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N-Strongly Projective Injective and Flat Moudles over Upper Traingular Matrix Artian Algebras

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ABSTRACT: In this article we determine all the n- Strongly Complete Projective Injective and Flat resolutions and all the n-Strongly Gorenstein Projective, Injective and Flat Modules over upper Triangular Matrix artin algebras.

Key words: Gorenstein Projective, Injective and Flat Modules, Strongly Gorenstein Projective, Injective and Flat Modules, n- Strongly Gorenstein Projective, Injective and Flat Modules, Upper Triangular Matrix artin algebras.

I. INTRODUCTION

Throughout this article R is a commutative ring with unit element, and all R modules are unital. If M is any R-Module, we use $pd_R(M)$, $id_R(M)$ and $fd_R(M)$ to denote the usual projective, injective and flat dimensions of M, resp.
Auslander and Bridger introduced the G dimension for finitely generated modules over Noetherian rings in 1967-69 denoted by G-dim(M) where G-dim(M)≤pd(M), G-dim(M)≤id(M) and G-dim(M)≤fd(M). If G-dim(M)=pd(M)=id(M)=fd(M) then it is finite.
The Gorenstein projective, injective and flat dimension of a module is defined in terms of resolutions by Gorenstein projective, injective and flat modules respectively.
Definition:
1. An R-mod M is said to b G-projective (Short of Gorenstein projective) if there exists an exact sequence of projective modules
P = ...$\rightarrow P_1 \rightarrow P_0 \rightarrow P^0 \rightarrow P^1 \rightarrow \cdots$ such that M $\cong Im(P_0 \rightarrow P^0)$ and such that Hom_R(- ,Q) leaes the sequence P exact whenever Q is a projective module. The exact sequence P is called a complete projective resolution.

2.An R-mod M is said to b G-injective (Short of Gorenstein injective) if there exists an exact sequence of projective modules
......$\rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots = I$such that M $\cong Im(I_0 \rightarrow I^0)$ and such that Hom_R(Q,-) leaves the sequence I exact whenever Q is a injective module. The exact sequence I is called a complete projective resolution.
3.An R-mod M is said to b G-flat (Short of Gorenstein flat) if there exists an exact sequence of projective modules
F= ......$\rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$ such that M $\cong Im(F_0 \rightarrow F^0)$ and such that - $\otimes I$ leaves the sequence F exact whenever I is a injective module. The exact sequence F is called a complete Flat resolution.

II. STRONGLY GORENSTEIN PROJECTIVE, INJECTIVE AND FLAT MODULES

In this section we introduce and study the strongly Gorenstein projective injective and flat modules which are defined as follows:
Definition: A complete projective resolution of the form
P = ...$\xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \ldots..$ is called strongly complete projective resolution and denoted by (P, f).
An R-mod M is called strongly Gorenstein projective if M$\cong Kerf$ for some strongly complete projective resolution (P, f).

A complete injective resolution of the form

$\ldots\ldots\overset{f}{\to} I \overset{f}{\to} I \overset{f}{\to} I \overset{f}{\to} \ldots\ldots = I$ is called strongly complete injective resolution and denoted by (I, f).

An R-mod M is called strongly Gorenstein injective if M$\cong Kerf$ for some strongly complete injective resolution (I, f).

A complete flat resolution of the form

$F = \cdots .\overset{f}{\to} F \overset{f}{\to} F \overset{f}{\to} F \overset{f}{\to} \ldots\ldots$ Is called strongly complete flat resolution and denoted by (F, f).

An R-mod M is called strongly Gorenstein injective if M$\cong Kerf$ for some strongly complete flat resolution (F, f).

III. n- STRONGLY GORENSTEIN PROJECTIVE, INJECTIVE AND FLAT MODULES

Let n be a positive integer. A module M $\in$ R-mod is called n-strongly Gorenstein projective if there exist an exact sequence $0\to M \overset{f_n}{\to} P_{n-1} \overset{f_{n-1}}{\longrightarrow} \ldots\ldots \overset{f_1}{\to} P_0 \overset{f_0}{\to} M \to 0$ in Mod R with P_i projective for $0\leq i \leq n-1$ such that Hom$_R$(-,P) leaves the sequence exact whenever $P \in$ Mod R is projective.

Let n be a positive integer. A module M $\in$ R-mod is called n-strongly Gorenstein injective if there exist an exact sequence

$0\to M \overset{f_0}{\to} I_0 \overset{f_1}{\to} \ldots\ldots \overset{f_{n-1}}{\longrightarrow} I_{n-1} \overset{f_n}{\to} M \to 0$ in Mod R with I_i injective for $0\leq i \leq n-1$ such that Hom$_R$(I,-) leaves the sequence exact whenever $I \in$ Mod R is injective.

Let n be a positive integer. A module M $\in$ R-mod is called n-strongly Gorenstein flat if there exist an exact sequence $0\to M \overset{f_n}{\to} F_{n-1} \overset{f_{n-1}}{\longrightarrow} \ldots\ldots \overset{f_1}{\to} F_0 \overset{f_0}{\to} M \to 0$ in Mod R with F_i flat for $0\leq i \leq n-1$ such that Hom$_R$(- $\otimes$ F) leaves the sequence exact whenever $P \in$ Mod R is flat.

On the basis of above following facts holds

1.A module is Gorenstein projective (resp. Injective) if and only if it is a direct summand of a n-strongly Gorenstein projective (resp. Injective) module.

2.For finite finitistic projective dimension every n-strongly Gorenstein projective module is n-strongly Gorenstein flat module.

Proposition 1: Every projective (resp. Injective) module is n- strongly Gorenstein projective (resp. Injective)

Proof: Since every projective module is strongly Gorenstein projective then it is n-strongly Gorenstein projective (resp. Injective).

$0\to M \overset{f_n}{\to} P_{n-1} \oplus P_{n-1} \overset{f_{n-1}}{\longrightarrow} \ldots\ldots \overset{f_1}{\to} P_0 \oplus P_0 \overset{f_0}{\to} M \to 0$

Where M$\cong Kerf$

Consider a projective module Q applying the functor Hom R(-, Q) to the above module M for P we get the following commutative diagram:

$\ldots\ldots\ldots\to Hom(M \oplus M, Q) \xrightarrow{Hom_R(f,Q)} Hom(M \oplus M, Q) \to\ldots\ldots\ldots$

$\ldots\ldots\ldots\to Hom(M, Q) \oplus HOM(M, Q) \to Hom(M, Q) \oplus Hom(M, Q) \to\ldots\ldots$

The n-strongly Gorenstein projective (resp. Injective) modules are not necessarily projective (resp. Injective).

Theorem 1: A module is Gorenstein projective (resp. Injective) if and only if it is a direct summand of a n-strongly Gorenstein projective (resp. Injective) module.

Proof: Let M be a Gorenstein projective. Then there exist a complete projective resolution

$0\to M \overset{f_n^P}{\to} P_{n-1} \overset{f_{n-1}^P}{\longrightarrow} \ldots\ldots \overset{f_1^P}{\to} P_0 \overset{f_{0\ 0}^P}{\longrightarrow} M \to 0$

Such that M$\cong Kerf_0^P$

Consider the exact sequence

$$0 \to \oplus M \xrightarrow{f_n^P} \oplus P_{n-1} \xrightarrow{f_{n-1}^P} \text{..........} \oplus P_0 \xrightarrow{f_0^P} M \to 0$$

Since Ker($\oplus f_i$)$\cong$ $\oplus$Kerf_i ,M is a direct summand of Ker($\oplus f_i$)

Moreover Hom $(\oplus_{i\in I} P_i\,, M) \cong \prod_{i\in I}(\oplus P_i\,, M)$

Which is an exact sequence for any projective (resp. Injective) module M. Thus M is a n-Strongly Gorenstein projective Module over direct summand.

IV. n- Strongly Projective, Injective and Flat Module over Upper Triangular Matrix

In this section determine the strongly complete projective (resp. Injective) resolutions and hence all the n-Strongly projective modules over an upper triangular matrix $\tau = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be an artin algebra of matrix.

Let X : = $\begin{pmatrix} P\oplus(M \otimes_B Q) \\ Q \end{pmatrix}$, f: =$\begin{pmatrix} \alpha & 0 \\ \beta & id_M \otimes g \\ & g \end{pmatrix}$: $X \to X$ with P a projective A- module and Q a projective B-module.

Lemma: If M is an A B bimodule such that $_AM$ and M_B are projective modules and Hom$_A$(M,A) is a projective B-module or injective A-module then X is n-SG-projective (resp. injective) left B-module, then $M \otimes_B X$ is a n-SG projective A-module.

Proof: Since X is n-SG projective left module there is a complete B-projective resolution

$0\to M \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \ldots.. \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$ such that M$\cong$ $Kerf_0$ since M_B is a projective module.

$$0\to M \otimes_B P_{n-1} \xrightarrow{id_M \otimes f_{n-1}} \ldots\ldots\ldots M \otimes_B P_0 \xrightarrow{id_M \otimes f_0} M \otimes_B P_1 \to 0$$

Is exact, we know that it is a complete projective resolution.

Theorem 2:

1. if n/ m then m- SG projective (R)$\cap$ n- SG projective (R) = n – SG projective (R)

2. if n∤ m and m = np+k where p is a positive integer and 0< k < n then m- SG projective (R) $\cap$ n- SG projective (R)$\subseteq$ j-SG projective (R)

Proof: 1 it is trival since n/m

3. by above m- SG projective (R) $\cap$ n- SG projective (R)$\subseteq$m- SG projective (R) $\cap$ np- SG projective (R). M $\in$m- SG projective (R) $\cap$ np - SG projective (R)

Then there exist an exact sequence

$0\to M \xrightarrow{f_m} P_{m-1} \xrightarrow{f_{m-1}} \ldots\ldots.. P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$............................(1)

In Mod R with P_i projective for any 0$\leq i \leq m-1$ put Li=Ker (Pi-1$\to$ Pi-2) for any 2$\leq i \leq m$ because M$\in$ pn-SG projective (R). It is easy to see that M and L_{kn} are projectively equivalent, that is there exist projective modules P and Q in Mod R, such that

M$\oplus$P$\cong$Q$\oplus L_{kn}$

Proposition 2: For any n$\geq$1 n-SG projective (R) is closed under direct sums.

Proof: Let {Mj}j$\in$J be a family of n-SG projective modules in Mod R then for any j$\in$J there exist an exact sequence

$0\to \oplus_{j\in J} Mj \xrightarrow{f_n} \oplus_{j\in J} P_{n-1}^j \xrightarrow{f_{n-1}} \ldots\ldots.. \oplus_{j\in J} P_1^j \xrightarrow{f_1} \oplus_{j\in J} P_0^j \xrightarrow{f_0} Mj \to 0$ in Mod R because $\oplus_{j\in J} P_{n-1}^j$............$\oplus_{j\in J} P_0^j$ are projective and the obtained exact sequence is still exact after applying the functor HomR(-, P) when P$\in$Mod R is projective $\oplus_{j\in J}$Mj is n-SG projective.

Proposition 3: For any module M, the following are equivalent:

1. M is n-Strongly Gorenstien Projective

2. There exist a short sequence 0→ $M \xrightarrow{f_n} P_{m-1} \xrightarrow{f_{n-1}} \dots\dots.. P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0$ where P is a projective module and

$Ext_1^n(M, Q)$ = 0 for any projective module Q

3.There exist a short exact sequence 0→ $M \xrightarrow{f_n} P_{m-1} \xrightarrow{f_{n-1}} \dots\dots.. P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0$ where P is a projective module such that for any projective module Q the short sequence

0→ $Hom(M, Q) \xrightarrow{f_n} Hom(P_{n-1,}Q) \xrightarrow{f_{n-1}} \dots\dots.. Hom(P_{1,}Q) \xrightarrow{f_1} Hom(P_0, Q) \xrightarrow{f_0} Hom(M, Q) \rightarrow 0$ is exact.

Theorem 3: If a module is M is n-Strongly Gorenstein flat then it is a direct summand of a n- Strongly Gorenstein flat modules.

Proof: A module is Gorenstein projective (resp. Injective) if and only if it is a direct summand of a n-strongly Gorenstein projective (resp. Injective) module.

Proof: Let M be a Gorenstein projective. Then there exist a complete projective resolution

0→ $M \xrightarrow{f_n^P} P_{n-1} \xrightarrow{f_{n-1}^P} \dots.. \xrightarrow{f_1^P} P_0 \xrightarrow{f_0^P{}_0} M \rightarrow 0$

Such that M≅ $Kerf_0^P$

Consider the exact sequence

$0 \rightarrow \oplus M \xrightarrow{f_n^P} \oplus P_{n-1} \xrightarrow{f_{n-1}^P} \dots\dots\dots \oplus P_0 \xrightarrow{f_0^P} M \rightarrow 0$

Since Ker($\oplus f_i$)≅ $\oplus$Kerf_i ,M is a direct summand of Ker($\oplus f_i$)

Moreover Hom $(\oplus_{i\in I} P_i, M) \cong \prod_{i\in I}(\oplus P_i, M)$

Which is an exact sequence for any projective (resp. Injective) module M. Thus M is a n-Strongly Gorenstein projective Module over direct summand.

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