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On Some Common Fixed Point Theorems with Rational expressions on Cone Metric Spaces over a Banach Algebra for Integral Type Mappings

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ABSTRACT: In the present paper Mahpeyker Ozturk and Metin Basarir have defined a new space called a BA –cone metric space by taking Banach algebra instead of a Banach space. Some common fixed point theorems involving rational expressions have been proved and some consequences obtained in these spaces. Also we have extended this work to four mappings with a weak commutatively property in BA – cone metric spaces for Integral type mappings.

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Keywords: Fixed point, Cone metric space, Metric space, Rational expression, Weak commutatively, Banach Algebra.

I. INTRODUCTION

Fixed point theory plays basic role in application of various branches of mathematics from elementary calculus and linear algebra to topology and analysis. Fixed point theory is not restricted to mathematics and this theory has many application in other disciplines. This theory is closely related to game theory, military, economics, statistics and medicine.

Much work has been done involving fixed points using the Banach contraction principle. This principle has been extended to other kinds of contraction principle, such as contractive conditions involving product, rational expressions and many others. The Banach contraction principle with rational expressions have been extended and some fixed and common fixed point theorems obtained in [4-5]. In [3], common fixed points for a pair of self mappings satisfying a rational expression have been obtained.

Quiet recently; Huang and Zhang[6] generalized the notion of metric space by replacing the real numbers by an ordered Banach space, thereby defining cone metric spaces. They have investigated convergence in cone metric spaces, introduced completeness of cone metric spaces, and proved a Banach contraction mapping theorem, and some other fixed points theorems involving contractive type mappings in cone metric spaces using the normality condition. Later, Various authors have proved some common fixed point theorems with normal and non-normal cones in these spaces.

The aim of this paper is to extend the result in [3] and Mahpeyker Ozturk and Metin Basarir to BA- cone metric spaces which we have defined using a Banach algebra instead of a Banach space. We get some consequences related to some special properties of mappings for Integral type.

II. PRELIMINARIES

Mahpeyker Ozturk and Metin Basarir give some facts and definitions which we need in the sequel.

Let B be a real Banach space and K a subset of B. Then K is called a cone if and only if

1. K is closed, nonempty and $K \neq \{0\}$,
2. $a,b \in R$, $a,b \geq 0$, $x,y \in K$, $\Rightarrow ax+by \in K$
3. $x \in K$ and $-x \in K \Rightarrow x = 0$.

If we take a Banach algebra instead of Banach space, then we say that K is a BA-cone. Given a cone $K \subset B$, we define a partial ordering $\leq$ with respect to K by $x \leq y$ if and only if $y-x \in K$ .We write $x < y$ if $x \leq y$ but $x \neq y$; $x \ll y$ if $y - x \in \text{int } K$, where int K is the interior of K.

We write $x < y$ if $x \leq y$ but $x \neq y$; $x \ll y$ if $y - x \in$ int K, where in t K is the interior of K. The cone K is called normal if there is a number $M > 0$ such that for all $x,y \in B$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq M\|y\| . \qquad (2.1)$$

2.1. Definition. Let X be non empty set,B a real Banach space and $K \subset B$ a cone. Suppose the mapping $d : X \times X \to B$ satisfies

d_1. $0 < d(x,y)$ for all $x,y \in K$ and $d(x,y) = 0$ if and only if $x = y$;

d_2. $d(x,y) = d(y,x)$ for all $x,y \in K$;

d_3. $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.It is obvious that the concept of a cone metric space is more general than that of a metric space.

If we replace the Banach space with a Banach algebra in Definition2.1 then we obtain a new space which is called a BA- cone metric space.

2.2 Example. Let $B = R^2$, $K = \{ (x,y) : x,y \geq 0\}$, X = R and let $d : X \times X \to B$ be defined by $d(x,y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant.Then (X,d) is a BA-cone metric space since B is a real commutative Banach algebra.

2.3 Example. (Mahpeyker Ozturk and Metin Basarir) Let C_R^2 ([0,1]) be the space of all real functions on [0,1] whose second derivative is continuous. We recall that for $a,b > 0$,the space C_R^2 ([0,1]) with the norm

$$\|f\| = \|f\|_\infty + a\|f'\|_\infty + b\|f''\|_\infty$$

Is a Banach space, where $\|f\|_\infty = \sup_{t\in[0,1]}|f(t)|$ .This space is a Banach algebra if and only if $2b \leq a^2$, If we take $X = B = C_R^2$ ([0,1]) with the above norm and $K = \{ u \in B: u \geq 0\}$, then (X,d) becomes a cone metric space where $d(x,y) = \left(\sup_{t\in[0,1]}|x(t) - y(t)|\right)f(t)$ and $f : [0,1] \to R$, $f(t) = e^t$.But if we take $2b > a^2$ then B is not Banach Algebra, hence (X,d) is not a BA-cone metric space.

2.4. Definition. Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. If for every $c \in B$ with $0 \ll c$,

1. there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n,x) \ll c$, then $\{x_n\}$ is said to be convergent,
2. there is $N \in \mathbb{N}$ such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n,x) \to 0$ as $n \to \infty$.

2.5 Remark. Let us recall that if X is a normal cone, $x \in K$, $a \in R$, $a \in [0,1)$ and $x \leq ax$, then x = 0.

Let $f: X \to X$ and $x_0 \in K$. The function f is continuous at x_0 if for any sequence $x_n \to x_0$ we have $f(x_n) \to f(x_0)$.

Throughout the paper, we take B to be a Banach commutative division algebra. Recall that, a division algebra is an algebra with identity e, in which every non –zero element is a unit, where the identity is a non-zero element such that xe = ex = x for all x and in any algebra with identity e, an element which has an inverse is called a unit , i.e x is a unit if and only if there exists an inverse y such that xy = yx = e. We write $y = x^{-1}$ and observe that x^{-1} is unique when it exists.

Also, throughout we will use a cone which has non empty interior. Therefore the uniqueness of the limit for a convergence sequence will be guaranteed.

2.6. Theorem (Banach's contraction principle) Let (X, d) be a complete metric space, c∈(0,1) and f: X→X be a mapping such that for each x, y ∈X,

d(fx, fy) ≤ cd (x, y) Then f has a unique fixed point a ∈X, such that for each

x∈ X, $\lim_{n\to\infty} f^n(x) = a.$

After the classical result, Kannan [7] gave a subsequently new contractive mapping to prove the fixed point theorem. Since then a number of mathematicians have been worked on fixed point theory dealing with mappings satisfying various type of contractive conditions.

In 2002, A. Branciari [2] analysed the existence of fixed point for mapping f defined on a complete metric space (X,d) satisfying a general contractive condition of integral type.

2.7 Theorem (Branciari) Let (X, d) be a complete metric space ,c $\in (0,1)$ and let f : X→ X be a mapping such that for each x, y ∈X,

$\int_0^{d(fx,fy)} \emptyset(t)dt \leq c \int_0^{d(x,y)} \emptyset(t)dt$ where $\emptyset:[0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,+\infty)$, non-negative ,and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \emptyset(t)dt$, then f has a unique fixed point a ∈X, such that for each x ∈ X, $\lim_{n\to\infty} f^n(x) = a.$

After the paper of Branciari, a lot of research works have been carried out on generalizing contractive condition of integral type for different contractive mappings satisfying various known properties. A fine work has been done by Rhoades [2] extending the result of Branciari by replacing the condition [1.2] by the following

$$\int_0^{d(fx,fy)} \emptyset(t)dt \leq \int_0^{\max\left\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\right\}} \emptyset(t)dt.$$

2.8 Theorem. Let (X,d) be a BA- complete cone metric space, K a BA-normal cone with normal constant M. Suppose the mappings S and T are two self- maps of X such that S and T satisfy the inequality

$$d(Sx,Ty) \leq \alpha \frac{d(x,Sx)d(x,Ty)+[d(x,y)]^2+d(x,Sx)d(x,y)}{d(x,Sx)+d(x,y)+d(x,Ty)}$$

for all x, y in X with x$\neq$ y, 0$< \alpha <$1 and $d(x,Sx) + d(x,y) + d(x,Ty) \neq 0$. Then S and T have a common fixed point. Further if $d(x,Sx) + d(x,y) + d(x,Ty) = 0$ implies d(Sx,Ty) = 0, then S and T have a unique common fixed point.

Our main results are extended and modified for above result.

III. MAIN RESULTS

In the following theorem we carry over to BA- cone metric spaces.

3.1 Theorem. Let (X,d) be a BA- complete cone metric space, K a BA-normal cone with normal constant M. Suppose the mappings S and T are two self- maps of X such that S and T satisfy the inequality

$$\int_0^{d(Sx,Ty)} \emptyset(t)dt \leq \alpha \int_0^{\frac{d(x,Sx)d(x,Ty)+[d(x,y)]^2+d(x,Sx)d(x,y)}{d(x,Sx)+d(x,y)+d(x,Ty)}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(x,Tx)d(x,y)+d(x,Ty)d(x,y)+[d(x,y)]^2}{d(x,Sx)+d(x,y)+d(x,Ty)}} \emptyset(t)dt$$

$$+\gamma \int_0^{\frac{d(x,Tx)d(x,Ty)+d(x,Sx)d(x,y)+[d(x,y)]^2}{d(x,Sx)+d(x,y)+d(x,Ty)}} \emptyset(t)dt \qquad (3.1.1)$$

For all x, y in X with $x \neq y$, $0 < \alpha + \beta + \gamma < 1$ and $\emptyset:[0,+\infty) \rightarrow [0,+\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,+\infty)$, non- negative ,and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \emptyset(t)dt$. Also $d(x,Sx) + d(x,y) + d(x,Ty) \neq 0$. Then S and T have a common fixed point. Further if $d(x,Sx) + d(x,y) + d(x,Ty) = 0$ implies $d(Sx,Ty) = 0$, then S and T have a unique common fixed point.

Proof: Let an x_0 be arbitrary point of X, and define x_n by

$$x_{2n+2} = Tx_{2n+1}, \quad x_{2n+1} = Sx_{2n}, \ n = 0, 1, 2 \ldots$$

Let $d(x,Sx) + d(x,y) + d(x,Ty) \neq 0$. Then using (3.1),

$$\int_0^{d(x_{2n+1},x_{2n+2})} \emptyset(t)dt = \int_0^{d(Sx_{2n},Tx_{2n+1})} \emptyset(t)dt$$

$$\leq \alpha \int_0^{\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tx_{2n+1})+[d(x_{2n},x_{2n+1})]^2+d(x_{2n},Sx_{2n})d(x_{2n},x_{2n+1})}{d(x_{2n},Sx_{2n})+d(x_{2n},x_{2n+1})+d(x_{2n},Tx_{2n+1})}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(x_{2n},Tx_{2n})d(x_{2n},x_{2n+1})+d(x_{2n},Tx_{2n+1})d(x_{2n},x_{2n+1})+[d(x_{2n},x_{2n+1})]^2}{d(x_{2n},Sx_{2n})+d(x_{2n},x_{2n+1})+d(x_{2n},Tx_{2n+1})}} \emptyset(t)dt$$

$$+\gamma \int_0^{\frac{d(x_{2n},Tx_{2n})d(x_{2n},Tx_{2n+1})+d(x_{2n},Sx_{2n})d(x_{2n},x_{2n+1})+[d(x_{2n},x_{2n+1})]^2}{d(x_{2n},Sx_{2n})+d(x_{2n},x_{2n+1})+d(x_{2n},Tx_{2n+1})}} \emptyset(t)dt$$

$$= \alpha \int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+2})+[d(x_{2n},x_{2n+1})]^2+d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+1})}{d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+2})}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+2})d(x_{2n},x_{2n+1})+[d(x_{2n},x_{2n+1})]^2}{d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+2})}} \emptyset(t)dt$$

$$+\gamma \int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+2})+d(x_{2n},x_{2n+1})d(x_{2n},x_{2n+1})+[d(x_{2n},x_{2n+1})]^2}{d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+1})+d(x_{2n},x_{2n+2})}} \emptyset(t)dt$$

Hence,

$$\int_0^{d(x_{2n+1},x_{2n+2})} \emptyset(t)dt \leq (\alpha + \beta + \gamma) \int_0^{d(x_{2n},x_{2n+1})} \emptyset(t)dt$$

Similarly;

$$\int_0^{d(x_{2n},x_{2n+1})} \emptyset(t)dt = \int_0^{d(Sx_{2n},Tx_{2n-1})} \emptyset(t)dt$$

$$\leq \alpha \int_0^{\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tx_{2n-1})+[d(x_{2n},x_{2n-1})]^2+d(x_{2n},Sx_{2n})d(x_{2n},x_{2n-1})}{d(x_{2n},\ Sx_{2n})+d(x_{2n},x_{2n-1})+d(x_{2n},Tx_{2n-1})}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(x_{2n},Tx_{2n})d(x_{2n},x_{2n-1})+d(x_{2n},Tx_{2n-1})d(x_{2n},x_{2n-1})+[d(x_{2n},x_{2n-1})]^2}{d(x_{2n},\ Sx_{2n})+d(x_{2n},x_{2n-1})+d(x_{2n},Tx_{2n-1})}} \emptyset(t)dt$$

$$+\gamma \int_0^{\frac{d(x_{2n},Tx_{2n})d(x_{2n},Tx_{2n-1})+d(x_{2n},Sx_{2n})d(x_{2n},x_{2n-1})+[d(x_{2n},x_{2n-1})]^2}{d(x_{2n},\ Sx_{2n})+d(x_{2n},x_{2n-1})+d(x_{2n},Tx_{2n-1})}} \emptyset(t)dt$$

$$= \alpha \int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n},x_{2n})+[d(x_{2n},x_{2n-1})]^2+d(x_{2n},x_{2n+1})d(x_{2n},x_{2n-1})}{d(x_{2n},\ x_{2n+1})+d(x_{2n},x_{2n-1})+d(x_{2n},x_{2n})}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n},x_{2n-1})+d(x_{2n},x_{2n})d(x_{2n},x_{2n-1})+[d(x_{2n},x_{2n-1})]^2}{d(x_{2n},\ x_{2n+1})+d(x_{2n},x_{2n-1})+d(x_{2n},x_{2n})}} \emptyset(t)dt$$

$$+\gamma \int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n},x_{2n})+d(x_{2n},x_{2n+1})d(x_{2n},x_{2n-1})+[d(x_{2n},x_{2n-1})]^2}{d(x_{2n},\ x_{2n+1})+d(x_{2n},x_{2n-1})+d(x_{2n},x_{2n})}} \emptyset(t)dt$$

Hence,

$$\int_0^{d(x_{2n},x_{2n+1})} \emptyset(t)dt \leq (\alpha+\beta+\gamma) \int_0^{d(x_{2n-1},x_{2n})} \emptyset(t)dt$$

$$\leq \delta \int_0^{d(x_{2n-1},x_{2n})} \emptyset(t)dt \qquad \text{where } \delta = \alpha+\beta+\gamma$$

By this way, if we continue, we get

$$\int_0^{d(x_{2n+1},x_{2n+2})} \emptyset(t)dt \leq \delta \int_0^{d(x_{2n},x_{2n+1})} \emptyset(t)dt$$

$$\leq \delta^2 \int_0^{d(x_{2n-1},x_{2n})} \emptyset(t)dt$$

…………………………..

$$\leq \delta^{2n+1} \int_0^{d(x_0,x_1)} \emptyset(t)dt .$$

It is obvious that the following inequality holds for $m > n$.

$$d(x_n,x_{n+m}) \leq \sum_{i=1}^{m} d(x_{n+i-1},x_{n+i})$$

$$\leq \sum_{i=1}^{m} \delta^{n+i-1} d(x_0,x_1)$$

$$\leq \frac{\delta^n}{1-\delta} d(x_0,x_1)$$

By (2.1)

$$\int_0^{\|d(x_n,x_{n+m})\|} \emptyset(t)dt \leq M\frac{\delta^n}{1-\delta}\int_0^{\|d(x_0,x_1)\|} \emptyset(t)dt \quad k=\frac{\delta^n}{1-\delta} \tag{3.1.2}$$

Which implies that $\int_0^{d(x_n,x_{n+m})} \varphi(t)dt = 0$ as $n \to \infty$. (3.1.3)

Now we prove that $\{x_n\}$ is a Cauchysequence. Suppose it is not.Then there exists an $\varepsilon > 0$ and sub sequence $\{y_{m(p)}\}$and $\{y_{n(p)}\}$such that

M(p) $<$ n(p) $<$ m(p+1) with

$d(x_{n(p)}, x_{m(p)})$ $\geq \varepsilon$,

$$d(x_{n(p)-1}, x_{m(p)} <\varepsilon \quad < \varepsilon \tag{3.1.4}$$

Now

$$d(x_{m(p)-1}, x_{n(p)-1}) \leq d(x_{m(p)-1}, x_{m(p)}) + d(x_{m(p)}, x_{n(p)-1})$$

$$< d(x_{m(p)-1}, x_{m(p)}) + \varepsilon \tag{3.1.5}$$

From (3.1.3), (3.1.5), we get

$$\lim_{p\to\infty} \int_0^{d(x_{m(p)-1}, x_{n(p)-1})} \varphi(t)dt \leq \int_0^{\varepsilon} \varphi(t)dt \tag{3.1.6}$$

Using (3.1.2), (3.1.4), and (3.1.6) we get ,

$$\int_0^{\varepsilon} \varphi(t)dt \leq \int_0^{d(x_{n(p)},x_{m(p)})} \varphi(t)dt$$

$$\leq k\int_0^{d(x_{n(p)-1},x_{m(p)-1})} \varphi(t)dt$$

$$\leq k\int_0^{\varepsilon} \varphi(t)dt$$

Which is contradiction, since k$\in$ (0, 1).

Hence, $\{x_n\}$ is a Cauchy sequence, so by the completeness of X this sequence must be convergent in X. Let z be the limit of $\{x_n\}$.

Now if we assume z $\neq$ Tz, then d(z,Tz)> 0. If we use the triangle inequality and Inequality (3.1) we have

$$\int_0^{d(z,Tz)} \emptyset(t)dt \leq \int_0^{d(z,x_{2n+1})} \emptyset(t)dt + \int_0^{d(x_{2n+1},Tz)} \emptyset(t)dt$$

$$= \int_0^{d(z,x_{2n+1})} \emptyset(t)dt + \int_0^{d(Sx_{2n},Tz)} \emptyset(t)dt$$

$$\leq \int_0^{d(z,x_{2n+1})} \emptyset(t)dt$$

$$+\alpha \int_0^{\frac{d(x_{2n},Sx_{2n})d(x_{2n},Tz)+[d(x_{2n},z)]^2+d(x_{2n},Sx_{2n})d(x_{2n},z)}{d(x_{2n},Sx_{2n})+d(x_{2n},z)+d(x_{2n},Tz)}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(x_{2n},Tx_{2n})d(x_{2n},z)+d(x_{2n},Tz)d(x_{2n},z)+[d(x_{2n},z)]^2}{d(x_{2n},Sx_{2n})+d(x_{2n},z)+d(x_{2n},Tz)}} \emptyset(t)dt$$

$$+\gamma \int_0^{\frac{d(x_{2n},Tx_{2n})d(x_{2n},Tz)+d(x_{2n},Sx_{2n})d(x_{2n},z)+[d(x_{2n},z)]^2}{d(x_{2n},Sx_{2n})+d(x_{2n},z)+d(x_{2n},Tz)}} \emptyset(t)dt$$

$$= \int_0^{d(z,x_{2n+1})} \emptyset(t)dt$$

$$+\alpha \int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n},Tz)+[d(x_{2n},z)]^2+d(x_{2n},x_{2n+1})d(x_{2n},z)}{d(x_{2n},x_{2n+1})+d(x_{2n},z)+d(x_{2n},Tz)}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n},z)+d(x_{2n},Tz)d(x_{2n},z)+[d(x_{2n},z)]^2}{d(x_{2n},x_{2n+1})+d(x_{2n},z)+d(x_{2n},Tz)}} \emptyset(t)dt$$

$$+\gamma \int_0^{\frac{d(x_{2n},x_{2n+1})d(x_{2n},Tz)+d(x_{2n},x_{2n+1})d(x_{2n},z)+[d(x_{2n},z)]^2}{d(x_{2n},x_{2n+1})+d(x_{2n},z)+d(x_{2n},Tz)}} \emptyset(t)dt$$

So using the condition of normal cone;

$$\int_0^{\|d(z,Tz)\|} \emptyset(t)dt \le M \left\{ \int_0^{\|d(z,x_{2n+1})\|} \emptyset(t)dt + \alpha \int_0^{\left\|\frac{d(x_{2n},x_{2n+1})d(x_{2n},Tz)+[d(x_{2n},z)]^2+d(x_{2n},x_{2n+1})d(x_{2n},z)}{d(x_{2n},x_{2n+1})+d(x_{2n},z)+d(x_{2n},Tz)}\right\|} \emptyset(t)dt + \beta \int_0^{\left\|\frac{d(x_{2n},x_{2n+1})d(x_{2n},z)+d(x_{2n},Tz)d(x_{2n},z)+[d(x_{2n},z)]^2}{d(x_{2n},x_{2n+1})+d(x_{2n},z)+d(x_{2n},Tz)}\right\|} \emptyset(t)dt + \gamma \int_0^{\left\|\frac{d(x_{2n},x_{2n+1})d(x_{2n},Tz)+d(x_{2n},x_{2n+1})d(x_{2n},z)+[d(x_{2n},z)]^2}{d(x_{2n},x_{2n+1})+d(x_{2n},z)+d(x_{2n},Tz)}\right\|} \emptyset(t)dt \right\}$$

As $n \to \infty$, we have

$$\int_0^{\|d(z,Tz)\|} \emptyset(t)dt \le 0,$$

Which is a contradiction. Hence, we get z = Tz; i.e z is a fixed point of T.

Similarly; let us suppose that $z \neq Sz$, then $d(z,Sz) > 0$.

$$\int_0^{d(z,Sz)} \emptyset(t)dt \leq \int_0^{d(z,x_{2n+2})} \emptyset(t)dt + \int_0^{d(x_{2n+2},Sz)} \emptyset(t)dt$$

$$= \int_0^{d(z,x_{2n+2})} \emptyset(t)dt + \int_0^{d(Sz,Tx_{2n+1})} \emptyset(t)dt$$

$$\leq \int_0^{d(z,x_{2n+2})} \emptyset(t)dt + \alpha \int_0^{\frac{d(z,Sz)d(z,Tx_{2n+1})+[d(z,x_{2n+1})]^2+d(z,Sz)d(z,x_{2n+1})}{d(z,\ Sz)+d(z,x_{2n+1})+d(z,Tx_{2n+1})}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(z,Tz)d(z,x_{2n+1})+d(z,Tx_{2n+1})d(z,x_{2n+1})+[d(z,x_{2n+1})]^2}{d(z,\ Sz)+d(z,x_{2n+1})+d(z,Tx_{2n+1})}} \emptyset(t)dt$$

$$+\gamma \int_0^{\frac{d(z,Tz)d(z,Tx_{2n+1})+d(z,Sz)d(z,x_{2n+1})+[d(z,x_{2n+1})]^2}{d(z,\ Sz)+d(z,x_{2n+1})+d(z,Tx_{2n+1})}} \emptyset(t)dt$$

$$= \int_0^{d(z,x_{2n+2})} \emptyset(t)dt + \alpha \int_0^{\frac{d(z,Sz)d(z,x_{2n+2})+[d(z,x_{2n+1})]^2+d(z,Sz)d(z,x_{2n+1})}{d(z,\ Sz)+d(z,x_{2n+1})+d(z,x_{2n+2})}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(z,Tz)d(z,x_{2n+1})+d(z,x_{2n+2})d(z,x_{2n+1})+[d(z,x_{2n+1})]^2}{d(z,\ Sz)+d(z,x_{2n+1})+d(z,x_{2n+2})}} \emptyset(t)dt$$

$$+\gamma \int_0^{\frac{d(z,Tz)d(z,x_{2n+2})+d(z,Sz)d(z,x_{2n+1})+[d(z,x_{2n+1})]^2}{d(z,\ Sz)+d(z,x_{2n+1})+d(z,x_{2n+2})}} \emptyset(t)dt$$

So by (2.1),

$$\int_0^{\|d(z,Tz)\|} \emptyset(t)dt \leq M\left\{\int_0^{\|d(z,x_{2n+2})\|} \emptyset(t)dt + \alpha \int_0^{\left\|\frac{d(z,Sz)d(z,x_{2n+2})+[d(z,x_{2n+1})]^2+d(z,Sz)d(z,x_{2n+1})}{d(z,\ Sz)+d(z,x_{2n+1})+d(z,x_{2n+2})}\right\|} \emptyset(t)dt + \right.$$

$$\beta \int_0^{\left\|\frac{d(z,Tz)d(z,x_{2n+1})+d(z,x_{2n+2})d(z,x_{2n+1})+[d(z,x_{2n+1})]^2}{d(z,\ Sz)+d(z,x_{2n+1})+d(z,x_{2n+2})}\right\|} \emptyset(t)dt +$$

$$\left.\gamma \int_0^{\left\|\frac{d(z,Tz)d(z,x_{2n+2})+d(z,Sz)d(z,x_{2n+1})+[d(z,x_{2n+1})]^2}{d(z,\ Sz)+d(z,x_{2n+1})+d(z,x_{2n+2})}\right\|} \emptyset(t)dt \right\}$$

Hence,

$$\int_0^{\|d(z,Tz)\|} \emptyset(t)dt \leq 0,$$

A contradiction. Therefore d(z ,Sz) = 0 and so z = Sz, i.e z is a fixed point of S.
Hence we find that z is a common fixed point of S and T.
For the uniqueness of z, let us suppose that d(x,Sx) + d(x,y) + d(x,Ty) = 0 implies d(Sx,Ty) = 0 and that w is another fixed point of T in X. Then,

d (z,Sz) + d (z,w) + d(z, Tw) = 0 implies d (Sz,Tw) = 0.

Therefore, we get

d (z,w) = d (Sz,Tw) = 0,

which implies that z = w, and this is the desired consequence.

3.2. Definition. Two self- mappings S and T of a cone metric space (X,d) are said to be weakly commuting if the following is satisfied for all x $\in$ X;

d (STx,TSx) $\leq$ d (Sx,Tx).

3.3. Definition. Let S and T be self – mappings of a cone metric space (X,d) with a normal cone K. Then { S,T} are said to be compatible if

$$\lim_{n\to\infty} d\,(STx_n, TSx_n) = 0$$

Whenever { x_n } is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n$ = w for some w in X.

3.4. Theorem. Let (X,d) be a BA- complete cone metric space, K a BA-normal cone with normal constant M. Suppose the mappings {S,I} and {T,J} be weakly commuting pairs of self- mappings satisfying the following:

(1) T(X) $\subset$ I (X), S(X) $\subset$ J (X).

(2) For all x,y in X ; either

$$\int_0^{d(Sx,Ty)} \emptyset(t)dt \leq \alpha \int_0^{\frac{d(Ix,Sx)d(Ix,Ty)+[d(Ix,Jy)]^2+d(Ix,Sx)d(Ix,Jy)}{d(Ix,Sx)+d(Ix,Jy)+d(Ix,Ty)}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(Ix,Ty)+d(Ix,Jy)}{2}} \emptyset(t)dt + \gamma \int_0^{d(Ix,Jy)} \emptyset(t)dt$$

for all x, y in X with x$\neq$ y, $0< \alpha+\beta+\gamma < 1$ and $\emptyset$:[0,+ ∞) $\rightarrow$[0,+ ∞) is a Lebesgue integrable mapping which is summable on each compact subset of [0,+ ∞) , non- negative ,and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \emptyset(t)dt$. Also $d(Ix,Sx)+d(Ix,Jy)+d(Ix,Ty) \neq 0$. Then S and T have a common fixed point. Further if $d(Ix,Sx)+d(Ix,Jy)+d(Ix,Ty) = 0$ implies d(Sx,Ty) = 0, If any of S, T, I, or J is continuous then S, T, I, and J have a unique common fixed point z. Furthermore, z is the unique common fixed point of S and I as well as of T and J.

Proof. Take x_0 as an arbitrary point of X. Since S(X) $\subset$ J (X) we can find a point x_1 in X such that $Sx_0 = Jx_1$. Also, since T(X) $\subset$ I (X) we can choose a point x_2 with $Tx_1 = Ix_2$. In general; for the point x_{2n} we can pick up a point x_{2n+1} such that $Sx_{2n} = Jx_{2n+1}$, and then a point x_{2n+2} with $Tx_{2n+1} = Ix_{2n+2}$. For n = 0,1, …….

Let us form D_{2n} = d (Sx_{2n}, Tx_{2n+1}) and D_{2n+1} = d (Sx_{2n+2}, Tx_{2n+1}).

Suppose D_{2n} = d (Sx_{2n}, Tx_{2n+1}) $\neq$ 0 and D_{2n+1} = d (Sx_{2n+2}, Tx_{2n+1}) $\neq$ 0 for n = 1,. …….

Now,

$$\int_0^{D_{2n+1}} \emptyset(t)dt = \int_0^{d\,(Sx_{2n+2},Tx_{2n+1})} \emptyset(t)dt$$

$$\leq \alpha \int_0^{\left\{\frac{d(Ix_{2n+2},Sx_{2n+2})d(Ix_{2n+2},Tx_{2n+1})+[d(Ix_{2n+2},Jx_{2n+1})]^2+d(Ix_{2n+2},Sx_{2n+2})d(Ix_{2n+2},Jx_{2n+1})}{d(Ix_{2n+2},\ Sx_{2n+2})+d(Ix_{2n+2},Jx_{2n+1})+d(Ix_{2n+2},Tx_{2n+1})}\right\}} \emptyset(t)dt$$

$$+\beta \int_0^{\left\{\frac{d(Ix_{2n+2},Tx_{2n+1})+d(Ix_{2n+2},Jx_{2n+1})}{2}\right\}} \emptyset(t)dt$$

$$+\gamma \int_0^{d(Ix_{2n+2},Jx_{2n+1})} \emptyset(t)dt$$

$$= \alpha \int_0^{\left\{\frac{d(Tx_{2n+1},Sx_{2n+2})d(Tx_{2n+1},Tx_{2n+1})+[d(Tx_{2n+1},Sx_{2n})]^2+d(Tx_{2n+1},Sx_{2n+2})d(Tx_{2n+1},Sx_{2n})}{d(Tx_{2n+1},\ Sx_{2n+2})+d(Tx_{2n+1},Sx_{2n})+d(Tx_{2n+1},Tx_{2n+1})}\right\}} \emptyset(t)dt$$

$$+\beta \int_0^{\left\{\frac{d(Tx_{2n+1},Tx_{2n+1})+d(Tx_{2n+1},Sx_{2n})}{2}\right\}} \emptyset(t)dt$$

$$+\gamma \int_0^{d(Tx_{2n+1},Sx_{2n})} \emptyset(t)dt$$

$$\leq \left(\alpha+\frac{\beta}{2}+\gamma\right) \int_0^{d(Tx_{2n+1},Sx_{2n})} \emptyset(t)dt$$

Which implies that

$$\int_0^{D_{2n+1}} \emptyset(t)dt \ \leq \lambda \int_0^{D_{2n}} \emptyset(t)dt \ \leq \lambda^2 \int_0^{D_{2n-1}} \emptyset(t)dt$$

$$\leq \ldots\ldots\ldots$$

$$\leq \lambda^{2n+1} \int_0^{D_0} \emptyset(t)dt$$

Where $\lambda = \alpha + \beta + \gamma < 1$. Using (2.1) ,

$$\int_0^{\|D_{2n+1}\|} \emptyset(t)dt \ \leq \mathrm{M}\,\lambda^{2n+1} \int_0^{\|D_0\|} \emptyset(t)dt .$$

In this inequality, $\int_0^{\|D_{2n+1}\|} \emptyset(t)dt \rightarrow 0$ as $n \rightarrow \infty$, so $\int_0^{d\,(Sx_{2n+2},Tx_{2n+1})} \emptyset(t)dt \rightarrow 0$ as $n \rightarrow \infty$. We get the following sequence

$$\{Sx_0, Tx_1, Sx_2, Tx_3, \ldots, Sx_{2n}, Tx_{2n+1}, \ldots\ldots\} \qquad \ldots\ldots\ldots\ldots (3.4.1)$$

Which is a Cauchy sequence in the complete cone metric space(X,d), and therefore converges a limit point $z \in X$. Therefore the sequences

$\{Sx_{2n}\} = \{Jx_{2n+1}\}$, $\{Tx_{2n-1}\} = \{Ix_{2n}\}$ which are subsequence of (3.4.1) and hence also converge to the same point $z \in X$.

Let assume that I is continuous so that the sequence $\{I^2x_{2n}\}$ and $\{ISx_{2n}\}$ converge to the same point Iz. We know that S and I are weakly commuting so we have;

$$\int_0^{d\,(SIx_{2n},ISx_{2n})} \emptyset(t)dt \ \leq \int_0^{d\,(Ix_{2n},Sx_{2n})} \emptyset(t)dt ,$$

And using (2.1)

$$\int_0^{\|d(SIx_{2n}, ISx_{2n})\|} \emptyset(t)dt \leq M \int_0^{\|d(Ix_{2n}, Sx_{2n})\|} \emptyset(t)dt$$

Hence the sequence $\{SIx_{2n}\}$ converges to the point Iz.

Now,

$$\int_0^{d(Iz,z)} \emptyset(t)dt \leq \int_0^{d(Iz,SIx_{2n})} \emptyset(t)dt + \int_0^{d(SIx_{2n},Tx_{2n+1})} \emptyset(t)dt$$

$$+ \int_0^{d(Tx_{2n+1},z)} \emptyset(t)dt$$

$$\leq \int_0^{d(Iz,SIx_{2n})} \emptyset(t)dt + \alpha$$

$$\int_0^{\frac{d(I^2x_{2n},SIx_{2n})d(I^2x_{2n},Tx_{2n+1})+[d(I^2x_{2n},Jx_{2n+1})]^2+d(I^2x_{2n},SIx_{2n})d(I^2x_{2n},Jx_{2n+1})}{d(I^2x_{2n},SIx_{2n})+d(I^2x_{2n},Jx_{2n+1})+d(I^2x_{2n},Tx_{2n+1})}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(I^2x_{2n},Tx_{2n+1})+d(I^2x_{2n},Jx_{2n+1})}{2}} \emptyset(t)dt$$

$$+\gamma \int_0^{d(I^2x_{2n},Jx_{2n+1})} \emptyset(t)dt + \int_0^{d(Tx_{2n+1},z)} \emptyset(t)dt$$

Which with Inequality (2.1), gives

$$\int_0^{\|d(Iz,z)\|} \emptyset(t)dt \leq M \left\{ \int_0^{\|d(Iz,SIx_{2n})\|} \emptyset(t)dt + \alpha \int_0^{\left\|\frac{d(I^2x_{2n},SIx_{2n})d(I^2x_{2n},Tx_{2n+1})+[d(I^2x_{2n},Jx_{2n+1})]^2+d(I^2x_{2n},SIx_{2n})d(I^2x_{2n},Jx_{2n+1})}{d(I^2x_{2n},SIx_{2n})+d(I^2x_{2n},Jx_{2n+1})+d(I^2x_{2n},Tx_{2n+1})}\right\|} \emptyset(t)dt + \right.$$

$$\beta \int_0^{\left\|\frac{d(I^2x_{2n},Tx_{2n+1})+d(I^2x_{2n},Jx_{2n+1})}{2}\right\|} \emptyset(t)dt + \gamma \int_0^{\|d(I^2x_{2n},Jx_{2n+1})\|} \emptyset(t)dt +$$

$$\left. \int_0^{\|d(Tx_{2n+1},z)\|} \emptyset(t)dt \right\}$$

So

$$\int_0^{\|d(Iz,z)\|} \emptyset(t)dt \leq M\left(\frac{\alpha}{2} + \beta + \gamma\right) \int_0^{\|d(Iz,z)\|} \emptyset(t)dt$$

Hence $\int_0^{\|d(Iz,z)\|} \emptyset(t)dt = 0$ and Iz = z. We want to show that Sz = z, too. Using the same inequality, we have

$$\int_0^{d(Sz,z)} \emptyset(t)dt \leq \int_0^{d(Sz,Tx_{2n+1})} \emptyset(t)dt + \int_0^{d(Tx_{2n+1},z)} \emptyset(t)dt$$

$$\leq \alpha \int_0^{\frac{d(Iz,Sz)d(Iz,Tx_{2n+1})+[d(Iz,Jx_{2n+1})]^2+d(Iz,Sz)d(Iz,Jx_{2n+1})}{d(Iz,Sz)+d(Iz,Jx_{2n+1})+d(Iz,Tx_{2n+1})}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(Iz,Tx_{2n+1})+d(Iz,Jx_{2n+1})}{2}} \emptyset(t)dt$$

$$+\gamma \int_0^{d(Iz,Jx_{2n+1})} \emptyset(t)dt + \int_0^{d(Tx_{2n+1},z)} \emptyset(t)dt$$

And again if (2.1) is used;

$$\int_0^{\|d(Sz,z)\|} \emptyset(t)dt \leq M$$

$$\left\{ \alpha \int_0^{\left\|\frac{d(Iz,Sz)d(Iz,Tx_{2n+1})+[d(Iz,Jx_{2n+1})]^2+d(Iz,Sz)d(Iz,Jx_{2n+1})}{d(Iz,Sz)+d(Iz,Jx_{2n+1})+d(Iz,Tx_{2n+1})}\right\|} \emptyset(t)dt + \right.$$

$$\beta \int_0^{\left\|\frac{d(Iz,Tx_{2n+1})+d(Iz,Jx_{2n+1})}{2}\right\|} \emptyset(t)dt +$$

$$\left. \gamma \int_0^{\|d(Iz,Jx_{2n+1})\|} \emptyset(t)dt + \int_0^{\|d(Tx_{2n+1},z)\|} \emptyset(t)dt \right\}$$

And, as n tends to infinity,

$$= M\left\{ \alpha \int_0^{\left\|\frac{d(z,Sz)d(z,z)+[d(z,z)]^2+d(z,Sz)d(z,z)}{d(z,Sz)+d(z,z)+d(z,z)}\right\|} \emptyset(t)dt + \beta \int_0^{\left\|\frac{d(z,Tz)+d(z,z)}{2}\right\|} \emptyset(t)dt + \right.$$

$$\left. \gamma \int_0^{\|d(z,z)\|} \emptyset(t)dt + \int_0^{\|d(z,z)\|} \emptyset(t)dt \right\}$$

Then, $\int_0^{\|d(Sz,z)\|} \emptyset(t)dt = 0$ and hence Sz = z.

We have seen that Sz = z, and we know that S(X) ⊂ J(X) so we can always find a point w such that Jw = z. Thus,

$$\int_0^{d(z,Tw)} \emptyset(t)dt = \int_0^{d(Sz,Tw)} \emptyset(t)dt$$

$$\leq \alpha \int_0^{\frac{d(Iz,Sz)d(Iz,Tw)+[d(Iz,Jw)]^2+d(Iz,Sz)d(Iz,Jw)}{d(Iz,Sz)+d(Iz,Jw)+d(Iz,Tw)}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(Iz,Tw)+d(Iz,Jw)}{2}} \emptyset(t)dt + \gamma \int_0^{d(Iz,Jw)} \emptyset(t)dt$$

$= \frac{\beta}{2} \int_0^{d(z,Tw)} \emptyset(t)dt$

So that $d(z, Tw) = 0$, Tw = z.

Since T and J weakly commute

$$d(Tz, Jz) = d(TJw, JTw) \leq d(Jw, Tw) = d(z,z) = 0,$$

which gives Tz = Jz, and so

$$\int_0^{d(z,Tz)} \emptyset(t)dt = \int_0^{d(Sz,Tz)} \emptyset(t)dt$$

$$\leq \alpha \int_0^{\frac{d(Iz,Sz)d(Iz,Tz)+[d(Iz,Jz)]^2+d(Iz,Sz)d(Iz,Jz)}{d(Iz,Sz)+d(Iz,Jz)+d(Iz,Tz)}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(Iz,Tz)+d(Iz,Jz)}{2}} \emptyset(t)dt + \gamma \int_0^{d(Iz,Jz)} \emptyset(t)dt$$

$$= \left(\frac{\alpha}{2} + \beta + \gamma\right) \int_0^{d(z,Tz)} \emptyset(t)dt$$

We get that z = Tz, consequently this yields Tz = Jz =z.
Thereby we have proved that the mappings S, T, I and J have a common fixed point. The proof is the same if one of the mappings S, T, J is continuous instead of I.
To show that z is unique, let u be another common fixed point of S and I. Then

$$\int_0^{d(u,z)} \emptyset(t)dt = \int_0^{d(Su,Tz)} \emptyset(t)dt$$

$$\leq \alpha \int_0^{\frac{d(Iu,Su)d(Iu,Tz)+[d(Iu,Jz)]^2+d(Iu,Su)d(Iu,Jz)}{d(Iu,Su)+d(Iu,Jz)+d(Iu,Tz)}} \emptyset(t)dt$$

$$+\beta \int_0^{\frac{d(Iu,Tz)+d(Iu,Jz)}{2}} \emptyset(t)dt + \gamma \int_0^{d(Iu,Jz)} \emptyset(t)dt$$

$$= \left(\frac{\alpha}{2} + \beta + \gamma\right) \int_0^{d(u,z)} \emptyset(t)dt$$

Again we get u = z. In the same way it can be show that z is the unique fixed point for the mapping T and J.
3.5. Remark. Weakly commuting mappings are obviously compatible, but the converse need not to be true. So, the condition weak commutativity can be replaced with compatibility with the same assumptions in the theorem.

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