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Approximation of common fixed points of Hemi-contractive mappings in Banach Space

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ABSTRACT: The main purpose of this paper is to study an iterative Mann-type schemes to find a common fixed point of a countable family of Hemi-contractive mapping and L-Lipschitzain mappings in Banach space. We study a Mann type iteration procedure then, we construct common fixed points of Hemi-contractive mappings in arbitrary Banach spaces.

I. INTRODUCTION

Let E be a real Banach space and E^* be its dual space. The normalized duality mapping: $E \rightarrow 2^{E^*}$ defined by $Jx = \{f \in E^*: \langle x,y \rangle = \|x^2\| = \|f^2\|\}$; for all $x \in E$ where $<.,.>$ denotes the duality paring between E and E^*.

Definition 1.1 A mapping T with domain $D(T)$ and range $R(T)$ in Banach space is called

(i) Pseudo contractive [2], if for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \| x-y \|^2 \quad (1.1)$$

Equivalently, for all $x, y \in D(T)$ and for all $s > 0$,

$$\| x-y \| \leq \| x-y+s[I-T)x-(I-T)y] \| \quad (1.2)$$

(ii) λ −Strictly pseudo contractive (in the terminology of Browder and Petryshyn) [2] for all $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \| x-y \|^2 - \lambda \,(x-Tx)-(y-Ty) \|^2 \quad (1.3)$$

(iii) Strongly pseudo-contractive if there exists $\lambda \in (0,1)$ for all $x, y \in D(T)$,there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \lambda \| x-y \|^2$$

(iv) L-lipschitizian if there exists L>0 such that for all $x, y \in D(T)$,

$$\| Tx - Ty \| \leq L \| x-y \|$$

In 1974, Ishikawa [2] introduced an iteration method for finding a fixed point of a Lipschitz - pseudo contractive mapping as follows:

Theorem 1.2 [1]. Let C be a nonempty compact convex subset of a Hilbert space H, $T: C \rightarrow C$ be a Lipschitz pseudocontractive mapping. For a fixed $x_0 \in C$, define a sequence $\{x_n\}$ by.

$$x_{n+1} = \propto_n x_n + (1-\propto_n)Ty_n;\ \ y_n = \beta_n x_n + (1-\beta_n)\,Tx_n, \quad (1.1)$$

Where $\{\propto_n\}$ and $\{\beta_n\}$ are real sequences in [0, 1] satisfying the following conditions;

(i) $\lim_{n\to\infty}\beta_n = 0$, (ii) $\sum_{n=1}^{\infty} \propto_{n\beta_n=\infty}$, (iii) $0 \leq \propto_n \leq \beta_n < 1$.

Then $\{x_n\}$ converges strongly to a fixed point of T.

It is natural to ask a question of whether or not the simple Mann iteration defined by $x_1 \in C$ and $x_{n+1} = \propto_n x_n + (1-\propto_n)Tx_n$ can be used to obtain the same conclusion as theorem above. Recently, this question was resolved in the negative by Chidume and Matangadura [4]. They constructed an example of a Lipschitzian pseudo contractive mapping defined on a compact convex subset of the Hilbert space R^2 for which no Mann sequence converges

II. PRELIMINARIES

In the sequel we shall make use of the following lemmas

Lemma 2.1 (Xu, [4]) Let $q > 1$ and X be a real Banach space. Then the following are equivalent.

X is q-uniformly smooth and for all $x, y \in X, j_p(x) \in J_p(x)$, the following inequalities holds:

$$||x + y||^q \geq ||x||^q + q\langle y, jq(x+y)\rangle + ||y||^q.$$

In the sequel we shall make use of the following lemmas.

Lemma 2.2[5, Lemma2.1]. Let $\{\sigma_n\}$ and $\{\beta_n\}$ be sequence of nonnegative real numbers satisfying the following inequality:

$$\{\beta_{n+1}\} \leq (1+\sigma_n)\,\beta_n \quad, n \geq 0$$

If $\sum_{n=1}^{\infty} \sigma_n < \infty$ then $\lim_{n\to\infty} \beta_n$ exists.

Definition 2.4 [6]. Let$\{T_n\}$ be a family of mappings from a subset C of a Banach space E into E with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ We say that$\{T_n\}$ satisfies the AKTT-condition if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} sup_{z\in B} \parallel T_{n+1}z - T_n z \parallel < \infty$$

Remark 2.5 [6, Lemma 3.2]. Suppose that $\{T_n\}$ satisfies the AKTT-condition. Then, for each y$\in C$, T_ny converges strongly to a point in C. Moreover, let T be defined by;

$$Ty = \lim_{n\to\infty} T_n y \text{ for all y} \in C,$$

Then for each bounded subset B of C, $\lim_{n\to\infty} sup_{z\in B} \parallel T_z - T_n z \parallel = 0$

III. MAIN RESULTS

Motivated by [8], we have the following lemma**.**

Lemma 3.1. Let C be a closed convex subset of a Banach space E. Let$\{T_n\}_{n=1}^{\infty}$: C → C be a family of Hemi contractive and L- Lipschitzian mappings such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ .define a sequence $\{x_n\}$ by $x_1 \in C$,

$$x_{n+1} = \propto_n x_n + (1-\propto_n) T_n x_n \quad \text{for all } n \geq 1,$$

Where $\{x_n\} \subset [0,1]$ satisfying $\sum_{n=1}^{\infty} \propto_n = \infty$ and $\sum_{n=1}^{\infty} {\propto_n}^p < \infty$ if $\{T_n\}_{n=1}^{\infty}$satisfies AKTT-condition, then

(1) $\lim_{n\to\infty} \parallel x_n - T_n x_{n+1} \parallel$ exits for all $p \in F$ and hence $\{x_n\}$ is bounded;
(2) $\lim_{n\to\infty} \parallel x_n - T_n x_n \parallel = 0$

Proof. (1) Let all $p \in F$. then $\parallel T_n x_n - T_n x_{n+1} \parallel \leq L \parallel x_n - T_n x_{n+1} \parallel$. Moreover,

$$\parallel x_{n+1} - T_n x_{n+1} \parallel \leq \propto_n \parallel x_n - T_n x_{n+1} \parallel + (1-\propto_n) \parallel T_n x_n - T_n x_{n+1} \parallel \leq (\propto_n + (1-\propto_n)\mathrm{L}) \parallel x_n - T_n x_{n+1} \parallel$$
$$\leq (1+L) \parallel x_n - T_n x_{n+1} \parallel \tag{3.1}$$

Consequently,

$$\parallel x_n - T_n x_n \parallel \leq \parallel x_n - T_n x_{n+1} \parallel + \parallel T_n x_{n+1} - T_n x_n \parallel \leq (1+\mathrm{L}) \parallel x_n - T_n x_{n+1} \parallel \tag{3.2}$$

From (3.1), we have, $\parallel x_{n+1} - T_n x_{n+1} \parallel \leq \parallel x_{n+1} - T_n x_{n+1} \parallel + \parallel T_n x_{n+1} - T_n x_{n+1} \parallel$

$$\leq (1+\mathrm{L}) \parallel x_{n+1} - T_n x_{n+1} \parallel \leq (1+\mathrm{L})^2 \parallel x_n - T_n x_{n+1} \parallel \tag{3.3}$$

It follows from (3.2) that

$$\parallel x_{n+1} - x_n \parallel = (1-\propto_n) \parallel T_n x_n - x_n \parallel \leq (1-\propto_n)(1+\mathrm{L}) \parallel x_n - T_n x_{n+1} \parallel \tag{3.4}$$

Since T_n is hemi contractive mapping, there exists $j\,(x_{n+1} - T_n x_{n+1}) \in J(x_{n+1} - T_n x_{n+1})$ such that

$\langle x_{n+1} - T_n x_{n+1,}\, \mathrm{j}\,(x_{n+1} - T_n x_{n+1,}\rangle \leq \parallel (x_{n+1} - \mathrm{T_n x_{n+1}}) \parallel^2$,

By Lemma 2.1, (3.1) and (3.4), we have

$$\| x_{n+1} - -T_n x_{n+1} \|^q = \| \left(x_n - -T_n x_{n+1,}\right) + (1-\propto_n)(T_n x_n - x_n) \|^q$$
$$= \| x_n - -T_n x_{n+1,} \|^q + q \propto_n \langle T_n x_n - x_n, j\left(x_{n+1} - -T_n x_{n+1,}\right)\rangle + ||(1-\propto_n)(T_n x_n - x_n) \|^q$$

$= \|$
$x_n - T_n x_{n+1,} \|^q + q(1-\propto_n)\langle T_n x_n - T_n x_{n+1}, j(x_{n+1} - T_n x_{n+1,})\rangle + q \propto_n \langle T_n x_{n+1} - x_{n+1}, j\left(x_{n+1,} - T_n x_{n+1,}\right)\rangle +$
$q \propto_n \langle x_{n+1} - x_n, j\left(x_{n+1}, T_n x_{n+1,}\right)\rangle T_n x_{n+1,} \|^q + q(1-\propto_n) \langle T_n x_n - T_n x_{n+1}, j\left(x_{n+1} - T_n x_{n+1,}\right)\rangle +$
$q \propto_n \langle T_n x_{n+1} - x_{n+1}, j\left(x_{n+1}, T_n x_{n+1,}\right)\rangle q \propto_n \langle x_{n+1} - x_n, j\left(x_{n+1} - -T_n x_{n+1,}\right)\rangle + ||(1-\propto_n)(T_n x_n - x_n) \|^q$
$\leq \| x_n - T_n x_{n+1,} \|^q + q(1-\propto_n) \| x_n -$
$x_{n+1} \| \| \left(x_{n+1} - T_n x_{n+1,}\right) \| - q \propto_n \| T_n x_{n+1} - x_{n+1} \|^q q \propto_n^q \| T_n x_n - x_n \| \| x_{n+1} - T_n x_{n+1,} \|$
$+ ||(1-\propto_n)(T_n x_n - x_n) \|^q$
$\leq \| x_n - T_n x_{n+1} \|^q + 2 \propto_n^q L(1+L)^2 \| x_n - T_n x_{n+1,} \|^q - 2(1-\propto_n) \| T_n x_{n+1} - x_{n+1} \|^q +$
$2 \propto_n^q (1+ L)^2 \| x_n - T_n x_{n+1,} \|^q \| x_n - T_n x_{n+1,} \|^q - 2(1-\propto_n) \| T_n x_{n+1} -$
$x_{n+1} \|^2 + 2 \propto_n^2 (1+ L)^2 \| + \left((1-\propto_n)L\right) \| x_n - T_n x_{n+1,} \|^q$

$\leq \|$
$x_n - T_n x_{n+1,} \|^q + 2 \propto_n^q L\left(1+L\right)^3 \| x_n - T_n x_{n+1} \|^q + \left((1-\propto_n)L\right) \| x_n - T_n x_{n+1} \|^q \; 2(1-\propto_n) \|$
$\| x_{n+1} - T_n x_{n+1} \|^2$ (3.5)

Consequently,

$\| x_{n+1} - T_n x_{n+1,} \|^q \leq$
$(1 + 2 \propto_n^q (1+L)^3) \| x_n - T_n x_{n+1,} \|^q + \left((1-\propto_n)L\right) \| x_n - T_n x_{n+1,} \|^q$

From Lemma 2.2 and $\Sigma_{n=1}^{\infty} \propto_n^2 < \infty$, we get that $\lim_{n\to\infty} \| x_n - T_n x_{n+1,} \|$ exists and hence $\{x_n\}$ is bounded.

(2) We first show that $\lim_{n\to\infty} \| x_{n+1} - T_n x_{n+1} \| = 0$. Suppose that $\lim_{n\to\infty} \| x_{n+1} - T_n x_{n+1} \| = \delta > 0$. There exists $N \in \mathbb{N}$such that $\| x_{n+1} - T_n x_{n+1} \| \geq \frac{\delta}{2}$ for all $n \geq N$ since $\{x_n\}$ is bounded, put $M = \sup_{n\in\mathbb{N}}\{\| x_n - T_n x_{n+1,} \|\}$ from (3.5),

$\| x_{n+1} - T_n x_{n+1,} \|^q \leq \| x_n - T_n x_{n+1,} \|^q - 2(1-\propto_n) \| x_{n+1} - T_n x_{n+1} \|^q \; 2 \propto_n^q (1+L)^3) \quad \| x_n - T_n x_{n+1,} \|^q +$
$\left((1-\propto_n)L\right) \| x_n - -T_n x_{n+1,} \|^q$

$$\leq \| x_n - T_n x_{n+1,} \|^q - (1-\propto_n)\lambda \frac{\delta^2}{2} + 2 \propto_n^q (1+L)^3) M^2 + (1-\propto_n) \frac{\delta^2}{2} \text{ for all } n \geq N$$

It follows that

$$\propto_n \frac{\delta^2}{2} \leq \| x_n - T_n x_{n+1,} \|^q - \| x_{n+1} - T_n x_{n+1,} \|^q + 2 \propto_n^q (1+L)^3) M^2 + \left((1-\propto_n)L\right) \| x_n - T_n x_{n+1,} \|^q$$

. For any $m \geq N$ we have,

$\frac{\delta^2}{2} \Sigma_{n=N}^m \propto_n \leq \Sigma_{n=N}^m (\| x_n - T_n x_{n+1,} \|^q - \| x_{n+1} - T_n x_{n+1,} \|^q) + 2(1+L)^2 M^2 \Sigma_{n=N}^m \propto_n^2$
$+ \Sigma_{n=N}^m (\| T_n x_n - T_n x_{n+1,} \|^q - \| T_{n+1} x_{n+1} - T_n x_{n+1,} \|^q) = \| x_n - T_n x_{n+1,} \|^q$
$- \| x_{m+1} - T_n x_{n+1,} \|^q + 2(1+L)^2 M^2 \Sigma_{n=N}^m \propto_n^2 + \| T_n x_n - T_n x_{n+1,} \| \; x_{m+1} - T_n x_{n+1,} \|^q$
$\leq \| x_n - T_n x_{n+1,} \|^2 + 2(1+L)^2 M^2 \Sigma_{n=N}^m \propto_n^2 \| T_n x_n - T_n x_{n+1,} \|^2 \; \Sigma_{n=N}^m (1-\propto_n^2)$

Because $\Sigma_{n=1}^{\infty} \propto_n^2 < \infty$ we have$\Sigma_{n=1}^{\infty} \propto_n < \infty$ witch is a contradiction. Hence
$\lim_{n\to\infty} inf \; \| x_{n+1} - T_n x_{n+1} \| = 0$ (3.6)

Consequently, since $\{x_n\}$ is bounded,
$\| x_{n+1} - T_{n+1} x_{n+1} \| \leq \| x_{n+1} - T_n x_{n+1} \| + \| T_n x_{n+1} - T_{n+1} x_{n+1} \| \leq \| x_{n+1} - T_n x_{n+1} \| + sup_{z\in\{x_n\}} \| T_z - T_n z \|$ Using (3.6) and AKTT-condition of $\{T_n\}$; we have
$\lim_{n\to\infty} inf \; \| x_{n+1} - T_n x_{n+1} \| = 0$

This completes the proof.

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