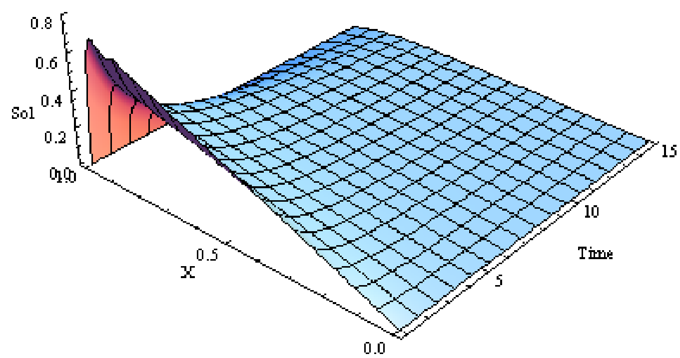
$$\delta_1(t) + 4\delta_2(t) + \delta_3(t) = 0, \qquad \ldots(14)$$

$$\delta_{n-1}(t) + 4\delta_n(t) + \delta_{n+1}(t) = 0. \qquad \ldots(15)$$

In Fig. 1, the 3D graph presents physical behavior of the approximate solution of system (12)-(15). It shows that as viscosity gets smaller and smaller, the graphs demonstrate a very sharp front near the left boundary and the steepness of the solution increases sharply near the right boundary. This steepness is controlled by taking small step for space variable *x* [20]. For different viscosity parameters, the numerical values of $U(x,0.1)$ obtained by using $\Delta t = 0.0001$, h = 0.0125, are presented in Table 1. A comparison of present method is shown with using different types of collocation points (i.e zeros of shifted Chebyshev polynomial and zeros of shifted Legendre polynomial) and with previous ones reported by [20,21,22,23, 24]. This shows that the CSCM gives a good agreement with the earlier work.



(a)



(b)

Fig. 1. Physical behavior of approximate solution for: (a) h = 0.0125, λ = 1, Δt = 0.0001 (b) h = 0.0125, λ = 0.005, Δt = 0.0001.

Table 1: Comparison of results at t = 0.1 for λ = 1, Δt = 0.00001, h = 0.0125.

x	Exact	CSCM-C	CSCM-L	CHCM-C [20]	CHCM-L [20]	[21]	[22]	EFDM [23]	EEFDM [23]	[24]	[25]
0.1	0.10954	0.10954	0.10958	0.10961	0.10954	0.10952	0.10965	0.10952	0.10955	0.10955	0.10954
0.2	0.20979	0.20979	0.20985	0.20989	0.20978	0.20975	0.20998	0.20975	0.20981	0.20981	0.2098
0.3	0.29190	0.29190	0.29201	0.29206	0.29190	0.29184	0.29213	0.29184	0.29192	0.29193	0.29191
0.4	0.34792	0.34798	0.34803	0.34811	0.34791	0.34785	0.34818	0.34786	0.34795	0.34796	0.34793
0.5	0.37158	0.37158	0.37161	0.37165	0.37158	0.37149	0.37185	0.37151	0.37161	0.37163	0.37158
0.6	0.35905	0.35903	0.35921	0.35931	0.35904	0.35896	0.35932	0.35898	0.35907	0.35910	0.35904
0.7	0.30991	0.30992	0.31015	0.31011	0.30990	0.30983	0.31017	0.30985	0.30993	0.30995	0.30989
0.8	0.22782	0.22782	0.22799	0.22809	0.22782	0.22776	0.22805	0.22778	0.22783	0.22786	0.22780
0.9	0.12069	0.12069	0.12078	0.12079	0.12069	0.12065	0.12083	0.12067	0.12070	0.12071	0.12067

IV. CONCLUSION

In this paper, the cubic B-spline collocation method is presented to solve Burger's equation which illustrates the validity and accuracy of the given method. The cubic B-spline collocation method is inherently smoother than other methods available in literature. The proposed method is simple, easy to implement and involves less computational effort. It has wide applicability to different engineering problems.

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