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On Common Fixed Point of Compatible Mappings of Type(P) for Six Self Maps

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ABSTRACT: The aim of this paper is to prove a common fixed point theorem using compatible mappings of type (P) for six self maps in a metric space which extends and improves some results in the literature. We also give an example to illustrate our result.

Keywords: Fixed point, self maps, compatible mappings of type (P) and associated sequence relative to six self maps.

AMS (2010) Mathematics Classification: 54H25, 47H10

I. INTRODUCTION AND PRELIMINARIES

In 1986, G. Jungck [1] introduced the concept of compatible maps as follows.

1.1. Compatible mappings [1]: Two self maps E and F of a metric space (X, d) are said to be compatible mappings if $\lim_{n\to\infty} d(EFx_n, FEx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ex_n = \lim_{n\to\infty} Fx_n = t$ for some $t \in X$.

In 1993 Jungck *et al* defined weaker class of maps called weakly compatible mappings of type (A) as follows.

1.2. Compatible mappings of type (A)[8]: Two self maps E and F of a metric space (X, d) are said to be compatible mappings of type (A) if $\lim_{n\to\infty} d(EFx_n, FFx_n) = 0$ and $\lim_{n\to\infty} d(FEx_n, EEx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ex_n = \lim_{n\to\infty} Fx_n = t$ for some $t \in X$.

In 1995, Pathak and Khan defined compatible mappings of type(B) as follows.

1.3. Compatible mappings of type (B) [13]: Two self maps E and F of a metric space (X,d) are said to be compatible mappings of type (B) if $\lim_{n\to\infty} d(EFx_n, FFx_n) \le \frac{1}{2}\left[\lim_{n\to\infty} d(EFx_n, Et) + \lim_{n\to\infty} d(Et, EEx_n)\right]$ and $\lim_{n\to\infty} d(FEx_n, EEx_n) \le \frac{1}{2}\left[\lim_{n\to\infty} d(FEx_n, Ft) + \lim_{n\to\infty} d(Ft, FFx_n)\right]$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ex_n = \lim_{n\to\infty} Fx_n = t$ for some $t \in X$.

In 1995, Pathak *et al.* introduced the concept of compatible mappings of type (P) as follows.

1.4. Compatible mappings of type (P) [14]: Two self maps E and F of a metric space (X,d) are said to be compatible mappings of type (P) if $\lim_{n\to\infty} d(EEx_n, FFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ex_n = \lim_{n\to\infty} Fx_n = t$ for some $t \in X$.

1.5. Associated sequence[6]: Suppose E, F, G, H, I and J are six self maps of a metric space (X, d) such that $E(X) \subseteq IJ(X)$ *and* $F(X) \subseteq GH(X)$. Then for an arbitrary $x_0 \in X$ we have $Ex_0 \in E(X)$. since $E(X) \subseteq IJ(X)$, there exists $x_1 \in X$ such that $Ex_0 = IJx_1$. for this point x_1, there is a point $x_2 \in X$ such that $Fx_1 = GHx_2$ and so on.

Repeating this process to obtain a sequence $\{y_n\}$ in X such that $y_{2n} = Ex_{2n} = IJx_{2n+1}$ *and* $y_{2n+1} = Fx_{2n+1} = GHx_{2n+2}$ for $n \geq 0$. we shall call this sequence $\{y_n\}$ an "associated sequence of x_0" relative to the six self maps E,F,G,H,I and J.

II. LEMMA

Let E, F, G, H, I and J are six self maps of a metric space (X,d) satisfying

$$E(X) \subseteq IJ(X) \text{ and } F(X) \subseteq GH(X) \tag{2.1}$$

$$d(Ex,Fy) \leq \alpha \frac{d(IJy,Fy)[1+d(GHx,Ex)]}{[1+d(GHx,IJy)]} + \beta d(GHx,IJy) \tag{2.2}$$

for all x,y in X where $\alpha, \beta \geq 0, \alpha + \beta < 1$.

Further if X is complete, then for any $x_0 \in X$ and for any of its associated sequence $Ex_0, Fx_1, Ex_2, Fx_3, ...Ex_{2n}, Fx_{2n+1}.....$ converges to some point p in X .

Proof: From the conditions (2.1) and (2.2) we have

$$d(y_{2n}, y_{2n+1}) = d(Ex_{2n}, Fx_{2n+1})$$

$$\leq \alpha \frac{d(IJx_{2n+1}, Fx_{2n+1})[1+d(GHx_{2n}, Ex_{2n})]}{[1+d(GHx_{2n}, IJy_{2n+1})]} + \beta d(GHx_{2n}, IJy_{2n+1})$$

$$= \alpha \frac{d(y_{2n}, y_{2n+1})[1+d(y_{2n-1}, y_{2n})]}{[1+d(y_{2n-1}, y_{2n})]} + \beta d(y_{2n-1}, y_{2n})$$

$$= \alpha d(y_{2n,} y_{2n+1}) + \beta d(y_{2n-1}, y_{2n}) \text{ and so that}$$

$$(1-\alpha) d(y_{2n}, y_{2n+1}) \leq \beta d(y_{2n-1}, y_{2n})$$

$$d(y_{2n}, y_{2n+1}) \leq \frac{\beta}{(1-\alpha)} d(y_{2n-1}, y_{2n}) = hd(y_{2n-1}, y_{2n}), \text{ where } h = \frac{\beta}{1-\alpha}$$

That is $d(y_{2n,} y_{2n+1}) \leq hd(y_{2n-1}, y_{2n})$ (2.3)

Similarly, we can prove that $d(y_{2n+1}, y_{2n+2}) \leq hd(y_{2n}, y_{2n+1})$. (2.4)

Hence, from (2.3) and (2.4), we get

$$d(y_n, y_{n+1}) \leq hd(y_{n-1}, y_n) \leq h^2 d(y_{n-2}, y_{n-1}) \leq .... \leq h^n d(y_0, y_1) \ . \tag{2.5}$$

Now for any positive integer k, we have

$$d(y_n, y_{n+k}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{n+k-1}, y_{n+k})$$

$$\leq h^n d(y_0, y_1) + h^{n+1} d(y_0, y_1) + ... + h^{n+k-1} d(y_0, y_1)$$

$$= (h^n + h^{n+1} + ... + h^{n+k-1}) d(y_0, y_1)$$

$$= h^n (1 + h + h^2 + ... + h^{k-1}) d(y_0, y_1)$$

$$< \frac{h^n}{1-h} d(y_0, y_1) \to 0 \quad \text{as } n \to \infty, \text{since } h<1.$$

So that $d(y_n, y_{n+k}) \to 0.$

Thus the sequence $\{y_n\}$ is a Cauchy sequence in X. Since X is a complete, it converges to some point p in X.

Remark: The converse of the above Lemma is not true. This can be seen from the following example.

Example: Let x=(0, 5] with the usual metric $d(x, y) = |x - y|$ for all x, y∈X. Define self mappings E, F, G, H, I and J of X by

$$E(x) = F(x) = \begin{cases} 1 & if\ 0 < x < 3 \\ 3 & if\ 3 \le x \le 5 \end{cases}, \quad J(x) = \begin{cases} x & if\ 0 < x < 3 \\ \frac{x+3}{2} & if\ 3 \le x \le 5 \end{cases}$$

$$I(x) = G(x) = x\ \ if\ 0 < x \le 5, \quad H(x) = \begin{cases} x & if\ 0 < x < 3 \\ \frac{2x+3}{3} & if\ 3 \le x \le 5 \end{cases}$$

Then

$$IJ(x) = \begin{cases} x & if\ 0 < x < 3 \\ \frac{x+3}{2} & if\ 3 \le x \le 5 \end{cases}, \quad GH(x) = \begin{cases} x & if\ 0 < x < 3 \\ \frac{2x+3}{3} & if\ 3 \le x \le 5 \end{cases}.$$

$E(x) = F(x) = \{1,3\}, \quad J(x) = (0,4], \quad IJ(x) = (0,4].$

and

H(x)=$\left(0, \frac{13}{3}\right]$, $GH(x) = \left(0, \frac{13}{3}\right]$

Clearly $E(X) \subseteq IJ(X)$, $F(X) \subseteq GH(X)$. Also the inequality (2.2) can easily be verified for appropriate values of $\alpha, \beta \ge 0, \alpha + \beta < 1$. Moreover if we take $x_n = 3 + \frac{1}{n}$ for $n \ge 1$ then the associated sequence $Ex_0, Fx_1, Ex_2, Fx_3, ...Ex_{2n}, Fx_{2n+1}.....$ converges to $3 \in X$. Note that (X, d) is not complete.

The following theorem was proved in [5].

Theorem: Let P, Q, S and T be self mappings from a complete metric space (X, d) into itself satisfying the following conditions

$$S(X) \subset Q(X)\ and\ T(X) \subset P(X) \tag{2.8.1}$$

$$d(Sx, Ty) \le \alpha \frac{d(Qy, Ty)[1 + d(Px, Sx)]}{[1 + d(Px, Qy)]} + \beta d(Px, Qy) \tag{2.8.2}$$

for all $x, y\ in\ X\ where\ \alpha, \beta \ge 0, \alpha + \beta < 1.$

one of P, Q, S and T is continuous (2.8.3)

and the pairs (S, P) and (T, Q) are compatible on X. (2.8.4)

Then P, Q, S and T have a unique common fixed point in X.

Now we extend and generalize the above Theorem to six self maps as follows.

III. MAIN RESULT

3.1 Theorem: If E, F, G, H, I and J are self maps of a metric space (X, d) satisfying the conditions

$$E(X) \subseteq IJ(X)\ and\ F(X) \subseteq GH(X) \tag{3.1.1}$$

$$d(Ex, Fy) \le \alpha \frac{d(IJy, Fy)[1 + d(GHx, Ex)]}{[1 + d(GHx, IJy)]} + \beta d(GHx, IJy) \tag{3.1.2}$$

for all $x, y\ in\ X\ where\ \alpha, \beta \ge 0, \alpha + \beta < 1.$

IJ=JI, GH=HG, HE=EH, FJ=JF, (GH)E=E(GH) and (IJ)F=F(IJ) (3.1.3)

G, H, I and J are continuous and (3.1.4)

the pairs (E, GH) and (F, IJ) are compatible mappings of Type (P) on X (3.1.5)

Further if there is a point $x_0 \in X$ and its associated sequence $\{y_n\} = \{Ex_0, Fx_1, Ex_2, Fx_3 ....\}$ relative to six self maps E, F, G, H, I and J converges to some point p∈X, then p is a unique common fixed point of E, F, G, H, I and J. (3.1.6)

Proof: From (3.1.6), we have $Ex_{2n} \to p,\ IJx_{2n+1} \to p,\ Fx_{2n+1} \to p,\ and\ GHx_{2n+2} \to p\ as\ n \to \infty.$ (3.1.7)

Since the pairs (E, GH) and (F, IJ) are compatible mappings of type (P),

we have $\lim_{n\to\infty}(GH)(GH)x_{2n} = \lim_{n\to\infty} EEx_{2n}\ \ and\ \ \lim_{n\to\infty}(IJ)(IJ)x_{2n} = \lim_{n\to\infty} FFx_{2n}$. (3.1.8)

Suppose GH is continuous. Then

$(GH)(GH)x_{2n},\ (GH)Ex_{2n} \to GHp\ as\ n \to \infty$ (3.1.9)

Now from (3.1.8) and (3.1.9), we get $EEx_{2n} \to GHp\ as\ n \to \infty$ (3.1.10)

Suppose IJ is continuous. Then

$(IJ)(IJ)x_{2n},\ (IJ)Fx_{2n} \to IJp\ as\ n \to \infty$ (3.1.11)

Now from (3.1.8) and (3.1.11), we get $FFx_{2n} \to IJp\ as\ n \to \infty$ (3.1.12)

Putting $x = Ex_{2n},\ \ y = Fx_{2n+1}$ in (3.1.2) and letting $n \to \infty$ and using (3.1.9), (3.1.10), (3.1.11), (3.1.12), we get

$$d(EEx_{2n}, FFx_{2n+1}) \le \alpha \frac{d((IJ)Fx_{2n+1}, FFx_{2n+1})[1 + d((GH)Ex_{2n}, EEx_{2n})]}{[1 + d((GH)Ex_{2n}, (IJ)Fx_{2n+1})]} + \beta d((GH)Ex_{2n}, (IJ)Fx_{2n+1})$$

$$d(GHp, IJp) \le \alpha \frac{d(IJp, IJp)[1 + d(GHp, GHp)]}{[1 + d(GHp, IJp)]} + \beta d(GHp, IJp)$$

$(1-\beta)d(GHp, IJp) \le 0$ which implies

$d(GHp, IJp) \le 0$ since 1-β>0 and so that

GHp=IJp (3.1.13)

Now putting $x = Ex_{2n},\ \ y = x_{2n+1}$ in (3.1.2) and letting $n \to \infty$ and using (3.1.7), (3.1.9), (3.1.10), we get

$$d(EEx_{2n}, Fx_{2n+1}) \le \alpha \frac{d((IJ)x_{2n+1}, Fx_{2n+1})[1 + d((GH)Ex_{2n}, EEx_{2n})]}{[1 + d((GH)Ex_{2n}, (IJ)x_{2n+1})]} + \beta d((GH)Ex_{2n}, (IJ)x_{2n+1})$$

$$d(GHp, p) \le \alpha \frac{d(p, p)[1 + d(GHp, GHp)]}{[1 + d(GHp, p)]} + \beta d(GHp, p)$$

$(1-\beta)d(GHp, p) \le 0$ which implies

$d(GHp, p) \le 0$ since 1-β>0, $\alpha+\beta < 1$ and so that

GHp = p

Therefore GHp = IJp = p (3.1.14)

Putting $x = p,\ \ y = x_{2n+1}$ in (3.1.2) and letting $n \to \infty$ and using (3.1.7), (3.1.14), we get

$$d(Ep, Fx_{2n+1}) \le \alpha \frac{d((IJ)x_{2n+1}, Fx_{2n+1})[1 + d((GH)p, Ep)]}{[1 + d((GH)p, (IJ)x_{2n+1})]} + \beta d((GH)p, (IJ)x_{2n+1})$$

$$d(Ep,p) \le \alpha \frac{d(p,p)[1+d(p,Ep)]}{[1+d(p,p)]} + \beta d(p,p)$$

$$d(Ep,p) = 0$$

Therefore Ep = p. (3.1.15)

Putting $x = x_{2n}$, $y = p$ in (3.1.2) and letting $n \to \infty$ and using (3.1.7), (3.1.14), we get

$$d(Ex_{2n},Fp) \le \alpha \frac{d((IJ)p,Fp)[1+d((GH)x_{2n},Ex_{2n})]}{[1+d((GH)x_{2n},(IJ)p)]} + \beta d((GH)x_{2n},(IJ)p)$$

$$d(p,Fp) \le \alpha \frac{d(p,Fp)[1+d(p,p)]}{[1+d(p,p)]} + \beta d(p,p)$$

$$\le \alpha d(p,Fp)$$

$(1-\alpha)d(p,Fp) \le 0$ which implies

$d(p,Fp) \le 0$ since 1-α>0 and so tha

Fp = p. (3.1.16)

Putting $x = Hp$, $y = p$ in (3.1.2) and using (3.1.3), (3.1.14), (3.1.15), (3.1.16), we get

$$d(EHp,Fp) \le \alpha \frac{d((IJ)p,Fp)[1+d((GH)Hp,EHp)]}{[1+d((GH)Hp,(IJ)p)]} + \beta d((GH)Hp,(IJ)p)$$

$$d(Hp,p) \le \alpha \frac{d(p,p)[1+d(Hp,Hp)]}{[1+d(Hp,p)]} + \beta d(Hp,p)$$

$$\le \beta d(Hp,p)$$

$(1-\beta)d(Hp,p) \le 0$ which implies

$d(Hp,p) \le 0$, since $1-\beta > 0$ and so that

Hp = p. (3.1.17)

Now from (3.1.14) we have Gp = p. (3.1.18)

Putting $x = p$, $y = Jp$ in (3.1.2) and using (3.1.3), (3.1.14), (3.1.15), (3.1.16), we get

$$d(Ep,FJp) \le \alpha \frac{d((IJ)Jp,FJp)[1+d((GH)p,Ep)]}{[1+d((GH)p,(IJ)Jp)]} + \beta d((GH)p,(IJ)Jp)$$

$$d(p,Jp) \le \alpha \frac{d(Jp,Pp)[1+d(p,p)]}{[1+d(p,Jp)]} + \beta d(p,Jp)$$

$$\le \beta d(p,Jp)$$

$(1-\beta)d(p,Jp) \le 0$ which implies

$d(p,Jp) \le 0$, since $1-\beta > 0$ and so that

Jp = p. (3.1.19)

Again from (3.1.14) we have Ip = p. (3.1.20)

Therefore Ep = Fp = Gp = Hp = Ip = Jp =p, showing that p is a common fixed point of E, F, G, H, I and J. The uniqueness of fixed point can be proved easily.

Remark: In the example (2.7), the self maps E, F, G, H, I and J satisfy all the conditions of the Theorem (3.1). It may be noted that '3' is the unique common fixed point of E, F, G, H, I and J.

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