N-Strongly Projective Injective and Flat Modules over Upper Triangular Matrix Artin Algebras

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ABSTRACT: In this article we determine all the n- Strongly Complete Projective Injective and Flat resolutions and all the n-Strongly Gorenstein Projective, Injective and Flat Modules over upper Triangular Matrix artin algebras.

Key words: Gorenstein Projective, Injective and Flat Modules, Strongly Gorenstein Projective, Injective and Flat Modules, n- Strongly Gorenstein Projective, Injective and Flat Modules, Upper Triangular Matrix artin algebras.

I. INTRODUCTION

Throughout this article R is a commutative ring with unit element, and all R modules are unital. If M is any R-Module, we use pd\(_R\)(M), id\(_R\)(M) and fd\(_R\)(M) to denote the usual projective, injective and flat dimensions of M, resp.

Auslander and Bridger introduced the G dimension for finitely generated modules over Noetherian rings in 1967-69 denoted by G-dim(M) where G-dim(M)\(\leq\)pd(M), G-dim(M)\(\leq\)id(M) and G-dim(M)\(\leq\)fd(M). If G-dim(M)=pd(M)=id(M)=fd(M) then it is finite.

The Gorenstein projective, injective and flat dimension of a module is defined in terms of resolutions by Gorenstein projective, injective and flat modules respectively.

Definition:
1. An R-mod M is said to b G-projective (Short of Gorenstein projective) if there exists an exact sequence of projective modules

\[ P = \cdots \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow P \rightarrow \cdots \]

such that M \cong Im(P_0 \rightarrow P), and such that Hom\(_R\)(- , Q) leaves the sequence P exact whenever Q is a projective module. The exact sequence P is called a complete projective resolution.

2. An R-mod M is said to b G-injective (Short of Gorenstein injective) if there exists an exact sequence of injective modules

\[ \cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I \rightarrow I^0 \rightarrow \cdots \]

such that M \cong Im(I_0 \rightarrow I^0), and such that Hom\(_R\)( Q, - ) leaves the sequence I exact whenever Q is an injective module. The exact sequence I is called a complete projective resolution.

3. An R-mod M is said to b G-flat (Short of Gorenstein flat) if there exists an exact sequence of flat modules

\[ \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F \rightarrow F^0 \rightarrow \cdots \]

such that M \cong Im(F_0 \rightarrow F), and such that \(- \otimes I \) leaves the sequence F exact whenever I is a injective module. The exact sequence F is called a complete Flat resolution.

II. STRONGLY GORENSTEIN PROJECTIVE, INJECTIVE AND FLAT MODULES

In this section we introduce and study the strongly Gorenstein projective and flat modules which are defined as follows:

Definition: A complete projective resolution of the form

\[ P = \cdots \rightarrow P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} \cdots \]

is called strongly complete projective resolution and denoted by \((P, f)\).

An R-mod M is called strongly Gorenstein projective if M \cong Ker f for some strongly complete projective resolution \((P, f)\).
A complete injective resolution of the form
\[ \cdots \xrightarrow{f} I \xrightarrow{f} I \xrightarrow{f} I \xrightarrow{f} \cdots = I \] is called strongly complete injective resolution and denoted by \((I, f)\).

An R-mod \( M \) is called strongly Gorenstein injective if \( M \cong \text{Ker} f \) for some strongly complete injective resolution \((I, f)\).

A complete flat resolution of the form
\[ F = \cdots \xrightarrow{g} F \xrightarrow{g} F \xrightarrow{g} F \xrightarrow{g} \cdots \] is called strongly complete flat resolution and denoted by \((F, g)\).

An R-mod \( M \) is called strongly Gorenstein injective if \( M \cong \text{Ker} f \) for some strongly complete flat resolution \((F, g)\).

### III. n-STRONGLY GORENSTEIN PROJECTIVE, INJECTIVE AND FLAT MODULES

Let\( n \) be a positive integer. A module \( M \in \text{R-mod} \) is called \( n \)-strongly Gorenstein projective if there exist an exact sequence
\[ 0 \rightarrow M \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0 \] in \( \text{Mod R} \) with \( P_i \) projective for \( 0 \leq i \leq n - 1 \) such that \( \text{Hom}_R(-, P) \) leaves the sequence exact whenever \( P \in \text{Mod R} \) is projective.

Let \( n \) be a positive integer. A module \( M \in \text{R-mod} \) is called \( n \)-strongly Gorenstein injective if there exist an exact sequence
\[ 0 \rightarrow M \xrightarrow{f_n} I_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} I_{n-1} \xrightarrow{f_n} I_n \rightarrow 0 \] in \( \text{Mod R} \) with \( I_i \) injective for \( 0 \leq i \leq n - 1 \) such that \( \text{Hom}_R(I, -) \) leaves the sequence exact whenever \( I \in \text{Mod R} \) is injective.

Let \( n \) be a positive integer. A module \( M \in \text{R-mod} \) is called \( n \)-strongly Gorenstein flat if there exist an exact sequence
\[ 0 \rightarrow M \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \rightarrow 0 \] in \( \text{Mod R} \) with \( F_i \) flat for \( 0 \leq i \leq n - 1 \) such that \( \text{Hom}_R( - \otimes F) \) leaves the sequence exact whenever \( P \in \text{Mod R} \) is flat.

On the basis of above following facts holds

1. A module is Gorenstein projective (resp. Injective) if and only if it is a direct summand of a \( n \)-strongly Gorenstein projective (resp. Injective) module.
2. For finite finitistic projective dimension every \( n \)-strongly Gorenstein projective module is \( n \)-strongly Gorenstein flat module.

**Proposition 1:** Every projective (resp. Injective) module is \( n \)-strongly Gorenstein projective (resp. Injective)

**Proof:** Since every projective module is strongly Gorenstein projective then it is \( n \)-strongly Gorenstein projective (resp. Injective).

\[ 0 \rightarrow M \xrightarrow{f_n} P_{n-1} \oplus P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \oplus P_0 \xrightarrow{f_0} M \rightarrow 0 \]

Where \( M \cong \text{Ker} f \)

Consider a projective module Q applying the functor \( \text{Hom}_R(-, Q) \) to the above module M for P we get the following commutative diagram:

\[ \cdots \rightarrow \text{Hom}(M \oplus M, Q) \xrightarrow{\text{Hom}_R(f, Q)} \text{Hom}(M \oplus M, Q) \rightarrow \cdots \]

\[ \cdots \rightarrow \text{Hom}(M, Q) \oplus \text{HOM}(M, Q) \rightarrow \text{Hom}(M, Q) \oplus \text{Hom}(M, Q) \rightarrow \cdots \]

The \( n \)-strongly Gorenstein projective (resp. Injective) modules are not necessarily projective (resp. Injective).

**Theorem 1:** A module is Gorenstein projective (resp. Injective) if and only if it is a direct summand of a \( n \)-strongly Gorenstein projective (resp. Injective) module.

**Proof:** Let M be a Gorenstein projective. Then there exist a complete projective resolution
\[ 0 \rightarrow M \xrightarrow{f_n^P} P_{n-1} \xrightarrow{f_{n-1}^P} \cdots \xrightarrow{f_1^P} P_0 \xrightarrow{f_0^P} M \rightarrow 0 \]

Such that \( M \cong \text{Ker} f_0^P \)
Consider the exact sequence
\[ 0 \to \bigoplus M \to \bigoplus P_{n-1} \xrightarrow{f_{n-1}^P} \ldots \xrightarrow{f_1^P} P_0 \to M \to 0 \]
Since \( \text{Ker}(\bigoplus f_i) \cong \bigoplus \text{Ker} f_i \). M is a direct summand of \( \bigoplus f_i \).
Moreover \( \text{Hom} \left( \bigoplus_{i \in J} P_i, M \right) \cong \prod_{i \in J} \left( \bigoplus P_i, M \right) \)
Which is an exact sequence for any projective (resp. injective) module M. Thus M is a n-Strongly Gorenstein projective Module over direct summand.

IV. n-Strongly Projective, Injective and Flat Module over Upper Triangular Matrix
In this section determine the strongly complete projective (resp. injective) resolutions and hence all the n-Strongly projective modules over an upper triangular matrix \( \tau = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \) be an artin algebra of matrix.

Let \( X := \left( P \bigoplus (M \otimes_B Q) \right) \), \( f := \left( \begin{array}{cc} \alpha & 0 \\ \beta & M \otimes g \end{array} \right) : X \to X \) with P a projective A-module and Q a projective B-module.

**Lemma:** If M is an A B bimodule such that , M and \( M_B \) are projective modules and \( \text{Hom}_R(M, A) \) is a projective B-module or injective A-module then X is n-SG-projective (resp. injective) left B-module, then \( M \otimes_B X \) is a n-SG projective A-module.

**Proof:** Since X is n-SG projective left module there is a complete B-projective resolution
\[ 0 \to P_{n-1} \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_1} P_0 \to M \to 0 \]
Assume \( n \leq \text{Ker} f_0 \) since \( M_B \) is a projective module.
\[ 0 \to M \otimes_B P_{n-1} \xrightarrow{id \otimes f_{n-1}} \ldots \xrightarrow{id \otimes f_1} P_0 \otimes_B M \xrightarrow{id \otimes f_0} M \xrightarrow{id \otimes f_0} P_1 \to 0 \]
Is exact, we know that it is a complete projective resolution.

**Theorem 2:**
1. if \( n/m \) then m- SG projective (R) \( \cap \) n- SG projective (R) = n – SG projective (R)
2. if \( n \nmid m \) and \( m = np + k \) where p is a positive integer and \( 0 < k < n \) then m- SG projective (R) \( \cap \) n- SG projective (R) \( \subseteq \) j-SG projective (R)

**Proof:** 1 it is trivial since \( n/m \)
3. by above m- SG projective (R) \( \cap \) n- SG projective (R) \( \subseteq \) m- SG projective (R) \( \cap \) np- SG projective (R). M \( \in \) n- SG projective (R) \( \cap \) np - SG projective (R)

Then there exist an exact sequence
\[ 0 \to M \xrightarrow{f_m} P_{m-1} \xrightarrow{f_{m-1}} \ldots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0 \]
In Mod R with \( P_i \) projective for any \( 0 \leq i \leq m - 1 \) put Li = Ker (Pi−1 → Pi) for any \( 2 \leq i \leq m \) because M \( \in \) n-SG projective (R). It is easy to see that M and L_i are projectively equivalent, that is there exist projective modules P and Q in Mod R, such that
\[ M \oplus P \cong Q \oplus L_i \]

**Proposition 2:** For any \( n \geq 1 \) n-SG projective (R) is closed under direct sums.

**Proof:** Let \( \{ M_j \} \) be a family of n-SG projective modules in Mod R then for any \( j \in I \) there exist an exact sequence
\[ 0 \to \bigoplus_{j \in J} M_j \xrightarrow{f_{n-1}} \bigoplus_{j \in J} P_{n-1}^j \xrightarrow{f_1^j} \ldots \xrightarrow{f_1^j} P_0^j \xrightarrow{f_0^j} M_j \to 0 \]
in Mod R because \( \bigoplus_{j \in J} P_{n-1}^j \) are projective and the obtained exact sequence is still exact after applying the functor \( \text{Hom}_R(\cdot, P) \) when \( P \in \text{Mod R} \) is projective \( \bigoplus_{j \in J} M_j \) is n-SG projective.

**Proposition 3:** For any module M, the following are equivalent:
1. M is n-Strongly Gorenstein Projective
2. There exist a short sequence \( 0 \rightarrow M \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \ldots \ldots P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0 \) where \( P \) is a projective module and 
\( \text{Ext}^1(M, Q) = 0 \) for any projective module \( Q \).

3. There exist a short exact sequence \( 0 \rightarrow M \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \ldots \ldots P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \rightarrow 0 \) where \( P \) is a projective module such that for any projective module \( Q \) the short sequence
\( 0 \rightarrow \text{Hom}(M, Q) \xrightarrow{f_n} \text{Hom}(P_{n-1}, Q) \xrightarrow{f_{n-1}} \ldots \ldots \text{Hom}(P_1, Q) \xrightarrow{f_1} \text{Hom}(P_0, Q) \xrightarrow{f_0} \text{Hom}(M, Q) \rightarrow 0 \) is exact.

**Theorem 3:** If a module is \( M \) is \( n \)-Strongly Gorenstein flat then it is a direct summand of a \( n \)-Strongly Gorenstein flat modules.

**Proof:** A module is Gorenstein projective (resp. Injective) if and only if it is a direct summand of a \( n \)-strongly Gorenstein projective (resp. Injective) module.

**Proof:** Let \( M \) be a Gorenstein projective. Then there exist a complete projective resolution
\( 0 \rightarrow M \xrightarrow{f_n^P} P_{n-1} \xrightarrow{f_{n-1}^P} \ldots \ldots P_1 \xrightarrow{f_1^P} P_0 \xrightarrow{f_0^P} M \rightarrow 0 \)

Such that \( M \cong \text{Ker} f_0^P \).

Consider the exact sequence
\( 0 \rightarrow \bigoplus M \xrightarrow{f_n^P} \bigoplus P_{n-1} \xrightarrow{f_{n-1}^P} \ldots \ldots \bigoplus P_0 \xrightarrow{f_0^P} M \rightarrow 0 \)

Since \( \text{Ker}(\bigoplus f_i) = \bigoplus \text{Ker} f_i \), \( M \) is a direct summand of \( \text{Ker}(\bigoplus f_i) \).

Moreover \( \text{Hom}(\bigoplus \bigoplus P_i, M) \cong \prod_{i \in I} (\bigoplus P_i, M) \)

Which is an exact sequence for any projective (resp. Injective) module \( M \). Thus \( M \) is a \( n \)-Strongly Gorenstein projective Module over direct summand.

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