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"Reflexivity and the dual $E'[\tau^o]$ of locally Convex Spaces"

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ABSTRACT: In this paper we consider a locally convex space $E[\tau]$ that holds a type of reflexivity from any of the eight types, namely, polar semi-reflexivity , polar reflexivity, semi-reflexivity, reflexivity, inductive semi-reflexivity, inductive reflexivity, B-semireflexivity, and B-reflexivity. We consider the statements "$E[\tau]$ holds a type of reflexivity imply $E'[\tau^o]$ holds some of the reflexivities. We discuss some results and investigate for the truth value of this statement.

Keywords: Bornological space, barreled, reflexive, polar reflexive, inductively reflexive, B-reflexive, strong dual.

AMS (2010) Mathematics Subject Classification: 46A25.

I. INTRODUCTION

Throughout the paper, $E[\tau]$ denotes a locally convex topological vector space, which is Huasdorff and abbreviated as locally convex space. The strong dual of $E[\tau]$ is $E'[\tau_b(E)]$ and the bidual of $E[\tau]$ is $E'' = (E'[\tau_b(E)])'$. If $E''= E$, then $E[\tau]$ is called semi-reflexive. A semi-reflexive locally convex space $E[\tau]$ is called reflexive provided $\tau = \tau_b(E')$. The space $E''[\tau^o]$ denotes the dual E' equipped with the topology τ^o of uniform convergence on the class of precompact sets in E. Let τ^{oo} be the topology on $(E'[\tau^o])'$ of uniform convergence on τ^o-precompact subsets of E'. If $(E'[\tau^o])' = E$, then $E[\tau]$ is called polar semi-reflexive, and polar reflexive if further $\tau = \tau^{oo}$. These reflexivities have been discussed in [4] as p-completeness and p-reflexivity, respectively. We also note that polar reflexivity is the t-reflexivity of [10]. Characterizations of polar semi-reflexivity and polar reflexivity are discussed in [3, 4, 7, 8, 9, 12]. The finest locally convex topology on E' for which all τ-equicontinuous subsets are bounded is denoted by τ^*, called inductive topology. The base of neighborhoods of 0 in $E'[\tau^*]$ is formed by the absolutely convex subsets of E' that absorbs all τ-equicontinuous subsets of E'. If $(E'[\tau^*])'$ coincides with E, then $E[\tau]$ is called inductively semi-reflexive. Moreover, if $\tau = \tau^{**}$ i.e. $(\tau^*)^*$, then $E[\tau]$ is called inductively reflexive [2]. Inductive (semi) reflexivity is also discussed in [1, 5, 11, 13]. Let $\boldsymbol{\tau_r}$ be the topology, called reflective topology, on E of uniform convergence over the class R of all the absolutely convex bounded subset B of the dual E' whose span space E'_B is a reflexive Banach space with B as unit ball. A locally convex space $E[\tau]$ is said to be B-semireflexive if it is barreled and $E = \tilde{E}[\tilde{\tau}_r]$ (completion of $E[\tau_r]$). If further, $\tau = \tau_r$, then $E[\tau]$ is called B-reflexive [13].

We recall some well known results on inter-relationship:

1. Every (semi) reflexive locally convex space is polar (semi) reflexive.
2. Every inductively (semi) reflexive locally convex space is (semi) reflexive.
3. B-semireflexive locally convex space is complete and reflexive. On the other hand, a reflexive locally convex space is B-semireflexive if and only if it is bornological.
4. B-semireflexive locally convex space is inductively semi-reflexive.

II. RESULTS

First we discuss the following result of [11]; its proof is given for completeness:

2.1 Theorem : Inductively reflexive locally convex space is B-semireflexive.

Proof: Let $E[\tau]$ be inductively reflexive. From $(E'[\tau^*])'= E$ we have $\tau^* \leq \tau_k(E)$. We always have $\tau_k(E) \leq \tau_b(E) \leq \tau^*$. Therefore $\tau_k(E) = \tau_b(E) = \tau^*$ on E'. Now $\tau = \tau^{**}$ implies that τ is the bornological topology for the $\tau_s(E)$-bounded sets in E' and therefore τ is the inductive limit topology on E for the class of E_B formed by the class of all absolutely convex $\tau_s(E')$ -bounded and $\tau_s(E')$-closed sets B in E. But $(E'[\tau_b(E)])' = (E'[\tau^*])' = E$, that is, $E[\tau]$ is semi-reflexive and so weakly quasi-complete. Therefore, each of the above E_B is a Banach space. Thus $E[\tau]$ is the inductive limit of the class of E_B of Banach spaces. So $E[\tau]$ is bornological and so $E[\tau]$ is quasi-barreled, and so it is reflexive. Again, since $\tau_b(E) =\tau^*$, the strong dual $E'[\tau_b(E)] = E'[\tau^*]$ is bornological. Now $E[\tau]$ is reflexive and the strong dual $E'[\tau_b(E)]$ is bornological, so $E[\tau]$ is B-semireflexive.

Now we investigate a result as under:

2.2 Theorem: If a locally convex space $E[\tau]$ is polar reflexive, then $E'[\tau^o]$ is polar reflexive.

Proof: If $E[\tau]$ is polar reflexive, then $(E'[\tau^o])' = E$ and $\tau = \tau^{oo}$. Consider $E'[\tau^o]$, its dual is E and on E the topology $(\tau^o)^o$ is τ and dual of $E[\tau]$ is E'. It means $E'[\tau^o]$ is polar semi-reflexive. Further, $(\tau^o)^{oo} = (\tau^{oo})^o = \tau^o$. Hence $E'[\tau^o]$ is polar reflexive.

From this theorem we obtain that if $E[\tau]$ is polar reflexive (and so if it is any of: reflexive, B-semireflexive, B-reflexive, inductively reflexive), then $E'[\tau^o]$ is polar reflexive (and so polar semi-reflexive).

We also have:

2.3 Theorem: If a locally convex space $E[\tau]$ is reflexive, then $E'[\tau^o]$ is semi-reflexive.

Proof: $E[\tau]$ is reflexive, then $E'[\tau_b(E)]$ is also reflexive. We know that $\tau_s(E) \le \tau^o \le \tau_b(E)$. But reflexivity of $E[\tau]$ implies that $\tau_b(E) = \tau_k(E)$, so $\tau_s(E) \le \tau^o \le \tau_k(E) = \tau_b(E)$.Since $E'[\tau_b(E)]$ is semi-reflexive, $E'[\tau^o]$ is semi-reflexive.

This theorem implies that if $E[\tau]$ is reflexive (and so if it is any of: B-semireflexive, B-reflexive, inductively reflexive), then $E'[\tau^o]$ is semi-reflexive (and so polar semi-reflexive).

Example-A: Consider the locally convex space $E[\tau] = \ell^1[\tau_k(c_0)]$. This space is inductively semi-reflexive (see[ ]). Its dual is $E' = c_0$. Therefore, $E'[\tau^o] = c_0[(\tau_k(c_0))^o]$. Now on c_0, $\tau_s(\ell^1) \le (\tau_k(c_0))^o \le \tau_b(\ell^1) = \tau_k(\ell^1)$. It means $(\tau_k(c_0))^o$ is compatible for the dual pair (c_0, ℓ^1), and so $(c_0[(\tau_k(c_0))^o])' = \ell^1$. Therefore, $c_0[(\tau_k(c_0))^o]$ is not semi-reflexive.

Thus we have an example that $E[\tau] = \ell^1[\tau_k(c_0)]$ is inductively semi-reflexive(and so also semi-reflexive, polar semi-reflexive) and its dual $E'[\tau^o] = c_0[(\tau_k(c_0))^o]$ is not semi-reflexive (and none of: reflexive, inductively semi-reflexive, inductively reflexive, B-semi-reflexive, B- reflexive).

Example-B: The Banach space ℓ^p, $1< p < \infty$, equipped with the norm topology τ_p is inductively reflexive as well as B-reflexive.([5]). Its dual is ℓ^q, where $1/p + 1/q = 1$. In the locally convex space $\ell^p[\tau_p]$, every precompact set is relatively compact but unit ball is not relatively compact and so not precompact. Therefore, the dual $\ell^q[(\tau_p)^o]$ is not barreled and so not reflexive.

In this example we have a locally convex space $E[\tau] = \ell^p[\tau_p]$ which is inductively reflexive and B-reflexive (and so also inductively semi-reflexive, B-semireflexive, reflexive, semi-reflexive, polar reflexive, polar semi-reflexive), that is, it holds all the eight types of reflexivity, but the dual $E'[\tau^o] = \ell^q[(\tau_p)^o]$ is not reflexive (and so none of: inductively reflexive, B-semireflexive, B-reflexive).

Example-C: We consider $E[\tau] = \ell^1[(\tau_s(c_0))^{oo}]$ and $E'[\tau^o] = c_0[((\tau_s(c_0))^{oo})^o]$

We note that $\ell^1[\tau_s(c_0)]$ is polar semi-reflexive ([5], Theorem 2.8). So it is polar semi-reflexive.

We also note that in the dual c_0, $(\tau_s(c_0))^o = \tau_b(\ell^1)$, the usual normed topology (barreled topology), and so $((\tau_s(c_0))^{oo})^o = (\tau_s(c_0))^o = \tau_b(\ell^1)$.

Now we consider the locally convex space $E[\tau] = \ell^1[(\tau_s(c_0))^{oo}]$. We have $E'[\tau^o] = c_0[((\tau_s(c_0))^{oo})^o] = c_0[\tau_b(\ell^1)]$, the Banach space c_0 which is not semi-reflexive. However $(E'[\tau^o])' = (c_0[((\tau_s(c_0))^{oo})^o])' = (c_0[\tau_b(\ell^1)])' = \ell^1 = E$, so $E[\tau]$ is polar semi-reflexive.

Thus we have an example that $E[\tau]$ is polar semi-reflexive but $E'[\tau^o]$ is not semi-reflexive (and so none of: reflexive, inductively semi-reflexive, inductively reflexive, B-semireflexive, B-reflexive).

To summarize the results, we use the following notations:

I-polar semi-reflexive, II-polar reflexive, III-semi-reflexive, IV-reflexive, V-inductively semi-reflexive, VI- inductively reflexive, VII- B-semireflexive, VIII-B-reflexive.

Using the results and illustrations discussed above, findings are given in the following table:

Table 1.

		I	II	III	IV	V	VI	VII	VIII
	VIII	⇒	⇒	⇒	□	*	□	□	□
	VII	⇒	⇒	⇒	□	*	□	□	□
$E[\tau]$	VI	⇒	⇒	⇒	□	*	□	□	□
	V	*	*	□	□	□	□	□	□
	IV	⇒	⇒	⇒	□	*	□	□	□
	III	*	*	□	□	□	□	□	□
	II	⇒	⇒	□	□	□	□	□	□
	I	*	*	□	□	□	□	□	□

$E'[\tau^o]$

* : Not decided.

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