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Fixed Point Theorems for Six Weakly Compatible Mappings in D^* - Metric Spaces for Integral type Mappings

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ABSTRACT: In the present paper, we give some new definitions of D^*- metric spaces and we prove a common fixed point theorem for six mappings under the condition of weakly compatible mappings in complete D^*- metric spaces. We get some improved versions of several fixed point theorems in complete D^*- metric spaces.

AMS: 54H25, 54E40, 54E35.
Keywords: D – metric, contractive mappings; complete D^*- metric space; common fixed point theorems.

I. INTRODUCTION AND PRELIMINARIES

In 1922, the Polish mathematician, Banach, proved a theorem which ensures, under appropriate conditions, the existences and uniqueness of a fixed point. His result is called Banach's Fixed point Theorem or the Banach Contraction principle. This theorems provides a technique for solving a variety of problems of applied nature in mathematical science and engineering. Many authors have extended, generalized and improved Banach's Fixed point Theorem in Different ways. In [17], Jungck introduced the notion of compatible mappings which are more general than commuting and weakly commuting mappings. This concept has been useful for obtaining more comprehensive fixed point theorems. Dhage [7] introduced the concept of generalized metric or D – metric spaces and claimed that D – metric convergence defines a Hausdorff topology and that D – metric is sequentially continuous in all the three variables. Many authors have taken these claims for granted and used them in proving fixed point theorems in D – metric spaces. Rhoades[17] generalized Dhage's contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self-maps in D – metric space. Recently, motivated by the concept of compatibility for metric space, Singh and Sharma[23] introduced the concept of D – compatibility of maps in D –metric space and proved some fixed point theorems using a contractive condition. Unfortunately, almost all theorems in D –metric spaces are not valid [14,15,16]. In this paper, we introduce D^*- metric which is a probable modification of the definition of D – metric introduced by Dhage[7] and prove some basic properties in D^*- metric spaces.
In what follows (X, D^*) will denotes D^*- metric space.

Definition1.1. Let X be a non- empty set. A generalized metric or D^*- metric on X is a function $D^*: X^3 \to \mathbb{R}^+$ that satisfies the following conditions for each x,y,z ,a $\in$ X.

(1) $D^*(x, y, z) \geq 0$,
(2) $D^*(x, y, z) = 0$ if and only if x = y = z,
(3) $D^*(x, y, z) = D^*(p\{x, y, z\})$, where p is a permutation function,
(4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D^*) is called a generalized metric space.
Immediate examples of such a function are the following :
(a) $D^*(x, y, z) = \max\{d(x,y), d(y,z), d(z,x)\}$,
(b) $D^*(x, y, z) = d(x,y) + d(y,z) + d(z,x)$.

Here, d is the ordinary metric on X.

Definition1.2. Let (X, D^*) be a D^*- metric space and A$\subset X$.

(1) If for every x$\in$ A there exist r > 0 such that B_{D^*}(x,r) $\subset$ A, then subset A is called open subset of X.

(2) Subset A of X is said to be D^*- bounded if there exists $r > 0$ such that D^*(x, y, y) < r for all x,y $\in$ A.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $D^*(x_n, x_n, x) = D^*(x, x, x_n) \to 0$ as n $\to \infty$.

(4) Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $D^*(x_n, x_n, x_m) < \varepsilon$ for each n,m $\geq n_0$. The D^*- metric space is said to be complete if every Cauchy sequence is convergent.

Definition1.3. Let (X, D^*) be a D^*- metric space. D^* is said to be continuous function on $X^3 \times (0,\infty)$ if $\lim_{n\to\infty} D^*(x_n, y_n, z_n) = D^*(x, y, z)$.

whenever a sequence $\{(x_n, y_n, z_n)\}$ in $X^3 \times (0,\infty)$ converges to a point (x,y,z) $\in X^3 \times (0,\infty)$ i.e.

$\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} z_n = z$.

Definition1.4. Let A and S be mappings from a D^*- metric space (X, D^*) into itself.Then the mappings are said to be weak compatible if they commute at their coincidence point, that is Ax = Sx implies that ASx = SAx.

Definition1.5. The pair (A,S) satisfies the property (E.A) [1], if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} D^*(Ax_n, u, u) = \lim_{n\to\infty} D^*(Sx_n, u, u) = 0 \text{ for some } u \in X.$$

Definition1.6. The pairs (A,S) and (B,T) of a D^*- metric space (X, D^*) satisfy a common property (E.A) if there exists two sequence $\{x_n\}$ and $\{y_n\}$ such that for some u $\in$ X

$$\lim_{n\to\infty} D^*(Ax_n, u, u) = \lim_{n\to\infty} D^*(Sx_n, u, u) = \lim_{n\to\infty} D^*(By_n, u, u)$$

$$= \lim_{n\to\infty} D^*(Ty_n, u, u) = 0.$$

II. MAIN RESULTS

Theorem2.1. Let S and T be self – mappings of a complete D^* - metric space (X, D^*) satisfying the following conditions :

$$\int_0^{D^*(Tx, TSy, Sz)} \phi(s)ds \leq \phi\left(\int_0^{L(x,y,z)} \phi(s)ds\right) \quad (2.1.1)$$

Where

$$L(x,y,z) = \alpha \max\begin{Bmatrix} D^*(x, Sy, z), D^*(x, Sy, Tx), \\ D^*(Tx, x, x), D^*(Tx, Sz, Sz) \end{Bmatrix} + \beta\left[\frac{D^*(x, Ty, z) + D^*(x, Sy, Sx)}{2}\right] + \gamma\left[\frac{D^*(Tx, Sy, Ty) + D^*(Sx, Sy, Ty)}{2}\right] + \delta D^*(y, z, z)$$

For all x,y $\in$ X, Let Φ be the set of all increasing and continuous function $\phi : R_+ \to R_+$ such that $\phi(s) < s$ for every s $\in (0,\infty)$, $\phi(0) = 0$. Also $\alpha, \beta, \gamma, \delta \in [0,1]$ with $\alpha + \beta + \gamma + \delta \leq 1$ .Then S and T have a unique common fixed point in X.

Proof : Let $x_0 \in$ X be an arbitrary point. Then there exist $x_1, x_2 \in$ X such that $Tx_0 = x_1$ and $Sx_1 = x_2$.

Inductively, construct sequence $\{x_n\}$ in X such that

$Tx_{2n} = x_{2n+1}$ and $Sx_{2n+1} = x_{2n+2}$, for n = 0,1,2,…………

Now, we prove that $\{x_n\}$ is a Cauchy sequence. Let $d_m = D^*(x_m, x_m, x_{m+1})$.

Replacing $x_{2n}, x_{2n-1}, x_{2n+1}$ by x, y, z respectively in (2.1.1), then we have

$$\int_0^{D^*(x_{2n+1},x_{2n+1},x_{2n+2})} \emptyset(s)ds = \int_0^{D^*(Tx_{2n},TSx_{2n-1},Sx_{2n+1})} \emptyset(s)ds \leq \emptyset\left(\int_0^{L(x_{2n},x_{2n-1},x_{2n+1})} \emptyset(s)ds\right) \quad (2.1.2)$$

Where

$$L(x_{2n},x_{2n-1},x_{2n+1}) = \alpha\max\left\{\begin{matrix} D^*(x_{2n},Sx_{2n-1},x_{2n+1}), D^*(x_{2n},Sx_{2n-1},Tx_{2n}), \\ D^*(\mathrm{T}x_{2n},x_{2n},x_{2n}), D^*(\mathrm{T}x_{2n},Sx_{2n+1},Sx_{2n+1}) \end{matrix}\right\} + \beta\left[\frac{D^*(x_{2n},Tx_{2n-1},x_{2n+1})+D^*(x_{2n},Sx_{2n-1},Sx_{2n})}{2}\right] + \gamma\left[\frac{D^*(\mathrm{T}x_{2n},Sx_{2n-1},\mathrm{T}x_{2n-1})+D^*(\mathrm{S}x_{2n},Sx_{2n-1},Tx_{2n-1})}{2}\right] + \delta D^*(x_{2n-1},x_{2n+1},x_{2n+1})$$

$$= \alpha \max\left\{\begin{matrix} D^*(x_{2n},x_{2n},x_{2n+1}), D^*(x_{2n},x_{2n},x_{2n+1}), \\ D^*(x_{2n+1},x_{2n},x_{2n}), D^*(x_{2n+1},x_{2n+2},x_{2n+2}) \end{matrix}\right\} + \beta\left[\frac{D^*(x_{2n},x_{2n},x_{2n+1})+D^*(x_{2n},x_{2n},x_{2n+1})}{2}\right] + \gamma\left[\frac{D^*(x_{2n+1},x_{2n},x_{2n})+D^*(x_{2n+1},x_{2n},x_{2n})}{2}\right] + \delta D^*(x_{2n-1},x_{2n+1},x_{2n+1})$$

Hence, we get

$$L(x_{2n},x_{2n-1},x_{2n+1}) = \alpha \max\{d_{2n}, d_{2n}, d_{2n}, d_{2n+1}\} + \beta\left[\frac{d_{2n+1}+d_{2n+1}}{2}\right] + \gamma\left[\frac{d_{2n+1}+d_{2n+1}}{2}\right] + \delta\ d_{2n+1}$$

We now prove that $d_{2n+1} \leq d_{2n}$ for every $n \in \mathbb{N}$. If $d_{2n+1} > d_{2n}$ for some $n \in \mathbb{N}$, by inequality (2.1.2), we have

$$\int_0^{d_{2n+1}} \emptyset(s)ds \leq \emptyset\left(\begin{matrix}\alpha\int_0^{d_{2n+1}} \emptyset(s)ds + \beta\int_0^{d_{2n+1}} \emptyset(s)ds + \gamma\int_0^{d_{2n+1}} \emptyset(s)ds \\ +\delta\int_0^{d_{2n+1}} \emptyset(s)ds\end{matrix}\right)$$

$$< \alpha\int_0^{d_{2n+1}} \emptyset(s)ds + \beta\int_0^{d_{2n+1}} \emptyset(s)ds + \gamma\int_0^{d_{2n+1}} \emptyset(s)ds + \delta\int_0^{d_{2n+1}} \emptyset(s)ds$$

$$= (\alpha+\beta+\gamma+\delta)\int_0^{d_{2n+1}} \emptyset(s)ds$$

Which is a contradiction. (as $\alpha+\beta+\gamma+\delta \leq 1$.)

Hence $d_{2n+1} \leq d_{2n}$

Now, replacing x, y, z by $x_{2n}, x_{2n-1}, x_{2n-1}$ respectively in (2.1.1), we obtain

$$\int_0^{D^*(x_{2n+1},x_{2n+1},x_{2n})} \emptyset(s)ds = \int_0^{D^*(Tx_{2n},TSx_{2n-1},Sx_{2n-1})} \emptyset(s)ds \leq \emptyset\left(\int_0^{L(x_{2n},x_{2n-1},x_{2n-1})} \emptyset(s)ds\right)$$

Where

$$L(x_{2n},x_{2n-1},x_{2n-1}) = \alpha\max\left\{\begin{matrix} \mathrm{D}^*(\mathrm{x}_{2n},\mathrm{Sx}_{2n-1},\mathrm{x}_{2n-1}), \mathrm{D}^*(\mathrm{x}_{2n},\mathrm{Sx}_{2n-1},\mathrm{Tx}_{2n}), \\ \mathrm{D}^*(\mathrm{Tx}_{2n},\mathrm{x}_{2n},\mathrm{x}_{2n}), \mathrm{D}^*(\mathrm{Tx}_{2n},\mathrm{Sx}_{2n-1},\mathrm{Sx}_{2n-1}) \end{matrix}\right\} + \beta\left[\frac{\mathrm{D}^*(\mathrm{x}_{2n},\mathrm{Tx}_{2n-1},\mathrm{x}_{2n-1})+\mathrm{D}^*(\mathrm{x}_{2n},\mathrm{Sx}_{2n-1},\mathrm{Sx}_{2n})}{2}\right] + \gamma\left[\frac{D^*(\mathrm{T}x_{2n},Sx_{2n-1},\mathrm{T}x_{2n-1})+D^*(\mathrm{S}x_{2n},Sx_{2n-1}T,x_{2n-1})}{2}\right] + \delta D^*(x_{2n-1},x_{2n-1},x_{2n-1})$$

$$= \alpha \max\left\{\begin{matrix} D^*(x_{2n},x_{2n},x_{2n-1}), D^*(x_{2n},x_{2n},x_{2n+1}), \\ D^*(x_{2n+1},x_{2n},x_{2n}), D^*(x_{2n+1},x_{2n},x_{2n}) \end{matrix}\right\} + \beta\left[\frac{D^*(x_{2n},x_{2n},x_{2n-1})+D^*(x_{2n},x_{2n},x_{2n+1})}{2}\right] + \gamma\left[\frac{D^*(x_{2n+1},x_{2n},x_{2n})+D^*(x_{2n+1},x_{2n},x_{2n})}{2}\right]$$

Hence, we get

$$L(x_{2n},x_{2n-1},x_{2n-1}) = \alpha \max\{d_{2n-1},d_{2n},d_{2n},d_{2n}\} + \beta\left[\frac{d_{2n}+d_{2n}}{2}\right] + \gamma\left[\frac{d_{2n}+d_{2n}}{2}\right]$$

$$= \alpha \max\{d_{2n-1},d_{2n},d_{2n},d_{2n}\} + \beta\ d_{2n} + \gamma\ d_{2n}$$

We prove that $d_{2n} \leq d_{2n-1}$, for every $n \in \mathbb{N}$. If $d_{2n} > d_{2n-1}$ for some $n \in \mathbb{N}$, by inequality (2.1.2), we have

$$\int_0^{d_{2n}} \emptyset(s)ds \leq \emptyset\left(\alpha \int_0^{d_{2n}} \emptyset(s)ds + \beta \int_0^{d_{2n}} \emptyset(s)ds + \gamma \int_0^{d_{2n}} \emptyset(s)ds\right)$$
$$< \alpha \int_0^{d_{2n}} \emptyset(s)ds + \beta \int_0^{d_{2n}} \emptyset(s)ds + \gamma \int_0^{d_{2n}} \emptyset(s)ds$$
$$= (\alpha + \beta + \gamma) \int_0^{d_{2n+1}} \emptyset(s)ds$$

Which is a contradiction. (as $\alpha + \beta + \gamma + \delta \leq 1$.)

Hence $d_{2n} \leq d_{2n-1}$.

Hence for each $n \in \mathbb{N}$ we have $d_n \leq d_{n-1}$. Thus sequence $\{d_n\}$ is lower bounded and decreasing sequence, hence it is lead to 0. It follows

$$\lim_{n\to\infty} \int_0^{D^*(x_n, x_n, x_{n+1})} \emptyset(s)ds = 0.$$

Therefore

$$\lim_{n\to\infty} D^*(x_n, x_n, x_{n+1}) = 0. \tag{2.1.3}$$

Now, we prove that $\{x_{2n}\}$ is Cauchy sequence. Suppose that $\{x_{2n}\}$ is not a Cauchy sequence in X. Then there is an $\epsilon > 0$ such that for each integer k, there exist integers 2m(k) and 2n(k) with m(k) $> n(k) \geq k$ such that

$$D^*\left(x_{2n(k)}, x_{2m(k)}, x_{2m(k)}\right) \geq \epsilon \text{ and}$$
$$D^*\left(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1}\right) < \epsilon \tag{2.1.4}$$

From (2.1.4), we have

$$\epsilon \leq D^*\left(x_{2n(k)}, x_{2m(k)}, x_{2m(k)}\right)$$
$$\leq D^*\left(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1}\right) + D^*\left(x_{2m(k)-1}, x_{2m(k)}, x_{2m(k)}\right)$$
$$\leq \epsilon + d_{2m(k)-1}$$

Letting $k \to \infty$ and using (2.1.3), we get

$$\lim_{k\to\infty} D^*\left(x_{2n(k)}, x_{2m(k)}, x_{2m(k)}\right) = \epsilon \tag{2.1.5}$$

Similarly, using (2.3) and (2.5), we can show that

$$\lim_{k\to\infty} D^*\left(x_{2n(k)+1}, x_{2m(k)}, x_{2m(k)}\right) = \lim_{k\to\infty} D^*\left(x_{2n(k)}, x_{2m(k)-1}, x_{2m(k)-1}\right) = \epsilon \tag{2.1.6}$$

Replacing x, y, z by $x_{2m(k)}$, $x_{2n(k)+1}, x_{2m(k)}$ in (2.1.1), we have

$$\int_0^{D^*(x_{2m(k)}, x_{2n(k)+1}, x_{2n(k)+1})} \emptyset(s)ds \leq \emptyset\left(\int_0^{L(x_{2m(k)}, x_{2n(k)+1}, x_{2m(k)})} \emptyset(s)ds\right)$$

Where

$$L(x_{2m(k)}, x_{2n(k)+1}, x_{2m(k)})$$
$$= \alpha \max\left\{\begin{matrix} D^*\left(x_{2m(k)}, Sx_{2n(k)+1}, x_{2m(k)}\right), D^*\left(x_{2m(k)}, Sx_{2n(k)+1}, Tx_{2m(k)}\right), \\ D^*\left(Tx_{2m(k)}, x_{2m(k)}, x_{2m(k)}\right), D^*\left(Tx_{2m(k)}, Sx_{2m(k)}, Sx_{2m(k)}\right) \end{matrix}\right\}$$
$$+ \beta\left[\frac{D^*(x_{2m(k)}, Tx_{2n(k)+1}, x_{2m(k)}) + D^*(x_{2m(k)}, Sx_{2n(k)+1}, Sx_{2m(k)})}{2}\right]$$
$$+ \gamma\left[\frac{D^*(Tx_{2m(k)}, Sx_{2n(k)+1}, T x_{2n(k)+1}) + D^*(Sx_{2m(k)}, Sx_{2n(k)+1} T, x_{2n(k)+1})}{2}\right]$$
$$+ \delta\, D^*\left(x_{2n(k)+1}, x_{2m(k)}, x_{2m(k)}\right)$$
$$= \alpha \max\left\{\begin{matrix} D^*\left(x_{2m(k)}, x_{2n(k)+2}, x_{2m(k)}\right), D^*\left(x_{2m(k)}, x_{2n(k)+2}, x_{2m(k)+1}\right), \\ D^*\left(x_{2m(k)+1}, x_{2m(k)}, x_{2m(k)}\right), D^*\left(x_{2m(k)+1}, x_{2m(k)+1}, x_{2m(k)+1}\right) \end{matrix}\right\}$$
$$+ \beta\left[\frac{D^*(x_{2m(k)}, x_{2n(k)+2}, x_{2m(k)}) + D^*(x_{2m(k)}, x_{2n(k)+2}, x_{2m(k)+1})}{2}\right]$$
$$+ \gamma\left[\frac{D^*(x_{2m(k)+1}, x_{2n(k)+2}, x_{2n(k)+2}) + D^*(x_{2m(k)+1}, x_{2n(k)+2}, x_{2n(k)+2})}{2}\right]$$
$$+ \delta\, D^*\left(x_{2n(k)+1}, x_{2m(k)}, x_{2m(k)}\right)$$

Making $k \to \infty$ and using (2.1.3), (2.1.5) and (2.1.6), we obtain

$$\int_0^{\epsilon} \emptyset(s)ds \leq \emptyset\left(\alpha \int_0^{\epsilon} \emptyset(s)ds + \beta \int_0^{\epsilon} \emptyset(s)ds + \gamma \int_0^{\epsilon} \emptyset(s)ds + \delta \int_0^{\epsilon} \emptyset(s)ds\right)$$

$$< (\alpha+\beta+\gamma+\delta)\int_0^{\epsilon} \emptyset(s)ds$$

Which is a contradiction. (as $\alpha+\beta+\gamma+\delta \le 1$.)

This establishes the fact that $\{x_{2n}\}$ is a Cauchy sequence.

$$D^*(x_{2n+1}, x_{2m+1}, x_{2m+1}) \le D^*(x_{2n+1}, x_{2n}, x_{2n}) + D^*(x_{2n}, x_{2m}, x_{2m}) + D^*(x_{2m}, x_{2m+1}, x_{2m+1})$$

Making n, m → ∞ we get $\lim_{n,m\to\infty} D^*(x_{2n+1}, x_{2m+1}, x_{2m+1}) = 0$. Similarly,

We get

$$\lim_{n,m\to\infty} D^*(x_{2n+1}, x_{2m}, x_{2m}) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence, and due to the completeness of X, $\{x_n\}$ converges to some x in X. That is $\lim_{n\to\infty} x_n$ = x. Hence

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = x$$

Now we show that Sx = x. From the inequality (2.1.1), we get

$$\int_0^{D^*(Tx_{2n},TSx_{2n+1},Sx)} \emptyset(s)ds = \int_0^{D^*(x_{2n+1},x_{2n+2},Sx)} \emptyset(s)ds \le \emptyset\left(\int_0^{L(x_{2n},x_{2n+1},x)} \emptyset(s)ds\right)$$

Where

$$\begin{aligned} L(x_{2n}, x_{2n+1}, x) &= \alpha \max\left\{\begin{matrix} D^*(x_{2n}, Sx_{2n+1}, x), D^*(x_{2n}, Sx_{2n+1}, Tx_{2n}), \\ D^*(Tx_{2n}, x_{2n}, x_{2n}), D^*(Tx_{2n}, Sx, Sx) \end{matrix}\right\} \\ &+ \beta\left[\frac{D^*(x_{2n},Tx_{2n+1},x)+D^*(x_{2n},Sx_{2n+1},Sx_{2n})}{2}\right] \\ &+ \gamma\left[\frac{D^*(Tx_{2n},Sx_{2n+1},Tx_{2n+1})+D^*(Sx_{2n},Sx_{2n+1}T,x_{2n+1})}{2}\right] \\ &+ \delta D^*(x_{2n+1}, x, x) \\ &= \alpha \max\left\{\begin{matrix} D^*(x_{2n}, x_{2n+2}, x), D^*(x_{2n}, x_{2n+2}, x_{2n+1}), \\ D^*(x_{2n+1}, x_{2n}, x_{2n}), D^*(x_{2n+1}, Sx, Sx) \end{matrix}\right\} \\ &+ \beta\left[\frac{D^*(x_{2n},x_{2n+2},x)+D^*(x_{2n},x_{2n+2},x_{2n+1})}{2}\right] \\ &+ \gamma\left[\frac{D^*(x_{2n+1}\, x_{2n+2},x_{2n+2})+D^*(x_{2n+1},x_{2n+2},x_{2n+2})}{2}\right] \\ &+ \delta D^*(x_{2n+1}, x, x) \end{aligned}$$

On making n → ∞ , we get

$$\int_0^{D^*(x,x,Sx)} \emptyset(s)ds \le \emptyset\left(\alpha\int_0^{D^*(x,x,Sx)} \emptyset(s)ds\right) < \alpha\int_0^{D^*(x,x,Sx)} \emptyset(s)ds,$$

Which is a contradiction. Therefore, it follows that Sx = x. Next we prove that Tx = x. For this, replacing x, y, z by x_{2n}, x, x in inequality (2.1.1), we have

$$\int_0^{D^*(Tx_{2n},TSx,Sx)} \emptyset(s)ds = \int_0^{D^*(Tx_{2n},Tx,x)} \emptyset(s)ds \le \emptyset\left(\int_0^{L(x_{2n},x,x)} \emptyset(s)ds\right)$$

Where

$$\begin{aligned} L(x_{2n}, x, x) &= \alpha \max\left\{\begin{matrix} D^*(x_{2n}, Sx, x), D^*(x_{2n}, Sx, Tx_{2n}), \\ D^*(Tx_{2n}, x_{2n}, x_{2n}), D^*(Tx_{2n}, Sx, Sx) \end{matrix}\right\} \\ &+ \beta\left[\frac{D^*(x_{2n},Tx,x)+D^*(x_{2n},Sx,Sx_{2n})}{2}\right] + \gamma\left[\frac{D^*(Tx_{2n},Sx,Tx)+D^*(Sx_{2n},Sx,Tx)}{2}\right] \\ &+ \delta D^*(x, x, x) \\ &= \alpha \max\left\{\begin{matrix} D^*(x_{2n}, x, x), D^*(x_{2n}, x, Tx_{2n}), \\ D^*(Tx_{2n}, x_{2n}, x_{2n}), D^*(Tx_{2n}, x, x) \end{matrix}\right\} \\ &+ \beta\left[\frac{D^*(x_{2n},Tx,x)+D^*(x_{2n},x,x)}{2}\right] + \gamma\left[\frac{D^*(Tx,x,Tx)+D^*(x,x,Tx)}{2}\right] \\ &+ \delta D^*(x, x, x) \end{aligned}$$

As n → ∞, we have

$$\int_0^{D^*(x,Tx,x)} \emptyset(s)ds \le \emptyset\left(\begin{matrix} \alpha\int_0^{D^*(x,x,Tx)} \emptyset(s)ds + \frac{\beta}{2}\int_0^{D^*(x,x,Tx)} \emptyset(s)ds + \\ \gamma\int_0^{D^*(x,x,Tx)} \emptyset(s)ds \end{matrix}\right)$$

$$< \left(\alpha + \frac{\beta}{2} + \gamma\right) \int_0^{D^*(x,x,Tx)} \emptyset(s)ds$$

Which is a contradiction. So it follows that Tx = x. Hence Tx = Sx = x, that is x is a common fixed point of T,S. The uniqueness of x follows from the inequality (2.1.1).

Theorem 2.2. Let (X, D^*) be a D^*- metric space and A,B, C, R, S and T be self- mappings of X s atisfying the following conditions:

A(X) $\subseteq$ T(X) and B(X) $\subseteq$ R(X) and C(X) $\subseteq$ S(X)

$$\int_0^{D^*(Ax,By,Cz)} \emptyset(s)ds \leq \emptyset\left(\int_0^{L(x,y,z)} \emptyset(s)ds\right) \qquad (2.2.1)$$

Where L(x,y,z) = $\alpha \max\left\{\begin{matrix} D^*(Sx,Ty,Rz), D^*(Ax,Ty,Rz), \\ D^*(Sx,By,Rz), D^*(Sx,Ty,Cz) \end{matrix}\right\} + \beta\left[\frac{D^*(Ty,By,Rz)+D^*(Sx,Ax,Rz)}{2}\right] + \gamma \left[\frac{D^*(Cz,Rz,Sx)+D^*(Cz,By,Sx)}{2}\right] + \delta\, D^*(Ax,By,Cz)$

For all x,y $\in$ X, Let Φ be the set of all increasing and continuous function $\emptyset : R_+ \to R_+$ such that $\emptyset(s) <$ s for every s $\in (0,\infty)$, $\emptyset(0) = 0$. Also $\alpha, \beta, \gamma, \delta \in [0,1]$ with $\alpha + \beta + \gamma + \delta \leq 1$. Suppose that two of the pairs (A,S), (C,R) and (B,T) satisfy the common property (E.A); pairs (A,S), (C,R) and (B,T) are weakly compatible, and one of the R(X), T(X) and S(X) is a closed subset of X. Then A, B, C, R, S and T have a unique common fixed point in X.

Proof. Suppose that (A,S),and (B,T) satisfy a common property (E.A). Then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for some u$\in$ X.

$$\lim_{n\to\infty} D^*(Ax_n,u,u) = \lim_{n\to\infty} D^*(Sx_n,u,u)$$
$$\lim_{n\to\infty} D^*(By_n,u,u) = \lim_{n\to\infty} D^*(Ty_n,u,u) = 0$$

As B(X) $\subseteq$ R(X), there exists a sequence $\{z_n\}$ in X such that $By_n = Rz_n$.

Thus $\lim_{n\to\infty} Rz_n$ = u. Now we prove that $\lim_{n\to\infty} Cz_n$ = u. Replacing x_n, y_n, z_n by x, y, z respectively in (2.2.1), we obtain

$$\int_0^{D^*(Ax_n,By_n,Cz_n)} \emptyset(s)ds \leq \emptyset\left(\int_0^{L(x_n,y_n,z_n)} \emptyset(s)ds\right)$$

Where $L(x_n,y_n,z_n) = \alpha \max\left\{\begin{matrix} D^*(Sx_n,Ty_n,Rz_n), D^*(Ax_n,Ty_n,Rz_n), \\ D^*(Sx_n,By_n,Rz_n), D^*(Sx_n,Ty_n,Cz_n) \end{matrix}\right\}$

$$+ \beta\left[\frac{D^*(Ty_n,By_n,Rz_n)+D^*(Sx_n,Ax_n,Rz_n)}{2}\right] + \gamma\left[\frac{D^*(Cz_n,Rz_n,Sx_n)+D^*(Cz_n,By_n,Sx_n)}{2}\right]$$
$$+ \delta\, D^*(Ax_n,By_n,Cz_n)$$

Hence $\lim_{n\to\infty} L(x_n,y_n,z_n) = \alpha \max\left\{0,0,0,D^*\left(u,u,\lim_{n\to\infty} Cz_n\right)\right\}$

$$+ \gamma\left[\frac{D^*\left(\lim_{n\to\infty} Cz_n,u,u\right)+D^*\left(\lim_{n\to\infty} Cz_n,u,u\right)}{2}\right] + \delta\, D^*\left(u,u,\lim_{n\to\infty} Cz_n\right)$$
$$= (\alpha+\gamma+\delta)\, D^*\left(u,u,\lim_{n\to\infty} Cz_n\right)$$

On making $n \to \infty$ in above inequality, we get

$$\int_0^{D^*\left(u,u,\lim_{n\to\infty} Cz_n\right)} \emptyset(s)ds \leq \emptyset\left((\alpha+\gamma+\delta)\int_0^{D^*\left(u,u,\lim_{n\to\infty} Cz_n\right)} \emptyset(s)ds\right)$$
$$< (\alpha+\gamma+\delta)\int_0^{D^*\left(u,u,\lim_{n\to\infty} Cz_n\right)} \emptyset(s)ds,$$

Which is a contradiction. (as $\alpha + \beta + \gamma + \delta \leq 1$.)

Hence $\lim_{n\to\infty} Cz_n = u$. Assume that S(X) is a closed subset of X.. Then there exists v ∈ X such that Sv = u.

If u $\neq$ Av, then using (2.2.1) we obtain

$$\int_0^{D^*(Av,By_n,Cz_n)} \emptyset(s)ds \leq \emptyset\left(\int_0^{L(v,y_n,z_n)} \emptyset(s)ds\right)$$

Where $L(v,y_n,z_n) = \alpha \max\left\{\begin{matrix} D^*(Sv,Ty_n,Rz_n), D^*(Av,Ty_n,Rz_n), \\ D^*(Sv,By_n,Rz_n), D^*(Sv,Ty_n,Cz_n) \end{matrix}\right\}$

$$+ \beta\left[\frac{D^*(Ty_n,By_n,Rz_n)+D^*(Sv,Av,Rz_n)}{2}\right] + \gamma\left[\frac{D^*(Cz_n,Rz_n,Sv)+D^*(Cz_n,By_n,Sv)}{2}\right] + \delta\, D^*(Av,By_n,Cz_n).$$

As $n \to \infty$, it follows that

Hence

$$\int_0^{D^*(Av,u,u)} \emptyset(s)ds \leq \emptyset\left(\left(\alpha+\frac{\beta}{2}+\delta\right)\int_0^{D^*(Av,u,u).} \emptyset(s)ds\right)$$
$$< \left(\alpha+\frac{\beta}{2}+\delta\right)\int_0^{D^*(Av,u,u).} \emptyset(s)ds,$$

Which is a contradiction. (as $\alpha+\beta+\gamma+\delta \leq 1$.)

Therefore Av = Sv = u. Since A(X) ⊆ T(X), there exists w ∈ X such that Av = Tw = u. If u ≠ Bw, using (2.2.1) we obtain

$$\int_0^{D^*(Av,Bw,Cz_n)} \emptyset(s)ds \leq \emptyset\left(\int_0^{L(v,w,z_n)} \emptyset(s)ds\right)$$

Where $L(v,w,z_n) = \alpha\max\begin{Bmatrix} D^*(Sv,Tw,Rz_n), D^*(Av,Tw,Rz_n), \\ D^*(Sv,Bw,Rz_n), D^*(Sv,Tw,Cz_n) \end{Bmatrix}$

$$+\beta\left[\frac{D^*(Tw,Bw,Rz_n)+D^*(Sv,Av,Rz_n)}{2}\right]+\gamma\left[\frac{D^*(Cz_n,Rz_n,Sv)+D^*(Cz_n,Bw,Sv)}{2}\right]$$
$$+\delta D^*(Av,Bw,Cz_n).$$

As $n \to \infty$, it follows that

Hence

$$\int_0^{D^*(u,Bw,u)} \emptyset(s)ds \leq \emptyset\left(\left(\alpha+\frac{\beta}{2}+\frac{\gamma}{2}+\delta\right)\int_0^{D^*(u,Bw,u).} \emptyset(s)ds\right)$$
$$< \left(\alpha+\frac{\beta}{2}+\frac{\gamma}{2}+\delta\right)\int_0^{D^*(u,Bw,u).} \emptyset(s)ds,$$

Which is a contradiction. (as $\alpha+\beta+\gamma+\delta \leq 1$.)

Therefore, Bw = u. Since B(X) ⊆ R(X), there exists e ∈ X such that Re = Bw = u. If e ≠ Re, using (2.2.1) we obtain

$$\int_0^{D^*(Av,Bw,Ce)} \emptyset(s)ds \leq \emptyset\left(\int_0^{L(v,w,e)} \emptyset(s)ds\right)$$

Where $L(v,w,e) = \alpha\max\begin{Bmatrix} D^*(Sv,Tw,Re), D^*(Av,Tw,Re), \\ D^*(Sv,Bw,Re), D^*(Sv,Tw,Ce) \end{Bmatrix}$

$$+\beta\left[\frac{D^*(Tw,Bw,Re)+D^*(Sv,Av,Re)}{2}\right]+\gamma\left[\frac{D^*(Ce,Re,Sv)+D^*(Ce,Bw,Sv)}{2}\right]$$
$$+\delta D^*(Av,Bw,Ce)$$

As $n \to \infty$, it follows that

Hence

$$\int_0^{D^*(u,u,Ce)} \emptyset(s)ds \leq \emptyset\left((\alpha+\gamma+\delta)\int_0^{D^*(u,u,Ce).} \emptyset(s)ds\right)$$
$$< (\alpha+\gamma+\delta)\int_0^{D^*(u,u,Ce).} \emptyset(s)ds,$$

Which is a contradiction. (as $\alpha+\beta+\gamma+\delta \leq 1$.)

Hence Ce = u. That is,

Av = Sv = Bw = Tw = Re = Ce = u.

By weak compatibility of the pairs (A,S), (B,T), and (R,C), we get Au = Su, Bu = Tu and Ru = Cu. If u ≠ Au, then using (2.2.1), we have

$$\int_0^{D^*(Au,Bw,Ce)} \emptyset(s)ds \leq \emptyset\left(\int_0^{L(u,w,e)} \emptyset(s)ds\right)$$

Where $L(u,w,e) = \alpha\max\begin{Bmatrix} D^*(Su,Tw,Re), D^*(Au,Tw,Re), \\ D^*(Su,Bw,Re), D^*(Su,Tw,Ce) \end{Bmatrix}$

$$+\beta\left[\frac{D^*(Tw,Bw,Re)+D^*(Su,Au,Re)}{2}\right]+\gamma\left[\frac{D^*(Ce,Re,Su)+D^*(Ce,Bw,Su)}{2}\right]$$
$$+\delta D^*(Au,Bw,Ce)$$

As $n \to \infty$, it follows that

Hence

$$\int_0^{D^*(Au,u,u)} \emptyset(s)ds \leq \emptyset\left(\left(\alpha+\frac{\beta}{2}+\delta\right)\int_0^{D^*(Au,u,u).} \emptyset(s)ds\right)$$
$$< \left(\alpha+\frac{\beta}{2}+\delta\right)\int_0^{D^*(Au,u,u).} \emptyset(s)ds,$$

Which is a contradiction. (as $\alpha + \beta + \gamma + \delta \leq 1$.)

Hence Au = Su =u. Similarly, we can prove that Bu = Tu = u and Ru = Cu = u. Thus u is a common fixed point of A, B, C, R, S and T. The uniqueness of u follows from inequality (2.1.1).

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