International Journal of Theoretical & Applied Sciences, Special Issue-NCRTAST 8(1): 109-112(2016)

ISSN No. (Print): 0975-1718
ISSN No. (Online): 2249-3247

Some Common Fixed Point Theorems Using Faintly Compatible Maps in Fuzzy 3- Metric Space

Kamal Wadhwa and Ashlekha Dubey
Govt. Narmada P.G. College Hoshangabad, (Madhya Pradesh), INDIA

(Corresponding author: Ashlekha Dubey)
(Received 11 April, 2016 Accepted 20 May, 2016)
(Published by Research Trend, Website: www.researchtrend.net)

ABSTRACT: In present paper we proved Some Common Fixed Point theorems for four self-mappings by using the general contractive condition given by Patel *et al.* [6] and improvises the result by replacing the occasionally weakly compatible (owc) mappings by the faintly compatible pair of mapping in Fuzzy 3-Metric Space.

Keywords: Fuzzy 3-Metric Space, Common Fixed Point, Property (E.A), Sub Sequentially Continuity, Faintly Compatible maps.
Mathematics Subject Classification: 52H25, 47H10.

I. INTRODUCTION

In 1988, Jungck *et al.* [4] introduce the notion of weakly compatible mappings and showed that compatible mappings are weakly compatible but not conversely. Al-Thagafi et al. [2] introduced the concept of occasionally weakly compatible (owc) mappings which is more general than the concept of weakly compatible mappings. Aamri *et al.* [1] generalized the concepts of non-compatibility by defining the notion of (E.A) property in metric space. Pant *et al.* [5] introduced the concept of conditional compatible maps. Bisht *et al.* [3] criticize the concept of occasionally weakly compatible (owc) as follows "Under contractive conditions the existence of a common fixed point and occasional weak compatibility are equivalent conditions, and consequently, proving existence of fixed points by assuming occasional weak compatibility is equivalent to proving the existence of fixed points by assuming the existence of fixed points". Therefore use of occasional weak compatibility is a redundancy for fixed point theorems under contractive conditions to removes this redundancy we used faintly compatible mapping in our paper which is weaker than weak compatibility or semi compatibility. Faintly compatible maps introduced by Bisht *et al.* [3] is an improvement of conditionally compatible maps .Using these concepts Wadhwa *et al.* [7,8] proved some common fixed point theorems. In this paper we prove some common fixed point for four mappings using the concept of faintly compatible pair of mappings in Fuzzy 3-Metric space.

II. PRELIMINARY NOTES

Definition 2.1:[9] A binary operation $*: [0, 1]^4 \rightarrow [0, 1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 * d_1 \leq a_2 * b_2 * c_2 * d_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, $d_1 \leq d_2$ for all $a_1, a_2, b_1, b_2, c_1, c_2$ and d_1, d_2 are in [0, 1].

Definition 2.2:[9] The 3- tupple (X, M,$*$) is said to be a Fuzzy 3- Metric Space if X is an arbitrary set , $*$ is a continuous t-norm and M is a fuzzy set on $X^4 \times [0,\infty)$ satisfying the following conditions, for all x,y,z,w,u $\in$ X and $t_1, t_2, t_3, t_4 > 0$.

(i) M(x,y,z,w,0)= 0

(ii) M(x,y,z,w,t)=1 for all t>0 (only when the three simplex <x,y,z,w>degenerate)

(iii) M(x,y,z,w,t)= M(x,w,z,y,t)= M(y,z,w,x,t) = M(z,w,x,y,t)=…

(iv) $M(x,y,z,w,t_1+t_2+t_3+t_4) \geq M(x,y,z,u,t_1) * M(x,y,u,w,t_2) * M(x,u,z,w,t_3) * M(u,y,z,w,t_4)$

(v) M(x ,y,z,w,.): [0, 1) $\rightarrow$[0,1] is left continuous.

Definition 2.3: Let (X, M,$*$) is a Fuzzy 3- Metric Space then

(i) A sequence $\{x_n\}$in X is said to converse to x in X if for each $\epsilon > 0$ and each $t > 0$ $\exists\, n_0 \in N$ such that $M(x_n, x, a, b, t) > 1-\epsilon$ for all $n \geq n_0$.

(ii) A sequence $\{x_n\}$in X is said to cauchy to if for each $\epsilon > 0$ and each $t > 0$ $\exists\, n_0 \in N$ such that $M(x_n, x_m, a, b, t) > 1-\epsilon$ for all $n,m \geq n_0$.

(iii) A fuzzy -3 metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.4: Let A and B be mappings from Fuzzy 3- Metric Space (X, M,∗) into itself.The maps A and B are said to be compatible if, for all t> 0, $\lim_{n\to\infty} M(ABx_n, BAx_n, a, b, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} A x_n = \lim_{n\to\infty} B x_n = x$ for some x ∈ X.

Definition 2.5: A pair of self-maps (A, B) on a Fuzzy- 3 metric Space (X, M,∗) is said to be ***Sub sequentially continuous***: iff there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X$ and satisfy $\lim_{n\to\infty} ABx_n = Ax$, $\lim_{n\to\infty} BAx_n = Bx$.

Satisfy the property (E.A.): if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some t ∈X.

Semi-compatible: if $\lim_{n\to\infty} M(ABx_n, Bx_n, a, b, t) = 1$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$, for some x ∈X.

Conditionally compatible: iff whenever the set of sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n$ is non-empty , there exists a sequence $\{z_n\}$ in X Such that $\lim_{n\to\infty} Az_n = \lim_{n\to\infty} Bz_n = u$, for some u ∈ X and $\lim_{n\to\infty} M(ABz_n, BAz_n, a, b, t) = 1$, for all t > 0.

Faintly compatible: iff (A, B) is conditionally compatible and A and B commute on a non-empty subset of the set of coincidence points, whenever the set of coincidence points is nonempty.

Definition 2.6 : Two self mappings A and B of a Fuzzy 3- Metric Space (X, M,∗) is said to be non-compatible if there exists at list one sequence $\{x_n\}$ such that
$\lim_{n\to\infty} A x_n = \lim_{n\to\infty} B x_n = z$ for some z in X but neither $\lim_{n\to\infty} M(ABx_n, BAx_n, a, b, t) \neq 1$ or the limit does not exists.

Definition 2.7: Let (X, M,∗) be Fuzzy 3- Metric Space. Let A and B be self-maps on X. Then a point x in X is called a coincidence point of A and B iff Ax=Bx. In this case, w=Ax=Bx is called a point of coincidence of A and B.

Definition 2.8 : A pair of self-mappings (A, B) of a Fuzzy 3- Metric Space (X, M,∗) is said to be weakly compatible if they commute at their Coincidence points i.e Ax=Bx for some x in X, then ABx=BAx.

Definition 2.9 :A pair of self-mappings (A, B) of a Fuzzy 3- Metric Space (X, M,∗) is said to be occasionally weakly compatible(owc) iff there is a point x in X which is a coincidence point of A and B at which A and B commute.

Lemma 1. Let (X, M,∗) be a Fuzzy 3-Metric Space. If there exists a number q ∈ (0, 1)
$M(x, y, a, b, qt) \geq M(x, y, a, b, t)$ for all x,y,a,b ∈X with a≠x,a≠y,b≠x,b≠y and t>0 then x=y

III. MAIN RESULTS

Theorem 3.1 Let (X, M, ∗) be a complete fuzzy 3-metric space and let P, R, S, T be Self mappings of X.if there exists q∈(0,1) such that

$$M(Px, Ry, a, b, qt) \geq \min\{ M(Sx, Ty, a, b, t), M(Sx, Px, a, b, t), M(Ry, Ty, a, b, t), M(Px, Ty, a, b, t), M(Ry, Sx, a, b, t), M(Px, Ry, a, b, t), M(Sx, Ty, a, b, t) * M(Ry, Ty, a, b, t)\} \quad \ldots(1)$$

For all x, y ∈ X and for all t >0. If pairs (P, S) and (R, T) satisfies E.A. property with sub sequentially continuous faintly compatible map then P, S, R, T have a unique common fixed point in X.

Proof :(P, S) and (R, T) satisfy E.A property which implies that there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Sx_n = t_1$ for some t_1 ∈X also $\lim_{n\to\infty} Rx_n = \lim_{n\to\infty} Tx_n = t_2$ for some t_2 ∈X.
Since pairs (P, S) and (R, T) are faintly compatible therefore conditionally compatibility of (P, S) and (R, T) implies that there exist sequences $\{z_n\}$ and $\{z_{n'}\}$ in X satisfying $\lim_{n\to\infty} Pz_n = \lim_{n\to\infty} Sz_n = u$ for some u ∈X, such that $M(PSz_n, SPz_n, a, b, t) = 1$, also $\lim_{n\to\infty} Rz_{n'} = \lim_{n\to\infty} Tz_{n'} = v$ for some v ∈X, such that $M(RTz_n', TRz_n', a, b, t) = 1$.
As the pairs (P, S) and (R, T) are sub sequentially continuous, we get $\lim_{n\to\infty} PSz_n = Pu$, $\lim_{n\to\infty} SPz_n = Su$ and so Pu = Su also $\lim_{n\to\infty} RTz_{n'} = Rv$, $\lim_{n\to\infty} TRz_{n'} = Tv$
so Rv = Tv. Since pairs (P, S) and (R, T) are faintly compatible, we get PSu = SPu & So PPu = PSu =SPu = SSu also RTv = TRv & So RRv=RTv=TRv=TTv.
Now we show that Pu=Rv. Let x=u and y=v in equation (1) we have

$$M(Pu, Rv, a, b, qt) \geq \min\{ M(Su, Tv, a, b, t), M(Su, Pu, a, b, t), M(Rv, Tv, a, b, t), M(Pu, Tv, a, b, t), M(Rv, Su, a, b, t), M(Pu, Rv, a, b, t), M(Su, Tv, a, b, t) * M(Rv, Tv, a, b, t)$$

$$\geq \min\{ M(Su, Tv, a, b, t), 1, 1, M(Pu, Rv, a, b, t), M(Rv, Pu, a, b, t), M(Pu, Rv, a, b, t), M(Pu, Rv, a, b, t) * 1\}$$

$$\geq \min\left\{ \begin{matrix} M(Pu, Rv, a, b, t), 1, 1, M(Pu, Rv, a, b, t), M(Rv, Pu, a, b, t), \\ M(Pu, Rv, a, b, t), M(Pu, Rv, a, b, t) * 1 \end{matrix} \right\}$$

$$\geq M(Pu, Rv, a, b, t)$$

Hence from the Lemma it is clear that Pu=Rv.

Now we have to show that PPu=Pu. Let x=Pu and y=v in equation (1) we have

$$M(PPu, Rv, a, b, qt) \geq \min\{ M(SPu, Tv, a, b, t), M(SPu, PPu, a, b, t), M(Rv, Tv, a, b, t), M(PPu, Tv, a, b, t), M(Rv, SPu, a, b, t), M(PPu, Rv, a, b, t), M(SPu, Tv, a, b, t) * M(Rv, Tv, a, b, t)\}$$

$$\geq \min\{ M(SPu, Tv, a, b, t), 1,1, M(PPu, Pu, a, b, t), M(Pu, PPu, a, b, t), M(PPu, Pu, a, b, t), M(SPu, Tv, a, b, t) * 1\}$$

$$\geq \min\{ M(PPu, Pu, a, b, t), 1,1, M(PPu, Pu, a, b, t), M(Pu, PPu, a, b, t), M(PPu, Pu, a, b, t), M(PPu, Tv, a, b, t) * 1\}$$

$$\geq M(PPu, Pu, a, b, t)$$

Hence by the lemma it is clear that PPu=Pu

Now we have to show that Pu=RRv.Let x=u and y=Rv in equation (1) we have

$$M(Pu, RRv, a, b, qt) \geq \min\{ M(Su, TRv, a, b, t), M(Su, Pu, a, b, t), M(RRv, TRv, a, b, t), M(Pu, TRv, a, b, t), M(RRv, Su, a, b, t), M(Pu, RRv, a, b, t), M(Su, TRv, a, b, t) * M(RRv, TRv, a, b, t)\}$$

$$\geq \min\{ M(Pu, RRv, a, b, t), M(Pu, Pu, a, b, t), M(RRv, TRv, a, b, t), M(Pu, RRv, a, b, t), M(RRv, Pu, a, b, t), M(Pu, RRv, a, b, t), M(Su, TRv, a, b, t) * M(RRv, RRv, a, b, t)\}$$

$$\geq \min\{ M(Pu, RRv, a, b, t), 1, 1, M(Pu, RRv, a, b, t), M(RRv, Pu, a, b, t), M(Pu, RRv, a, b, t), M(Pu, RRv, a, b, t) * 1\}$$

$$\geq M(Pu, RRv, a, b, t)$$

Hence from the lemma it is clear that Pu=RRv.

PPu=SPu=Pu and Pu=RRv=TRv=TPu

Since Rv=Pu hence we have P(Pu)=S(Pu)=R(Pu)=T(Pu)

Let Pu=z then P(Z)=S(Z)=R(Z)=T(Z) where z is a common fixed point of P, S, R and T .Hence the uniqueness of the fixed point holds from equation (1)

Hence Proved.

Theorem 3.2 Let (X, M, ∗) be a complete fuzzy 3-metric space and let P, S, R, T be Self-mappings of X. If there exists $q \in (0,1)$ such that

$$M(Px, Ry, a, b, qt) \geq \phi(\min\{ M(Sx, Ty, a, b, t), M(Sx, Px, a, b, t), M(Ry, Ty, a, b, t), M(Px, Ty, a, b, t), M(Ry, Sx, a, b, t), M(Px, Ry, a, b, t)\}) \quad (2)$$

For all x, y ∈ X and $\phi: [0,1] \rightarrow [0,1]$ such that $\phi(t)>t$ for all $0<t<1$. If pairs (P, S) and (R, T) satisfies E.A property with sub sequentially continuous faintly compatible map then P, S, R, T have a unique common fixed point in X.

Proof :(P, S) and (R, T) satisfy E.A property which implies that there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Sx_n = t_1$ for some $t_1 \in X$ also $\lim_{n\to\infty} Rx_n = \lim_{n\to\infty} Tx_n = t_2$ for some $t_2 \in X$.

Since pairs (P, S) and (R, T) are faintly compatible therefore conditionally compatibility of (P, S) and (R, T) implies that there exist sequences $\{z_n\}$ and $\{z_{n'}\}$ in X satisfying $\lim_{n\to\infty} Pz_n = \lim_{n\to\infty} Sz_n = u$ for some u ∈X, such that $M(PSz_n, SPz_n, a, b, t) = 1$, also $\lim_{n\to\infty} Rz_{n'} = \lim_{n\to\infty} Tz_{n'} = v$ for some v ∈X, such that $M(RTz_n', TRz_n', a, b, t) = 1$.

As the pairs (P, S) and (R, T) are sub sequentially continuous, we get $\lim_{n\to\infty} PSz_n = Pu$, $\lim_{n\to\infty} SPz_n = Su$ and so Pu = Su also $\lim_{n\to\infty} RTz_{n'} = Rv$, $\lim_{n\to\infty} TRz_{n'} = Tv$ and so Rv = Tv. Since pairs (P, S) and (R, T) are faintly compatible, we get PSu = SPu So PPu = PSu =SPu = SSu also RTv = TRv So RRv=RTv=TRv=TTv.

Now we show that Pu=Rv.Let x=u and y=v in equation (2) we have

$$M(Pu, Rv, a, b, qt) \geq \phi(\min\{ M(Su, Tv, a, b, t), M(Su, Pu, a, b, t), M(Rv, Tv, a, b, t), M(Pu, Tv, a, b, t), M(Rv, Su, a, b, t), M(Pu, Rv, a, b, t)\})$$

$$\geq \phi(\min\{ M(Su, Tv, a, b, t), 1, 1, M(Pu, Rv, a, b, t), M(Rv, Pu, a, b, t), M(Pu, Rv, a, b, t)\})$$

$$\geq \phi(\min\{ M(Pu, Rv, a, b, t), 1, 1, M(Pu, Rv, a, b, t), M(Rv, Pu, a, b, t), M(Pu, Rv, a, b, t)\})$$

$$\geq \phi(M(Pu, Rv, a, b, t))$$

$$\geq M(Pu, Rv, a, b, t)$$

Hence from the Lemma it is clear that Pu=Rv.

Now we have to show that PPu=Pu .Let x=Pu and y=v in equation (2) we have

$$M(PPu, Rv, a, b, qt) \geq \phi(\min\{ M(SPu, Tv, a, b, t), M(SPu, PPu, a, b, t), M(Rv, Tv, a, b, t), M(PPu, Tv, a, b, t), M(Rv, SPu, a, b, t), M(PPu, Rv, a, b, t)\})$$

$$\geq \phi(\min\{ M(SPu, Tv, a, b, t), 1,1, M(PPu, Pu, a, b, t), M(Pu, PPu, a, b, t), M(PPu, Pu, a, b, t)\})$$

$$\geq \phi(\min\{ M(PPu, Pu, a, b, t), 1, 1, M(PPu, Pu, a, b, t), M(Pu, PPu, a, b, t), M(PPu, Pu, a, b, t)\})$$
$$\geq \phi(M(PPu, Pu, a, b, t))$$
$$\geq M(PPu, Pu, a, b, t)$$

Hence by the lemma it is clear that PPu=Pu

Now we have to show that Pu=RRv

Let x=u and y=Rv in equation (2) we have.

$$M(Pu, RRv, a, b, qt) \geq \phi(\min\{ M(Su, TRv, a, b, t), M(Su, Pu, a, b, t), M(RRv, TRv, a, b, t), M(Pu, TRv, a, b, t), M(RRv, Su, a, b, t), M(Pu, RRv, a, b, t)\})$$
$$\geq \phi(\min\{ M(Pu, RRv, a, b, t), M(Pu, Pu, a, b, t), M(RRv, TRv, a, b, t), M(Pu, RRv, a, b, t), M(RRv, Pu, a, b, t), M(Pu, RRv, a, b, t)\})$$
$$\geq \phi(\min\{ M(Pu, RRv, a, b, t), 1, 1, M(Pu, RRv, a, b, t), M(RRv, Pu, a, b, t), M(Pu, RRv, a, b, t)\})$$
$$\geq \phi(M(Pu, RRv, a, b, t))$$
$$\geq M(Pu, RRv, a, b, t)$$

Hence from the lemma it is clear that Pu=RRv

PPu=SPu=Pu and Pu=RRv=TRv=TPu

Since Rv=Pu hence we have P(Pu)=S(Pu)=R(Pu)=T(Pu).

Let Pu=z then P(Z)=S(Z)=R(Z)=T(Z) where z is a common fixed point of P, S, R and T .Hence the uniqueness of the fixed point holds from equation (2)

Hence Proved.

REFERENCE

[1]. Aamri, M. and Moutawakil, D. El. *"Some new common fixed point theorems under stric contractive conditions"*, *J. Math. Anal. Appl.*, **270** (2002), 181-188.

[2]. Al-Thagafi , M. A and Shahzad, N. *"Generalized I-non expansive self maps and invariant Approximations"*, *Acta. Math. Sinica*, **24**(5) (2008), 867-876.

[3]. Bisht, R.K and Shahzad, N. *"Faintly compatible mappings and common fixed points"*, *Fixed point theory and applications*, **2013**, 2013:156.

[4]. Jungck, G. and Rhodes, B.E *"Fixed Point for occasionally weakly compatible mappings"*, *Fixed Point Theory*, **7**(2006)280-296.

[5]. Pant, R.P and Bisht, R.K. "*Occasionally weakly compatible mappings and fixed points". Bull. Belg. Math. Soc. Simon Stevin*, **19** (2012), 655-661.

[6]. Patel, Shailesh T, Makwana, Vijay C and Soni Vijay P *"Fixed Point Theorems & Common in Fuzzy 3-Metric Spaces". International Journal of Emerging Research in Management & Technology*. Volume **4**, Issue 1,46-49 (2015)

[7]. Wadhwa, Kamal and Bharadwaj, Ved Prakash. "*Some common Fixed Point Theorems Using Faintly Compatible Maps in Fuzzy Metric space". International Journal of Fuzzy Mathematics and systems.* 2248-9940 Volume **4**, Number 2(2014), pp.155-160.

[8]. Wadhwa, Kamal and Bharadwaj, Ved Prakash. *"Common Fixed Point Theorems Using Faintly Compatible Mappings in Fuzzy Metric Spaces"*. Computer Engineering and Intelligent Systems.2222-2863.Vol.**5**, No.7, 2014.

[9]. Wadhwa, Kamal, Panthi, Jyoti and Bharadwaj, Ramakant. *"Fixed Point Theorem in Complete Fuzzy 3-Metric Space Through Rational Expression". International Journal of Theoretical & Applied Science,* **4**(2): 206-210(2012).