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Common Fixed Point Theorem in Probabilistic 2-Metric Space by Weak Compatibility

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ABSTRACT: The object of this paper is to extend and generalize the result of Vasuki [8] from fuzzy metric space to probabilistic 2-metric space using the concept of weak compatibility.

Keywords: Common fixed point, Menger space, Probabilistic 2-metric space, compatible maps, semi-compatible maps, weak compatible maps.
AMS Subject Classification: Primary 47H10, Secondary 54H25.

I. INTRODUCTION

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [3]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [5] studied this concept and gave some fundamental results on this space.

The notion of compatible mapping in a Menger space has been introduced by Mishra [4]. Sessa [7] initiated the tradition of improving commutativity in fixed point theorems by introducing the notion of weakly commuting maps in metric space. Jungck [1] soon enlarged this concept by introducing the concept of compatible maps. Recently, Jungck and Rhoades [2] termed a pair of self maps to be coincidentally commuting or equivalently weak-compatible if they commute at their coincidence points. The concept of R-weakly commuting maps in fuzzy metric space has been introduced by Vasuki [8].

The main object of this paper is to extend and generalize the result of Vasuki [8] from fuzzy metric space to probabilistirc 2-metric space in the following ways :

(i) To increase the number of maps from 2 to 4.

(ii) To relax the continuity requirement of the maps completely.

II. PRELIMINARIES

Definition 2.1. [4] A mapping $F : R \to R^+$ is called a *distribution* if it is non-decreasing left continuous with $\inf \{ F(t) \mid t \in R \} = 0$ and $\sup \{ F(t) \mid t \in R\} = 1$.

We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0 & , \quad t \le 0 \\ 1 & , \quad t > 0 \end{cases}.$$

Definition 2.2. [9] A *probabilistic 2-metric space (2-PM space)* is an ordered pair (X, F) where X is an abstract set and F is a function defined on X × X × X into L, the collection of all distribution functions. The value of F at (x, y, z) ∈ X × X × X is generally represented by $F_{x,y,z}$ or F(x, y, z). The distribution function F(x, y, z) satisfy the following conditions:

(1) F(x, y, z; 0) = 0,

(2) For all distinct x, y in X there exists a point z in X such that
F(x, y, w; t) < 1 for some t > 0.

(3) F(x, y, z; t) = 1 for all t > 0 if and only if at least two of the three points are equal.

(4) F(x, y, z; t) = F(x, z, y ; t) = F(y, z, x; t)(Symmetry)

(5) If $F(x, y, z; t_1) = F(x, z, y; t_2) = F(z, y, x; t_3) = 1$ then
$F(x, y, z,; t_1 + t_2 + t_3) = 1$.

Definition 2.3. [9] The mapping t: [0, 1] × [0, 1] × [0, 1] → [0, 1] is a *t-norm* if t satisfies the following conditions:

(1) t(x, 1, 1) = x , t(0, 0, 0) = 0;

(2) t(x, y, z) = t(x, z, y) = T(z, y, x);

(3) $t(x_1, y_1, z_1) \ge t(x_2, y_2, z_2)$ for $x_1 \ge x_2$, $y_1 \ge y_2$, $z_1 \ge z_2$;

(4) t(t(x, y, z), p, q) = t(x, t(y, z, p), q) = t(x, y, t(z, p, q)).

Definition 2.4. [9] A *Menger probabilistic 2-metric space* is a triplet (X, F, t) where (X, F) is a 2-PM space and t is a t-norm satisfying the following triangle inequality :

$F(x, y, z; t_1 + t_2 + t_3) \geq y(F(x, y, p; t_1), F(x, p, z; t_2), F(p, y, z; t_3)$ for all x, y, z, p ∈ X and $t_1, t_2, t_3 \geq 0$.

Definition 2.5. [9] A sequence $\{x_n\}$ in a 2-Menger space (X, F, t) is said to *converge* to a point x ∈ X if for each ε > 0 and λ > 0 there exists a positive integer M(ε,λ) such that

$F(x_n, x, a; \varepsilon) > 1 - \lambda$, for all a ∈ X and n ≥ M(ε, λ).

The sequence $\{x_n\}$ converges to x if and only if

$$F(x_n, x, a; t) = H(t) \quad \text{for all } a,$$

where H is the distribution function defined as above.

Definition 2.6. [9] A sequence $\{x_n\}$ in a 2-Menger space (X, F, t) is said to be *Cauchy* if, for each ε > 0 and λ > 0 there exists a positive integer M(ε, λ) such that

$F(x_n, x_m, a; \varepsilon) > 1 - \lambda$, for all a ∈ X and n, m ≥ M(ε, λ).

Lemma 2.1. [9] Let $\{x_n\}$ be a sequence in a 2-Menger space (X, **F**, t) where t is continuous and satisfies t(x, x, x) ≥ x for all x ∈ (0, 1). If there exists a positive number h < 1 such that

$F(x_{n+1}, x_n, a; hu) \geq F(x_n, x_{n-1}, a; u)$, n = 1, 2, 3, ...

for all a ∈ X and u ≥ 0 then $\{x_n\}$ is a Cauchy sequence in X.

Definition 2.7. Self mappings A and S of a Menger probabilistic 2-metric space (X, F, t) are said to be *compatible* if $F_{ASx_n, SAx_n, a}(x) \to 1$ for all a ∈ X, x > 0, whenever $\{x_n\}$ is a sequence in X such that Ax_n, $Sx_n \to u$ for some u in X, as n → ∞.

Definition 2.7. Self maps S and T of a Menger probabilistic 2-metric space (X, F, t) are said to be *weak-compatible* if they commute at their coincidence points, i.e. Sx = Tx for x ∈ X implies STx = TSx.

Remark 2.1. It is obvious that the concept of weak compatibility is more general than that of compatibility.

Lemma 2.1. [9] Let $\{p_n\}$ be a sequence in a Menger probabilistic 2-metric space (X, F, t) with continuous t-norm and t(x, x) ≥ x. Suppose, for all x ∈ [0, 1], there exists k ∈ (0, 1) such that for all x > 0 and n ∈ N,

$$F_{p_n, p_{n+1}, a}(kx) \geq F_{p_{n-1}, p_n, a}(x)$$

Or $$F_{p_n, p_{n+1}, a}(x) \geq F_{p_{n-1}, p_n, a}(k^{-1}x).$$

Then $\{p_n\}$ is a Cauchy sequence in X.

In [8], Vasuki proved the following result:

Theorem 2.1. Let (X, M,*) be a complete fuzzy metric space and f and g be R-weakly commuting self mappings of X satisfying the condition

$$M(fx, fy, t) \geq r[M(gx, gy, t)],$$

where r : [0, 1] → [0,1] is a continuous function such that r(t)>t for each 0 < t < 1. If f(x) ⊂ g(x) and either f or g is continuous then f and g have a unique common fixed point.

III. MAIN RESULT

Theorem 3.1. Let A, B, S and T be self mappings of a complete probabilistic 2-metric space (X, F, min) satisfying

(3.1) A(X) ⊂ T(X), B(X) ⊂ S(X);

(3.2) One of A(X), B(X), T(X) or S(X) is complete;

(3.3) Pairs (A, S) and (B, T) are weak compatible;

(3.4) for all x, y ∈ X and t > 0,

$$F_{Ax,By,a}(t) \geq rF_{Sx,Ty,a}(t)$$

where r : [0, 1] → [0, 1] is some continuous function such that r(t) > t, for each 0 < t < 1.

Then A, B, S and T have a unique common fixed point in X.

Proof. Let x_0 ∈ X be any arbitrary point.

As A(X) ⊂ T(X) and B(X) ⊂ S(X), there exists x_1, x_2 ∈ X such that

$$Ax_0 = Tx_1 \quad \text{and} \quad Bx_1 = Sx_2.$$

Inductively, construct sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$y_{2n+1} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$ for n = 0, 1, 2, ...

Now, using (3.4) with $x = x_{2n}$ and $y = x_{2n+1}$, we obtain that

$$\begin{aligned} F_{y_{2n+1}, y_{2n+2}, a}(t) &= F_{Ax_{2n},Bx_{2n+1}, a}(t) \\ &\geq rF_{Sx_{2n},Tx_{2n+1}, a}(t) \\ &= rF_{y_{2n}, y_{2n+1},a}(t) \\ &> F_{y_{2n}, y_{2n+1},a}(t) \quad \text{for } t \in (0, 1). \end{aligned}$$

Similarly,

$$F_{y_{2n+2}, y_{2n+3}, a}(t) > F_{y_{2n+1}, y_{2n+2}, a}(t).$$

In general,

$$F_{y_{n+1}, y_n, a}(t) > F_{y_n, y_{n-1}, a}(t).$$

Thus, $\{F_{y_{n+1}, y_n, a}(t), n > 0\}$ is a increasing sequence of positive real numbers in [0, 1] and therefore tends to a limit L ≤ 1.

If L < 1, then $F_{y_{n+1}, y_n, a}(t) = L > r(1) > 1$,

which is a contradiction.

Hence, L = 1.

Hence, for all n and p,

$$F_{y_n, y_{n+p}, a}(t) = 1.$$

Thus, $\{y_n\}$ is a Cauchy sequence in X. By completeness of X, $\{y_n\}$ converges to $z \in X$.

Hence, its subsequences

$\{Ax_{2n}\} \to z$, $\{Sx_{2n}\} \to z$, $\{Tx_{2n+1}\} \to z$ and $\{Bx_{2n+1}\} \to z$. …(3.1)

Case I. T(X) is complete.

In this case $z \in T(X)$.

Hence, there exists $u \in X$ such that z = Tu. …(3.2)

Step 1. By putting $x = x_{2n}$ and y = u in (3.4), we obtain

$$F_{Ax_{2n},Bu, a}(t) \geq rF_{Sx_{2n},Tu, a}(t). \quad \text{…(3.3)}$$

Taking limit as $n \to \infty$ and using (1), we get

$$F_{z,Bu, a}(t) \geq rF_{z,Tu, a}(t)$$
$$= rF_{z,z, a}(t) = r(1) = 1 \quad \text{…(3.4)}$$

which gives z = Bu = Tu.

As (B, T) is weak compatible, we get

TBu = BTu,

i.e. Tz = Bz. …(3.5)

Step II. By putting $x = x_{2n}$ and y = z in (3.4), we obtain that

$$F_{Ax_{2n},Bz, a}(t) \geq rF_{Sx_{2n},Tz, a}(t).$$

Taking limit as $n \to \infty$ and using (1), (2) and (3.5), we get

$$F_{z, Bz, a}(t) \geq rF_{z, z, a}(t)$$
$$= r(1) = 1,$$

which gives z = Bz and we get

Tz = Bz = z. …(3.6)

Step III. As $B(X) \subset S(X)$, there exists $v \in X$ such that

z = Bz = Sv.

By putting x = v, y = z in (3.4), we get

$$F_{Av, Bz, a}(t) \geq rF_{Sv, Tz, a}(t)$$

i.e. $F_{Av, z, a}(t) \geq rF_{z,z}(t) = 1$

which gives Av = z = Sv and weak compatibility of (A, S) gives

ASv = SAv,

i.e. Az = Sz.

Step IV. By putting x = z, y = z in (3.4) and assuming $Az \neq Bz$, we get

$$F_{Az, Bz, a}(t) \geq rF_{Sz, Tz, a}(t)$$
$$= rF_{Az, Bz, a}(t)$$
$$> F_{Az, Bz, a}(t),$$

which is a contradiction and we get Az = Bz.

Combining all the results, we get

z = Az = Bz = Sz = Tz,

i.e., z is a common fixed point of the four self maps A, B, S and T.

Case II. S(X) is complete.

In this case $z \in S(X)$. Hence there exists $w \in X$ such that z = Sw.

Step I. By putting x = w, $y = x_{2n+1}$ in (3.4), we get

$$F_{Aw,Bx_{2n+1}, a}(t) \geq rF_{Sw,T_{2n+1}, a}(t).$$

Taking limit as $n \to \infty$ and using (3) and (4), we obtain that

$$F_{Aw, z, a}(t) \geq rF_{z, z, a}(t)$$
$$= r(1) = 1.$$

Hence, z = Aw = Sw and weak comatibility of (A, S) gives

ASw = SAw,

i.e. Az = Sz. (3.7)

Step II. Put x = z, $y = x_{2n+1}$ in (3.4) and we get

$$F_{Az, Bx_{2n+1}, a}(t) \geq rF_{Sz, Tx_{2n+1}, a}(t).$$

Taking limit as $n \to \infty$ and using (3) and (4), we obtain that

$$F_{Az, z, a}(t) \geq rF_{Az, z, a}(t)$$
$$> F_{Az, z, a}(t), \text{ if } F_{Az, z, a}(t) > 0,$$

which is a contradiction, hence z = Az = Sz.

Step III. As $A(X) \subset T(X)$, there exists some $u_1 \in X$, such that

$$z = Az = Tu_1.$$

By putting $x = x_{2n}$, $y = u_1$ in (3.4), we have

$$F_{Ax_{2n}, Bu_1, a}(t) \geq rF_{Sx_{2n},Tu_1, a}(t).$$

Taking limit as $n \to \infty$ and using (1) and (2), we get

$$F_{z, Bu_1, a}(t) \geq rF_{z, z, a}(t)$$
$$= r(1) = 1.$$

Thus $z = Bu_1 = Tu_1$.

As (B, T) is weakly compatible, we get

TBu = BTu,

i.e. Tz = Bz.

Step IV. By putting x = z, y = z in (3.4) and assuming $Az \neq Bz$, we have

$$F_{Az, Bz, a}(t) \geq rF_{Sz, Tz, a}(t)$$
$$= rF_{Az, Bz, a}(t)$$
$$> F_{Az, Bz, a}(t),$$

which is a contradiction and we suppose Az = Bz = z.

Combining all the results, we get

z = Az = Bz = Sz = Tz,

i.e. z is a common fixed point of the maps A, B, S and T in this case also.

Case III. As A(X) or B(X) is complete.

As $A(X) \subset T(X)$ and $B(X) \subset S(X)$, the result follows from case I and case II respectively.

Uniqueness. Let z and z' be the two common fixed points of the maps A, B, S and T then z = Az = Bz = Sz = Tz and z' = Az' = Bz' = Sz' = Tz'.

On assuming $z \neq z'$ and using (3.4), we get

$$F_{z, z'}(t) = F_{Az, Bz', a}(t) \geq rF_{Sz, Tz', a}(t) = rF_{z, z', a}(t) > F_{z, z', a}(t), \text{ if } F_{z, z', a}(t) > 0$$

which is a contradiction hence $z = z'$ and we get z is the unique common fixed point of the four self maps.

If we take A = B = f and S = T = g in theorem 3.1., we get

Corollary 3.2. Let (X, F, min) be a complete probabilistic 2-metirc space and f and g are weak compatible self mappings of X satisfying the conditions :

$$F_{fx, fy, a}(t) \geq rF_{gx, gy, a}(t)$$

where, $r : [0, 1] \to [0, 1]$ is a continuous function such that $r(t) > t$ for each $0 < t < 1$.

If $f(x) \subset g(x)$ and either f(x) of g(x) is complete then f and g have a unique common fixed point in X.

Now, on taking S = I, the identity map on X, in theorem 3.1, we have the following result for three self maps none of which is continuous and just a pair of them is needed to be weak compatible only.

Corollary 3.3. Let A, B and T be self mappings of a complete probabilistic 2-metric space (X, F, min) satisfying :

$$A(X) \subset T(X); \quad \ldots(3.8)$$

$$(B, T) \text{ is weak compatible}; \quad \ldots(3.9)$$

$$\forall x, y \in X \text{ and } t > 0, \quad \ldots(3.10)$$

$$F_{Ax, By, a}(t) \geq rF_{x, Ty, a}(t),$$

where $r : [0, 1] \to [0, 1]$ is some continuous function such that $r(t) > t$ for each $0 < t < 1$.

Then A, B and T have unique common fixed point in X.

Again, taking A = I, the identity map on X, in theorem 3.1 we have another result for three self maps none of which is continuous and just a pair of them is needed to be weak compatible only.

Corollary 3.4. Let B, S and T be self mappings of complete probabilistic 2-metric space (X, F, min) satisfying :

$$B(X) \subset S(X), T \text{ is onto}; \quad \ldots(3.11)$$

$$(B, T) \text{ is weak compatible}; \quad \ldots(3.12)$$

$$\forall x, y \in X \text{ and } t > 0, \quad \ldots(3.13)$$

$$F_{x, By, a}(t) \geq rF_{Sx, Ty, a}(t),$$

where $r : [0, 1] \to [0, 1]$ is continuous function such that $r(t) > t$ for each $0 < t < 1$.

Then B, S and T have a unique common fixed point in X.

Again on taking S = T = I the identity map in theorem 3.1, the conditions (3.1), (3.2) and (3.3) are satisfied trivially and we get the following important result.

Corollary 3.5. Let A and B be self mappings of complete probabilistic 2-metric space (X, F, min) satisfying :

$$F_{Ax, By, a}(t) \geq rF_{x, y, a}(t)$$

$\forall x, y \in X$, where $r : [0, 1] \to [0, 1]$ is continuous function such that $r(t) > t$ for each $0 < t < 1$.

Then A and B have a unique common fixed point in X.

Now, taking A = I and B = I in theorem 3.1, the conditions (3.2), (3.3) are satisfied trivially and we get an important result for surjective maps as follows.

Corollary 3.6. Let S and T be self mappings of complete probabilistic 2-metric space (X, F, min) satisfying :

$$F_{x, y}(t) \geq rF_{Sx, Ty}(t)$$

$\forall x, y \in X$, where $r : [0, 1] \to [0, 1]$ is continuous function such that $r(t) > t$ for each $0 < t < 1$.

Then S and T have a unique common fixed point in X.

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