Coupled Fixed Point Theorem in Menger Space

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ABSTRACT: The purpose of this paper is to introduce the new concept of occasionally weakly compatible mappings for coupled maps and prove a coupled fixed point theorem under more general t-norm (H-type norm) in Menger space. Finally, we also give an application.

Keywords. Menger Spaces, Occasionally Weakly Compatible Maps.

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I. INTRODUCTION

Menger [10] introduced the notion of a probabilistic metric space (PM-space) which was, in fact, a generalization of metric space in the year 1942. The idea behind this is to associate a distribution function with a pair of points, say (p,q), denoted by F_p,q(t) where t > 0 and interpret this function as the probability that distance between p and q is less than t, whereas in the metric space, the distance function is a single positive number. Sehgal [13] initiated the study of fixed points in probabilistic metric spaces. The study of these spaces was expanded rapidly with the pioneering works of Schweizer and Sklar [12]. In 1991, Mishra [11] introduced the notion of compatible mappings in the setting of probabilistic metric space. In 1996, Jungck [8] introduced the notion of weakly compatible. Further, Singh and Jain [14] proved some results for weakly compatible in Menger spaces. Cho, Murthy and Stojakovik [2] proposed the concept of compatible maps of type (A) in Menger space and gave some fixed point theorems. Recently, using the concept of compatible mappings in Menger space and gave some fixed point theorems. Finally, we also give an application.

II. PRELIMINARIES

Definition 2.1. [15] A mapping F : [0, ∞) → [0,1] is called a distribution function if it is non-decreasing and left-continuous with \( \inf_{x \in R} F(x) = 0 \). If in addition F(0) = 0, then F is called a distance distribution function. A distance distribution function F satisfying \( \lim_{t \to 0} F(t) = 1 \) is called a Menger distance distribution function. The set of all Menger distance distribution functions is denoted by \( D^* \). This space \( D^* \) is partially ordered by the usual pointwise ordering of functions, that is, \( F \leq G \) if and only if \( F(t) \leq G(t) \) for all \( t \in [0, \infty) \). The maximal element for \( D^* \) in this order is the distance distribution function \( \varepsilon_{0} \) given by

\[
\varepsilon_{0}(t) = \begin{cases} 
0 & , \ t = 0 \\
1 & , \ t > 0.
\end{cases}
\]

Definition 2.2. [15] A triangular norm (shortly, t-norm) is a binary operation \( \Delta \) on \([0,1] \) satisfying the following conditions:

1. \( \Delta \) is associative and commutative;
2. \( \Delta \) is continuous;
3. \( \Delta(a, 1) = a \) for all \( a \in [0, 1] \);
4. \( \Delta(a, b) \leq \Delta(c, d) \) whenever \( a \leq c \) and \( b \leq d \) for all \( a, b, c, d \in [0,1] \).

Two typical examples of the continuous t-norm are \( \Delta_s(a,b) = ab \), \( \Delta_m(a,b) = \min\{a, b\} \) for all \( a, b \in [0,1] \).

Now, the t-norm is recursively defined by \( \Delta^* = \Delta \) and

\[
\Delta^n(x_1, \ldots, x_{n+1}) = \Delta(\Delta^{n-1}(x_1, \ldots, x_n), x_{n+1})
\]

for all \( n \geq 2 \) and \( x_i \in [0,1], i = 1, 2, \ldots, n + 1 \).
A t-norm \( \Delta \) is said to be of Hadžić-type if the family \( \{\Delta^a\} \) is equicontinuous at \( x = 1 \), that is, for any \( \varepsilon \in (0, 1) \), there exists \( \delta \in (0, 1) \) such that

\[
1 - \delta \Rightarrow \Delta^a(1) > 1 - \varepsilon
\]

for all \( n \geq 1 \).

\( \Delta_M \) is a trivial example of a t-norm of Hadžić-type [11].

**Definition 2.3.** [15] A Menger probabilistic metric space (briefly, a Menger PM-space) is a triple \((X, \Phi, \Delta)\), where \( X \) is a nonempty set, \( \Phi \) is a continuous t-norm and \( \Phi \) is a mapping from \( X \times X \rightarrow D^+ \) (\( F_{x,y} \) denotes the value of \( \Phi \) at the pair \((x, y)\)) satisfying the following conditions:

- **(PM-1)** \( F_{x,y}(t) = 1 \) for all \( x, y \in X \) and \( t > 0 \) if and only if \( x = y \);
- **(PM-2)** \( F_{x,y}(t) = F_{y,x}(t) \) for all \( x, y \in X \) and \( t > 0 \);
- **(PM-3)** \( F_{x,y}(t + s) \geq \Delta(F_{x,y}(t), F_{x,y}(s)) \) for all \( x, y, z \in X \) and \( t, s \geq 0 \).

**Definition 2.4.** [15] Let \((X, \Phi, \Delta)\) be a Menger PM-space.

1. A sequence \( \{x_n\} \) in \( X \) is said to be convergent to a point \( x \in X \) (write \( x_n \rightarrow x \)) if, for any \( t > 0 \) and \( 0 < \varepsilon < 1 \), there exists a positive integer \( N \) such that

\[
F_{x_n, x}(t) > 1 - \varepsilon
\]

whenever \( n \geq N \);

2. A sequence \( \{x_n\} \) in \( X \) is called a Cauchy sequence if, for any \( t > 0 \) and \( 0 < \varepsilon < 1 \), there exists a positive integer \( N \) such that

\[
F_{x_n, x_m}(t) > 1 - \varepsilon \quad \text{whenever} \quad n, m \geq N.
\]

3. A Menger PM-space \((X, \Phi, \Delta)\) is said to be complete if every Cauchy sequence in \( X \) is convergent to a point in \( X \).

**Definition 2.5.** [1] Let \( X \) be a non-empty set and \( T : X \times X \rightarrow X \) be a mapping. An element \((x, y) \in X \times X\) is said to be a coupled fixed point of \( T \) if

\[
T(x, y) = x, \quad T(y, x) = y.
\]

**Definition 2.6.** [9] Let \( X \) be a non-empty set and \( T : X \times X \rightarrow X, h : X \rightarrow X \) be two mappings.

1. An element \((x, y) \in X \times X\) is said to be a coupled coincidence point of \( T \) and \( h \) if

\[
T(x, y) = h(x), \quad T(y, x) = h(y);
\]

2. An element \((x, y) \in X \times X\) is said to be a coupled common fixed point of \( h \) and \( T \) if

\[
T(x, y) = h(x) = x, \quad T(y, x) = h(y) = y.
\]

**Definition 2.7.** [15] Let \((X, \Phi, \Delta)\) be a Menger PM-space and \( T : X \times X \rightarrow X, h : X \rightarrow X \) be two mappings. The mappings \( T \) and \( h \) are said to be weakly compatible (or w-compatible) if they commute at their coupled coincidence points, i.e., if \((x, y)\) is a coupled coincidence point of \( T \) and \( h \), then \( g(F(x, y)) = F(gx, gy) \).

**Definition 2.8.** Let \((X, \Phi, \Delta)\) be a Menger PM-space and \( T : X \times X \rightarrow X, h : X \rightarrow X \) be two mappings. The mappings \( T \) and \( h \) are said to be occasionally weakly compatible if there is a point \( x \in X \) which is a coupled coincidence point of \( f \) and \( g \) at which \( f \) and \( g \) commute.

**Definition 2.9.** [12] Define \( \Phi = \{ \phi : R^+ \rightarrow R^+ \} \), where \( R^+ = [0, +\infty) \) and each \( \phi \in \Phi \) satisfies the following conditions:

- **(\phi-1)** \( \phi \) is non-decreasing;
- **(\phi-2)** \( \phi \) is upper semi-continuous from the right;
- **(\phi-3)** \( \sum_{n=0}^{\infty} \phi^n(t) < \infty \) for all \( t > 0 \), where \( \phi^{n+1}(t) = \phi(\phi^n(t)), n \in N \).

Clearly, if \( \phi \in \Phi \), then \( \phi(t) < t \) for all \( t > 0 \).

### III. MAIN RESULT

**Theorem 3.1.** Let \((X, \Phi, *)\) be Menger PM-Space, * being continuous t-norm of H-type. Let \( f : X \times X \rightarrow X \) and \( g : X \rightarrow X \) be two mappings and there exists \( \phi \in \Phi \) such that followings hold:

1. \( F_{x,y}(t) \geq 1 - \delta \) for all \( x, y \in X \) and \( t > 0 \);
2. Suppose that \( f(X \times X) \subset g(X) \);
3. pair \((f, g)\) is occasionally weakly compatible;
4. range space of one of the maps \( f \) or \( g \) is complete.

Then \( f \) and \( g \) have a coupled coincidence point. Moreover, there exists a unique point \( x \) in \( X \) such that \( f(x, y) = g(x) \).

**Proof.** Let \( x_0, y_0 \) be two arbitrary points in \( X \). Since \( f(X \times X) \subseteq g(X) \), we can choose \( x_1, y_1 \) in \( X \) such that \( g(x_1) = f(x_0, y_0) \), \( g(y_1) = f(y_0, x_0) \). Continuing in this way we can construct two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\[
g(x_{n+1}) = f(x_n, y_n) \quad \text{and} \quad g(y_{n+1}) = f(y_n, x_n) \quad \text{for all} \quad n \geq 0.
\]

**Step 1.** Firstly we show that \( \{gx_n\} \) and \( \{gy_n\} \) are Cauchy sequences.

Since * is a t-norm of H-type, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
(1 - \delta)^n \geq (1 - \varepsilon)^n \quad \text{for all} \quad n \in N.
\]

Since \( \lim_{n \to \infty} F_{x,y}(t) = 1 \), for all \( x, y \in X \), there exists \( t_0 > 0 \) such that

\[
F_{g^p, g^q}(t_0) \geq \Delta(1 - \delta) \quad \text{and} \quad F_{g^p, g^q}(t_0) \geq 1 - \delta.
\]

Since \( \phi \in \Phi \) and using condition \( (\phi-3) \), we have

\[
\sum_{n=0}^{\infty} \phi^n(t) < \infty. \quad \text{Then for any} \quad t > 0, \text{there exists} \quad n_0 \in N \text{such that}
\]

\[
t > \sum_{k=n_0}^{\infty} \phi^k(t_0).
\]
From (3.1), we have
\[ F_{x_1,y_1}(\psi(t_0)) = F_{(x_0,y_0), (x_1,y_1)}(\psi(t_0)) \]
\[ \geq F_{E_{x_1}(t_0)} * F_{E_{y_1}(t_0)} \]
\[ F_{E_{x_2}, E_{y_2}}(\psi(t_0)) = F_{(x_1,y_1), (x_2,y_2)}(\psi(t_0)) \]
\[ \geq F_{E_{x_1}(t_0)} * F_{E_{y_1}(t_0)}. \]
Similarly, we can also get
\[ F_{E_{x_2}, E_{y_2}}(\phi(t_0)) = F_{(x_1,y_1), (x_2,y_2)}(\phi(t_0)) \]
\[ \geq F_{E_{x_1}(t_0)}^2 * F_{E_{y_1}(t_0)}^2. \]
Continuing in this way, we can get
\[ F_{E_{x_n}, E_{y_n}}(\phi(t_0)) \geq F_{E_{x_1}(t_0)}^{2^{n-1}} \]
\[ F_{E_{x_{n+1}}, E_{y_{n+1}}}(\phi(t_0)) \geq F_{E_{x_1}(t_0)}^{2^{n-1}} \]
So, from (3.5) and (3.6), for \( m > n \), we have
\[ F_{E_{x_n} E_{y_n}}(t) \geq F_{x_n, y_n} \left( \sum_{k=n}^{\infty} \phi^k(t_0) \right) \]
\[ \geq F_{x_n, y_n} \left( \sum_{k=n}^{m-1} \phi^k(t_0) \right) \]
\[ \geq F_{E_{x_n} E_{y_n}}(\phi(t_0)) * F_{E_{x_{n+1}} E_{y_{n+1}}}(\phi^2(t_0)) * \cdots \]
\[ \geq \left[ F_{E_{y_1}(t_0)}^{2^{n-1}} * F_{E_{x_1}(t_0)}^{2^{n-1}} \right] \]
\[ \geq \left[ F_{E_{y_1}(t_0)}^{2^{n-1}} * F_{E_{x_1}(t_0)}^{2^{n-1}} \right] \]
\[ \geq (1 - \delta) * (1 - \delta) * \cdots * (1 - \delta) \geq (1 - \varepsilon), \]
which implies that
\[ F_{x_n, y_n}(t) \geq (1 - \varepsilon), \]
for all \( m, n \in N \) with \( m > n \).
So, \( \{gx_n\} \) is a Cauchy sequence. Similarly, we can get that \( \{gy_n\} \) is also a Cauchy sequence.

**Step 2.** Now we show that \( f \) and \( g \) have a coupled coincidence point.

Without loss of generality, we assume that \( g(X) \) is complete, then there exists points \( x, y \) in \( g(X) \) so that
\[ \lim_{n \to \infty} g(x_{n+1}) = x, \quad \lim_{n \to \infty} g(y_{n+1}) = y. \]
Again \( x, y \in g(X) \) implies the existence of \( p, q \) in \( X \) so that
\[ g(p) = x \quad \text{and hence} \quad \lim_{n \to \infty} g(x_{n+1}) = \lim_{n \to \infty} f(x_n, y_n) = g(p) = x, \]
\[ \lim_{n \to \infty} g(y_{n+1}) = \lim_{n \to \infty} f(y_n, x_n) = g(q) = y. \]
From (3.1),
\[ f(x, y) = g(p, q) \geq F_{E_{x_0} E_{y_0}}(t_0) * F_{E_{y_0} E_{x_0}}(t_0). \]
Taking limit as \( n \to \infty \), we get
\[ F_{g(t_0) g(t_0)}(\phi(t_0)) = 1 \]
which implies that \( f(p, q) = g(p, q) = x \).
Similarly, \( f(q, p) = g(q, p) = y \).
But \( f \) and \( g \) are occasionally weakly compatible, so that \( f(p, q) = g(p, q) = x \) and \( f(q, p) = g(q, p) = y \) implies \( g(p, q) = f(p, q) \) and \( f(q, p) = f(q, p) \).
Hence \( f \) and \( g \) have a coupled coincidence point.

**Step 3.** Now we show that \( g(x) = y \) and \( g(y) = x \).
Since \( * \) is a \( t \)-norm of \( H \)-type, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[ (1 - \delta) * (1 - \delta) * \cdots * (1 - \delta) \geq (1 - \varepsilon), \]
for all \( p \in N \).
Since \( \lim_{t \to \infty} F_{x, y}(t) = 1 \), for all \( x, y \) in \( X \), there exists \( t_0 \)
\[ t > \sum_{k=n}^{\infty} \phi^k(t_0). \]
Using (3.1), we have
\[ F_{E_{x_1} E_{y_1}}(\phi(t_0)) = F_{E_{x_1} E_{y_1}}(\phi(t_0)) \geq F_{E_{x_1} E_{y_1}}(t_0) \]
\[ F_{E_{y_1} E_{x_1}}(\phi(t_0)) \]
\[ \geq \left[ F_{E_{x_1}(t_0)}^{2^{n-1}} * F_{E_{y_1}(t_0)}^{2^{n-1}} \right] \]
\[ \geq (1 - \delta) * (1 - \delta) * \cdots * (1 - \delta) \geq (1 - \varepsilon), \]
which implies that
\[ F_{E_{x_n} E_{y_n}}(t) \geq (1 - \varepsilon), \]
for all \( m, n \in N \) with \( m > n \).
So, \( \{gx_n\} \) is also a Cauchy sequence.

**Step 4.** Next we shall show that \( x = y \).
Since \(*\) is a t-norm of H-type, for any \(\varepsilon > 0\), there exists \(\delta > 0\) such that
\[
\left(1 - \frac{\delta}{p}\right) \ast \left(1 - \frac{\delta}{p}\right) \ast \ldots \ast \left(1 - \frac{\delta}{p}\right) \geq (1 - \varepsilon),
\]
for all \(p \in \mathbb{N}\).
Since \[\lim_{t \to \infty} F_{x, y}(t) = 1, \text{ for all } x, y \in \mathbb{X}, \text{ there exists } t_0 > 0 \text{ such that } F_{x, z}(t_0) \geq (1 - \delta).
\] Since \(\phi \in \Phi\) and using condition \((\phi - 3)\), we have
\[
\sum_{n=1}^{\infty} \phi^n(t_0) < \infty.
\]
Then for any \(t > 0\), there exists \(n_0 \in \mathbb{N}\) such that
\[
t > \sum_{k=n_0}^{\infty} \phi^k(t_0).
\]
Using \((3.1)\), we have
\[
F_{x, z}(\phi(t_0)) = F_{x, z}([F_{x, z}(t_0)]^{\phi(t_0)}) \geq \sum_{k=0}^{\infty} \phi^k(t_0) \in \Phi.
\]
Thus, we have
\[
F_{x, z}(t) \geq F_{x, z} \left( \sum_{k=n_0}^{\infty} \phi^k(t_0) \right)
\]
\[
\geq F_{x, z} \left( \phi^{n_0}(t_0) \right)
\]
\[
\geq [F_{x, z}(t_0)]^{n_0} = [F_{x, z}(t_0)]^2
\]
\[
\geq (1 - \delta)^{n_0}
\]
\[
= 2^{\infty}.
\]
which implies that \(x = z\).

Hence, \(f\) and \(g\) have a unique common fixed point \(x\) in \(X\).

**IV. AN APPLICATION**

**Theorem 4.1.** Let \((X, \mathcal{F}, \ast)\) be a Menger PM-space, \(\ast\) being continuous t-norm defined by \(a \ast b = \min\{a, b\}\) for all \(a, b \in X\). Suppose \(P\) and \(Q\) be occasionally weakly compatible self maps on \(X\) satisfying the following conditions:
\[
(4.1) P(X) \subseteq Q(X),
\]
\[
(4.2) \text{there exists } \phi \in \Phi \text{ such that } F_{P, Q}(\phi(t)) \geq F_{Q, Q}(t) \text{ for all } x, y \in \mathbb{X} \text{ and } t > 0.
\]
If range space of any one of the maps \(P\) or \(Q\) is complete, then \(P\) and \(Q\) have a unique common fixed point in \(X\).

**Proof.** By taking \(f(x, y) = P(x)\) and \(g(x) = Q(x)\) for all \(x, y \in X\) in Theorem 3.1, we get the desired result.

Taking \(\phi(t) = kt, k \in (0, 1)\), we have the following:

**Corollary 4.1.** Let \((X, \mathcal{F}, \ast)\) be a Menger PM-space, \(\ast\) being continuous t-norm defined by \(a \ast b = \min\{a, b\}\) for all \(a, b \in X\). Suppose \(P\) and \(Q\) be occasionally weakly compatible self maps on \(X\) satisfying \((4.1)\) and the following condition:
\[
(4.3) \text{there exists } k \in (0, 1) \text{ such that } F_{P, Q}(kt) \geq F_{Q, Q}(t) \text{ for all } x, y \in X \text{ and } t > 0.
\]
If range space of any one of the maps P or Q is complete, then P and Q have a unique common fixed point in X. Taking $Q = I$, the identity map on $X$, we have the following:

**Corollary 4.2.** Let $(X, \mathcal{F}, *)$ be a Menger PM-space, * being continuous t-norm defined by $a*b = \min\{a, b\}$ for all $a, b$ in $X$. Suppose $\rho$ and $Q$ be occasionally weakly compatible self maps on $X$ satisfying (4.1) and the following condition:

(4.4) there exists $k \in (0, 1)$ such that

$$F_{P,P}(kt) \geq F_{x,y}(t)$$

for all $x, y$ in $X$ and $t > 0$. If range space of the map $P$ is complete, then $P$ and $Q$ have a unique common fixed point in $X$.

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