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A Common Fixed Point Theorem for Weakly Compatible Maps in Complex Valued Metric Spaces

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ABSTRACT: In this paper we prove a common fixed point theorem for weakly compatible maps in complex valued metric spaces without using the notion of continuity. Our result generalizes and extends the results of S. Bhatt, S. Chaukiyal and R.C. Dimri.

Keywords: Weakly compatible maps, common fixed point, complex valued metric spaces.

I. INTRODUCTION AND PRELIMINARIES

Metric spaces form a special class of cone metric spaces, yet this idea is intended to define rational expressions which are not meaningfull in cone metric spaces and thus many results of analysis cannot be generalized to cone metric spaces. Indeed the definition of a cone metric space banks on the underlying Banach space which is not a division Ring. However, in complex valued metric spaces, we can study improvements of a host of results of analysis involving divisions. In this paper we proved coincidence point common fixed point theorem involving two pair of compatible mappings satisfying complex inequality in complex valued metric space.

A. Azam, B. Fisher and M. Khan [2], introduced the concept of complex valued metric spaces and obtained sufficient conditions for the existence of common fixed points of a pairs of mappings satisfying contractive type condition. Plenty of material is also available in other generalized metric spaces, such as, rectangular metric spaces. Pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, probabilistic metric spaces, D-metric spaces and cone metric spaces [3-16]. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces; additionally, it offers numerous research activities in mathematical analysis. Consistent with A. Azam, B. Fisher and M. Khan [2], the following definitions and results will be needed in the sequel.

Let $\mathbb{C}$ be the set of complex numbers and let $z_1,z_2 \in \mathbb{C}$. Define a partial order $\leq$ on $\mathbb{C}$ as follows: $z_1 \leq z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im\, Re(z_2)$. It follows that $z_1 \leq z_2$ if one of the following conditions is satisfied:

$Re(z_1) = Re(z_2),\ Im(z_1) \leq Im\, Re(z_2)$
$Re(z_1) \leq Re(z_2),\ Im(z_1) = Im\, Re(z_2)$
$Re(z_1) < Re(z_2),\ Im(z_1) < Im\, Re(z_2)$
$Re(z_1) = Re(z_2),\ Im(z_1) = Im\, Re(z_2)$

In particular, we will write $z_1 \leq z_2$ if one of (1),(2) and (3) is satisfied and we write $z_1 < z_2$ if only (3) is satisfied.

Definition 1.1. [2] Let X be a non-empty set. Suppose that the mapping $d: X \times X \to \mathbb{C}$ satisfies:

$0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y)=0$ if and only if $x=y$;

(b) $d(x, y)=d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, z)+d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 1.1. [17] Let X = C be a set of complex number. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, by

$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X, d) is a complex valued metric space.

Example 1.2. [18] Let $X = \mathbb{C}$ be a set of complex number. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, by

$d(z_1, z_2) = e^{ik} |z_1 - z_2|$, where $k \in R$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X, d) is a complex valued metric space.

Definition 1.2. [2] Let (X, d) be a complex valued metric space and $A \subseteq X$

(i) $x \in X$ is called an interior point of a set B whenever there is $0 < r \in \mathbb{C}$ such that $B(x, r) \subseteq A$ where $B(b, r) = \{y \in X : d(x, y) < r\}$.

(ii) A point $x \in X$ is called a limit point of A whenever for every $0 < r \in \mathbb{C}$, $B(x, r) \cap (A \setminus X) \neq \emptyset$.

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.The family $F = \{B(x, r) : x \in X, 0 < r\}$ is a sub-basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Definition 1.3. [2] Let (X, d) be a complex valued metric space and $\{x_n\}_{n\geq1}$ be a sequence in X and x ∈ X. We say that

(i) The sequence $\{x_n\}_{n\geq1}$ converges to x if for every c ∈ ℂ, with 0 < c there is $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x) < c$,. We denote this by limn $x_n = x$, or $x_n \to x$, as $n \to \infty$,

(ii) The sequence $\{x_n\}_{n\geq1}$ is Cauchy sequence if for every c ∈ ℂ with 0 < c there is $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x_{n+m}) < c$,

(iii) The metric space (X, d) is a complete complex valued metric space, if every Cauchy sequence is convergent.

Definition 1.4. [1]. let S and T be two self-maps defined on set X. Then S and T are said to be weakly compatible if they commute at their coincidence points.

Lemma 1.1 [2]. Let (X, d) be a complex valued metric space and $\{x_n\}$a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n\to\infty$.

Lemma 1.2 [2]. Let (X, d) be a complex valued metric space and $\{x_n\}$a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if$|d(x_{n+m}, x)| \to 0$ as $n\to\infty$.

II. MAIN RESULT

Theorem 2.1. Let (X, d) be a complex valued metric space and let f, g, S and T are four self maps of X such that $T(X) \subseteq f(X)$ and $S(X) \subseteq g(X)$ and satisfying

$$d(Sx, Ty) \leq a\, d(fx, gx) + b\, [\, d(fx, Sx) + d(gy,Ty)\,] + c[\frac{d(fx,\ Ty)+d(gy,\ Sx)}{1+d(fx,\ Ty)\, d(gy,\ Sx)}] + e\, [d(fx, Sx) + d(fx, gy)\,] + h\, [d(fx, Ty) - d(gy, Ty)\,] \quad (2.1)$$

Where a, b, c, e, h ≥ 0 & a+2b+2c+2e+h<1 suppose that the pair {f, S} and {g, T}are weakly compatible. Then f, g, S and d T have a unique common fixe point.

Proof: suppose x_0 is an arbitrary point of X. Define the sequence $\{y_n\}$ such that

$Y_{2n} = Sx_{2n} = g\, x_{2n+1}$

$Y_{2n+1} = Tx_{2n+1} = f\, x_{2n+2}$

$$\begin{aligned}
d(y_{2n},y_{2n+1}) &= d(Sx_{2n},Tx_{2n+1}) \\
&\leq a\, d(fx_{2n},gx_{2n+1}) + b\, [\, d(fx_{2n},Sx_{2n}) + d(gx_{2n+1},Tx_{2n+1})\,] + c[\frac{d(fx_{2n},\ Tx_{2n+1})+(gx_{2n+1},\ Sx_{2n})}{1+d(fx_{2n},\ Tx_{2n+1})\, d(gx_{2n+1},\ Sx_{2n})}] + e\, [d(fx_{2n},Sx_{2n}) + d(fx_{2n},gx_{2n+1})\,] + h\, [d(fx_{2n},Tx_{2n+1}) - d(gx_{2n+1},Tx_{2n+1})\,] \\
&\leq a\, d(fx_{2n},gx_{2n+1}) + b\, [\, d(fx_{2n},Sx_{2n}) + d(gx_{2n+1},Tx_{2n+1})\,] + c[d(fx_{2n},Tx_{2n+1}) + d(gx_{2n+1},Sx_{2n})] + e\, [d(fx_{2n},Sx_{2n}) + d(fx_{2n},gx_{2n+1})\,] + h\, [d(fx_{2n},Tx_{2n+1}) - d(gx_{2n+1},Tx_{2n+1})\,] \\
&\leq a\, d(y_{2n-1},y_{2n}) + b\, [d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1})] + c[d(y_{2n-1},y_{2n+1}) + d(y_{2n},y_{2n})] + e[d(y_{2n-1},y_{2n}) + d(y_{2n-1},y_{2n})] + h[d(y_{2n-1},y_{2n+1}) - d(y_{2n},y_{2n+1})] \\
&\leq a\, d(y_{2n-1},y_{2n}) + b\, [d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1})] + c[d(y_{2n-1},y_{2n+1}) + d(y_{2n},y_{2n})] + e[d(y_{2n-1},y_{2n}) + d(y_{2n-1},y_{2n})] + h[d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1}) - d(y_{2n},y_{2n+1})] \\
&\leq (a+b+c+2e+h)\, d(y_{2n-1},y_{2n}) + (b + c)\, d(y_{2n},y_{2n+1}) \\
&\leq \frac{a+b+c+2e+h}{1-b-c} d(y_{2n-1}, y_{2n}) \\
&\leq k\, d(y_{2n-1},y_{2n})
\end{aligned}$$

Where $k = \frac{a+b+c+2e+h}{1-b-c}$

Similarly it can be shown that $d(y_{2n+1}, y_{2n+2}) \leq k\, d(y_{2n}, y_{2n+1})$

Therefore,

$d(y_{n+1}, y_{n+2}) \leq k\, d(y_n, y_{n+1}) \leq \ldots\ldots\ldots \leq k_{n+1}\, d(y_0, y_1)$

Now for all m>n

$$\begin{aligned}
d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots\ldots\ldots + d(y_{m-1}, y_m) \\
&\leq (k_n + k_{n+1} + \ldots\ldots\ldots\ldots + k_{m-1})\, d(y_1,y_0) \\
&\leq \frac{k^n}{1-k} d(y_1, y_0)
\end{aligned}$$

$$|\, d(y_n, y_m)\, | \leq \frac{k^n}{1-k} |\, d(y_1, y_0)|$$

Which implies that $|\, d(y_n, y_m)\, | \to 0$ as m, $n\to\infty$.

Hence $\{y_n\}$is a cauchy sequence, since X is complete , there exist a point z in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} fx_{2n+2} = z$.

Since $T(X) \subseteq f(X)$,there exist a point u ∈X such that z = fu.

Then by (2.1) we have

$$\begin{aligned}
d(Su, z) &\leq d(Su, Tx_{2n-1}) + d(Tx_{2n-1}, z) \\
&\leq a\, d(fu, gx_{2n-1}) + b\, [\, d(fu, Su) + d(gx_{2n-1}, Tx_{2n-1})\,] + c[\frac{d(fu,\ Tx_{2n-1})+d(gx_{2n-1},\ Su)}{1+d(fu,\ Tx_{2n-1})\, d(gx_{2n-1},\ Su)}] + e\, [d(fu, Su) + d(fu, gx_{2n-1})\,] + h\, [d(fu,Tx_{2n-1}) - d(gx_{2n-1},Tx_{2n-1})\,] + d(Tx_{2n-1}, z) \\
&\leq a\, d(fu, gx_{2n-1}) + b\, [\, d(fu, Su) + d(gx_{2n-1},Tx_{2n-1})\,] + c[d(fu, Tx_{2n-1}) + d(gx_{2n-1}, Su)] + e\, [d(fu, Su) + d(fu, gx_{2n-1})\,] + h\, [d(fu, Tx_{2n-1}) - d(gx_{2n-1}, Tx_{2n-1})\,] + d(Tx_{2n-1}, z)
\end{aligned}$$

Taking limit $n\to\infty$, yields,

$d(Su, z) \leq a\, d(z, z) + b[d(z, Su) + d(z, z)] + c[d(z, z) + d(z, Su)] + e[d(z, Su) + d(z, z)] + h[d(z, z) - d(z, z)] + d(z, z)$

$d(Su, z) \leq (b + c + e)\, d(Su, z)$, a contradiction.

Since $a + 2b + 2c + 2e + h < 1$. Therefore $Su = fu = z$, since $S(X) \subseteq g(X)$, there exists a point $v \in X$, such that $z = gv$. Then by (2.1) we have,

$$d(z, Tv) \leq d(Su, Tv)$$
$$\leq a\, d(fu, gv) + b[d(fu, gv) + d(gv, Tv)] + c\left[\frac{d(fu, Tv)+d(gv, Su)}{1+d(fu, Tv)d(gv, Su)}\right] + e[d(fu, Su) + d(fu, gv)] + h[d(fu, Tv) - d(gv, Tv)]$$
$$\leq a\, d(fu, gv) + b[d(fu, gv) + d(gv, Tv)] + c[[d(fu, Tv) + d(gv, Su)]] + e[d(fu, Su) + d(fu, gv)] + h[d(fu, Tv) - d(gv, Tv)]$$
$$d(z, Tv) \leq a\, d(z, z) + b[d(z, z) + d(z, Tv)] + c[d(z, Tv) + d(z, z)] + e[d(z, z) + d(z, z)] + h[d(z, Tv) - d(z, Tz)]$$

$d(z, Tv) \leq (b + c)\, d(z, Tv)$, a contradiction. Since $a + 2b + 2c + 2e + h < 1$. Therefore $Tv = gv = z$ and so $Su = fu = Tv = gv = z$. Since f and S are weakly compatible maps then $Sfu = fSu$ and so $Sz = fz$. Now we show that z is a fixed point of S, if $Sz \neq z$ from (2.1) we have,

$$d(Sz, z) \leq d(Sz, Tv)$$
$$\leq a\, d(fz, gv) + b[d(fz, Sz) + d(gv, Tv)] + c\left[\frac{d(fz, Tv)+d(gv, Sz)}{1+d(fz, Tv)d(gv, Sz)}\right] + e[d(fz, Sz) + d(fz, gv)] + h[d(fz, Tv) - d(gv, Tv)]$$
$$d(Sz, z) \leq a\, d(fz, gv) + b[d(fz, Sz) + d(gv, Tv)] + c[d(fz, Tv) + d(gv, Sz)] + e[d(fz, Sz) + d(fz, gv)] + h[d(fz, Tv) - d(gv, Tv)]$$
$$d(Sz, z) \leq a\, d(Sz, z) + b[d(Sz, Sz) + d(z, z)] + c[d(Sz, z) + d(z, Sz)] + e[d(Sz, Sz) + d(Sz, z)] + h[d(Sz, z) - d(z, z)]$$
$$\leq (a + 2c + e + h)\, d(Sz, z)$$

a contradiction. Since $a + 2b + 2c + 2e + h < 1$. Therefore $Sz = z$ and so $Sz = fz = z$. Similarly g and T are weakly compatible maps then $Tz = gz$. Now we show that z is a fixed point of T. If $Tz \neq z$, then by (2.1) we have

$$d(z, Tz) \leq d(Sz, Tz)$$
$$\leq a\, d(fz, gz) + b[d(fz, Sz) + d(gz, Tz)] + c\left[\frac{d(fz, Tz)+d(gz, Sz)}{1+d(fz, Tz)d(gz, Sz)}\right] + e[d(fz, Sz) + d(fz, gz)] + h[d(fz, Tz) - d(gz, Tz)]$$
$$\leq a\, d(fz, gz) + b[d(fz, Sz) + d(gz, Tz)] + c[d(fz, Tz) + d(gz, Sz)] + e[d(fz, Sz) + d(fz, gz)] + h[d(fz, Tz) - d(gz, Tz)]$$
$$d(z, Tz) \leq a\, d(z, Tz) + b[d(z, z) + d(Tz, Tz)] + c[d(z, Tz) + d(Tz, z)] + e[d(z, z) + d(Tz, z)] + h[d(z, Tz) - d(Tz, Tz)]$$
$$d(z, Tz) \leq (a + 2c + e + h)\, d(z, Tz)$$

a contradiction. Since $a + 2b + 2c + 2e + h < 1$. Therefore $Tz = z$ and so $Sz = Tz = fz = gz = z$.

Finally in order to prove that uniqueness of z, suppose that z and w are distinct common fixed points of f, g, S and T from (2.1) we have,

$$d(z, w) \leq d(Sz, Tw)$$
$$\leq a\, d(fz, gw) + b[d(fz, Sz) + d(gw, Tw)] + c\left[\frac{d(fz, Tw)+d(gw, Sz)}{1+d(fz, Tw)d(gw, Sz)}\right] + e[d(fz, Sz) + d(fz, gw)] + h[d(fz, Tw) - d(gw, Tw)]$$
$$\leq a\, d(fz, gw) + b[d(fz, Sz) + d(gw, Tw)] + c[d(fz, Tz) + d(gw, Sz)] + e[d(fz, Sz) + d(fz, gw)] + h[d(fz, Tw) - d(gw, Tw)]$$
$$\leq a\, d(z, w) + b[d(z, z) + d(w, w)] + c[d(z, w) + d(w, z)] + e[d(z, z) + d(z, w)] + h[d(z, w) - d(w, w)]$$

$d(z, w) \leq (a + 2c + e + h)\, d(z, w)$ a contradiction . And so $z = w$, proving that z is unique common fixed point off, g, S and T.

Corollary 2.2 . Let(X, d) be a complex valued metric space and let f, S and T are three self- maps of X such that $T(X) \subseteq f(X)$, $S(X) \subseteq f(X)$ and satisfying

$$d(Sx, Ty) \leq a\, d(fx, gx) + b[d(fx, Sx) + d(fy, Ty)] + c\left[\frac{d(fx, Ty)+d(fy, Sx)}{1+d(fx, Ty)\, d(fy, Sx)}\right] + e[d(fx, Sx) + d(fx, fy)] + h[d(fx, Ty) - d(fy, Ty)]$$

For all $x, y \in X$ where a, b and $c \geq 0$ and $a + 2b + 2c + 2e + h < 1$.

Suppose that the pairs {f, S} and {f, T} are weakly compatible then f, S and T have a unique common fixed point.

Proof: The result follows on putting f=g in theorem (2.1).

III. CONCLUSION

This article investigates common fixed point theorems for four self mappings. The concept of weakly compatible maps in complex valued metric spaces without using notion of continuity. Several Fixed point theorems in complex valued metric spaces such as fixed point theorems for three and two self mappings have been derived in the present study as particular cases.

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