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# Random Fixed Point Theorem in Polish Spaces with W-Distance

***Gourish Parashar[1], Anil Agrawal[2] and Manoj Kumar Shukla[3]***
*[1]Department of Mathematics,*
*TIT & Science Bhopal and Research Scholar M.G.C.G.V. Chitrakoot (Madhya Pradesh), India*
*[2]Department of Mathematics, M.G.C.G.V. Chitrakoot (Madhya Pradesh), India*
*[3]Department of Mathematics,*
*Institute for Excellence in Higher Education, Bhopal, (Madhya Pradesh), India*

*(Corresponding author: Gourish Parashar)*
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**ABSTRACT: In this present paper we have proved some fixed point results in generating Polish space (random space which is more general than the other spaces) with implicit relations, for non-commuting $JSR - mapping$ with $W - distance$ in Polish Space.**

**Keyword:** Fixed point, Random space, Polish spaces, $JSR - mapping$, Implicit relation.
**Mathematics Subject Classification:** 47H10, 54H25.

## I. INTRODUCTION

An importance area in probabilistic functional analysis is Random fixed point theorems for contraction mappings in polish spaces. Their study was initiated by the Prague school of probabilistic with work of Spacek [16] and Hans [5], [6].The survey article by Bharucha-Reid [4] in 1976 attracted attention of several mathematician and gave wings to this theory. Itoh [8] extended Spacek's and Hans's theorem to multi-valued contraction mappings and gave their applications to random differential equations in Banach Spaces.

Random fixed point theorems are stochastic generalization of classical fixed point theorem [5], [16] Itoh [8] extended several well known fixed point theorems, i.e. for contraction, non expansive and condemning, mappings to the random case. Thereafter various stochastic aspects of Schauder's fixed point theorem have been studied by Sehgal and Singh [15], Papageorgiou [13], Lin [11] and many authors. In a separable metric space, random fixed point theorems for contractive mappings were proved by Spacek [16], Hans [5],[6], Afterwards Beg and Shahzad [2],[3], studied the structure of common random fixed points and random coincidence points of a pair of compatible random operators and proved the random fixed point theorems for contraction random operators in Polish Spaces. Badshah and Gagrani proved existence of common Random fixed points of two Random multi-valued operators on Polish spaces also studied random version of fixed point theorems for increasing decreasing, and mixed monotone random mappings in ordered polish spaces. They also introduced order continuous random mapping and discuss its measure ability.

A bulk of literature exist with commuting pairs and its weaker forms such as weakly commuting, compatible, compatible of type (A), D-compatible, semi compatible, etc. we prove fixed point
theorem in polish space with $w - distance$ for $JSR - mappings$ introduced in [12] and [19] which is more improved than known mappings.

In 1996, Kada *et al.* [21] introduced the concept of $w - distance$ on a metrice space $(X, d)$. By using such a $w - distance$ concept, they improved some important theorems such as Caristi's fixed point theorem, Ekeland's variation principle and the non-convex minimization theorem. Recently Dhagat, V.B. and Thakur, [19] and Mehta, *et. al.* [12] did lot of work on fixed point theorems in generating space and Fixed point of multi-valued operators in Polish Space. Also they discussed the concept of generalization metric distance by using concepts of $w - distance$, he proved somr results on fixed point problems in their different articles.

## II. PRELIMINARIES

**Definition 2.1:** $w - distance$: Let $X$ be a Polish Space with metric $d$. Then the function $p: (\Omega \times X) \times: (\Omega \times X) \rightarrow [0, \infty)$ is called $w - distance$ on $X$ if

(2.1.1) $p\big(x(\omega), z(\omega)\big) \leq p\big(x(\omega), y(\omega)\big) + p\big(y(\omega), z(\omega)\big)$

For any $x, y, z \in X, \omega \in \Omega$ be a selector.

(2.1.2) For any $x \in p(x(\omega), .) \to [0, \infty)$ is a lower semi- continuous and

(2.1.3) For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$\text{p}(\text{z}(\omega), x(\omega)) \leq \delta$ and $\text{p}(\text{z}(\omega), y(\omega)) \leq \delta$ imply $\text{p}(\text{x}(\omega), y(\omega)) \leq \varepsilon$

For any $x, y, z \in X$,

$\omega \in \Omega$ be a selector.

**Definition 2.2:** Let $(X, d)$ be a Polish Space and let $S$ and $T$ be mappings from $\Omega \times X \to X$ and $\omega \in \Omega$ be a selector. The mappings $S$ and $T$ will be called $weakly$ commuting iff

$d(STx(\omega), TSx(\omega)) \leq d\big(Sx(\omega), Tx(\omega)\big)$ for each $x \in X$.

**Definition 2.3:** Let $(X, d)$ be a Polish Space with a $w - distance\ p$ and let $S$ and $T$ be mappings from $\Omega \times X \to X$ and $\omega \in \Omega$ be a selector. The mappings $S$ and $T$ will be called $p - weakly$ commuting iff

$\max\{p\big(STx(\omega), TSx(\omega)\big), p\big(TSx(\omega), STx(\omega)\big)\} \leq p\big(Sx(\omega), Tx(\omega)\big) \quad \forall x \in X$

**Definition 2.4:** Let $(X, d)$ be a Polish Space and let $S$ and $T$ be mappings from $\Omega \times X \to X$ and $\omega \in \Omega$ be a selector. The mappings $S$ and $T$ will be called compatible iff every sequence $\{x_n(\omega)\}$ such that

$\lim_{n\to\infty} Sx_n(\omega) = \lim_{n\to\infty} Tx_n(\omega) = t$ for some $t \in X$

Implies $\lim_{n\to\infty} d(STx_n(\omega), TSx_n(\omega)) = 0$

**Definition 2.5:** Let $(X, d)$ be a Polish Space with a $w - distance\ p$ and let $S$ and $T$ be mappings from $\Omega \times X \to X$ and $\omega \in \Omega$ be a selector. The mappings $S$ and $T$ will be called $(p)$ compatible iff every sequence $\{x_n(\omega)\}$ such that

$\lim_{n\to\infty} Sx_n(\omega) = \lim_{n\to\infty} Tx_n(\omega) = t$ for some $t \in X$

Implies $\lim_{n\to\infty} \max\{p\big(STx_n(\omega), TSx_n(\omega)\big), p\big(TSx_n(\omega), STx_n(\omega)\big)\} \leq 0$

**Definition 2.6:** $S - JSR\ mapping$ [12]: Let $S$ and $T$ are two self maps Metric Space $(X, d)$. the pair $\{S, T\}$ is said to be $S - JSR\ mappings$ iff

$\alpha d(STx_n(\omega), Tx_n) \leq \alpha d(SSx_n(\omega), Sx_n))$

Where $\alpha = \lim Sup$ or $\lim\ inf$ and $\{x_n\}$ is a sequence in $X$ such that

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Example: Let $X = [0,1]$ with $d(x, y) = |x - y|$ and $S, T$ are two self mapping on $X$ defined by

$Sx = \frac{x}{x+2,} Tx = \frac{1}{x+1}$ for $x \in X$. Now we have the sequence $\{x_n\}$ in $X$ is defined as

$x_n = \frac{1}{n}, n \in N$. Then we have $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 1$.

$d(STx_n, Tx_n) \to \frac{1}{3}$ and $d(SSx_n, Sx_n) \to \frac{2}{3}$ as $n \to \infty$.

Clearly we have

$d(TSx_n - Tx_n) < d(SSx_n, Sx_n)$

Thus pair $\{S, T\}$ is $S - JSR\ mapping$. But this pair is neither compatible nor weakly compatible or other non commuting mapping. Hence pair of $JSR - mapping$ is more general then others.

**Definition 2.7:** Let $(X, d)$ be a Polish Space and let $S$ and $T$ be mappings from $\Omega \times X \to X$ and $\omega \in \Omega$ be a selector. The pair $\{S, T\}$ is said to be $S - JSR\ mappings$ iff every sequence $\{x_n(\omega)\}$ such that

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$

$\Longrightarrow \alpha d(STx_n(\omega), Tx_n) \leq \alpha d(SSx_n(\omega), Sx_n))$

Where $\alpha = \lim\ sup$ or $\lim\ inf$.

**Definition 2.8:** Let $(X, d)$ be a Polish Space and let $S$ and $T$ be mappings from $\Omega \times X \to X$ and $\omega \in \Omega$ be a selector. The pair $\{S, T\}$ is said to be $S - JSR(p)\ mappings$ iff every sequence $\{x_n(\omega)\}$ such that

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$ implies

$\max\{\alpha p\big(STx_n(\omega), Tx_n(\omega)\big), \alpha p\big(Tx_n(\omega), STx_n(\omega)\big)\}$

$\leq \max\{\alpha p\big(SSx_n(\omega), Sx_n(\omega)\big), \alpha p\big(Sx_n(\omega), SSx_n(\omega)\big)\}$

Where $\alpha = \lim\ Sup$ or $\lim\ inf$.

## III. SOME CONCERNING RESULTS

**Lemma 3.1:** Let $X$ be a Polish Space with metric $d$ and $p$ be a $w - distance$ on $X$. Let $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ be sequence in $\Omega \times X$ and $\omega \in \Omega$ be a selector, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, \infty)$ converging to $0$. Then for $x, y, z \in X$ the following conditions hold:

(3.1.1) If $p(x_n(\omega), y(\omega)) \leq \alpha_n$ and $p(x_n(\omega), z(\omega)) \leq \beta_n$ for any $n \in N$, then

$x_n(\omega) = y(\omega)$.

In particular if $p(x(\omega), y(\omega)) = 0$ and $p(x(\omega), z(\omega)) = 0$ then $x_n(\omega) = y(\omega)$.

(3.1.2) If $p(x_n(\omega), y_n(\omega)) \leq \alpha_n$ and $p(x_n(\omega), z(\omega)) \leq \beta_n$ for any $n \in N$, then $y_n(\omega)$ converges to $z$.

(3.1.3) If $p(x_n(\omega), x_m(\omega))$ for any $n, m \in N$, then $\{x_n(\omega)\}$ is Cauchy sequence.

(3.1.4) If $p(y(\omega), x_n(\omega)) \leq \alpha_n$ for any $n, m \in N$, then $\{x_n(\omega)\}$ is Cauchy sequence.

**Lemma 3.2** Let $(X, d)$ be a metric space with a $w - distance\ p$ and let $S_j$ and $T_i$ be mappings from $\Omega \times X \to X$ satisfying $T_i x_n(\omega) = S_j x_{n+1}(\omega)$ for $n = 0,1,2, \dots\dots\dots\dots\dots\dots\dots$..

Assume that there exists a continuous self mapping $p$ of $[0, \infty]$ such that

(3.2.1) $p(T_i x(\omega), T_i y(\omega)) \leq \wp\left\{p\left(S_j x(\omega), S_j y(\omega)\right)\right\}$

(3.2.2) For all $x, y \in X$ and for each $t > 0$, $\wp(t) < t$

Then by (3.2.1), for an arbitrary $\varepsilon > 0$, there exists positive integer $m \leq n < s$

Implies $p(T_i x_n(\omega), T_i x_s(\omega)) < \varepsilon$. and (3.2.2), the sequence $\{T_i x_n(\omega)\}$ is a Cauchy sequence.

**Proof:** we have

$p(T_i x_n(\omega), T_i x_{n+1}(\omega)) \leq \wp\{p\left(S_j x_{n+1}(\omega)\right)\}$

$\leq \wp\{p(T_i x_{n-1}(\omega), T_i x_n(\omega))\}$

$< p(T_i x_{n-1}(\omega), T_i x_n(\omega))$

For $n = 1,2,3, \dots\dots\dots$.... Thus $\{p(T_i x_n(\omega), T_i x_{n+1}(\omega))\}$ is a decreasing sequence of non negative real number and there exists non negative real number $\lambda$ such that

$\lim_{n\to\infty} p(T_i x_n(\omega), T_i x_{n+1}(\omega)) = \lambda$

Let $\lambda > 0$, then the inequality

$p(T_i x_n(\omega), T_i x_{n+1}(\omega)) \leq \wp\{p(T_i x_{n-1}(\omega), T_i x_n(\omega))\}$

Now by the continuity of $\wp$ we have $\wp(\lambda) < \lambda$, which is contraction. Therefore $\lambda = 0$

So $p(T_i x_n(\omega), T_i x_{n+1}(\omega)) \to 0$ as $n \to \infty$.

To prove (3.2.1), suppose that (3.2.1) does not hold. Then there exists an $\varepsilon > 0$ such that for all sufficiently large positive integer $k$, there exists positive integers $s_k, n_k$ with $k \leq n_k < s_k$ such that

$p\left(T_i x_{n_k}(\omega), T_i x_{s_k}(\omega)\right) \geq \varepsilon$ and $p\left(T_i x_{n_k}(\omega), T_i x_{n_k-1}(\omega)\right) < \varepsilon \dots\dots\dots$ (3.2.3)

From the above result and (3.2.3), we have

$p\left(T_i x_{n_k}(\omega), T_i x_{s_k}(\omega)\right) \to \varepsilon$ and $p\left(T_i x_{n_k}(\omega), T_i x_{n_k-1}(\omega)\right) \to \varepsilon$ as $k \to \infty$

And $p\left(T_i x_{n_k}(\omega), T_i x_{s_k}(\omega)\right)$

$\leq p\left(T_i x_{n_k}(\omega), T_i x_{n_k+1}(\omega)\right) + p\left(T_i x_{n_k+1}(\omega), T_i x_{s_k}(\omega)\right)$

$\leq p\left(T_i x_{n_k}(\omega), T_i x_{n_k+1}(\omega)\right) + \wp\{p\left(S x_{n_k+1}(\omega), S x_{s_k}(\omega)\right)$

$\leq P\left(T_i x_{n_k}(\omega), T_i x_{n_k+1}(\omega)\right) + \wp\{p\left(T_i x_{n_k+1}(\omega), T_i x_{s_k}(\omega)\right)\}$

$\dots\dots\dots\dots\dots$ (3.2.4)

By the hypothesis and (3.2.4) we obtain $\varepsilon \leq \wp(\varepsilon) < \varepsilon$. this is a contradiction therefore (3.2.1) holds.

(3.2.2): By the condition (3.2.3) of the definition of a $w - distance\ (p)$ and

(3.2.1): We have that $\{(T_i x_n(\omega)\}$ is a Cauchy sequence.

**Lemma 3.3:** Let $(X, d)$ be a metric space with a $w - distance\ p$ and let $S_j$ and $T_i$ be self mappings from $\Omega \times X \to X$ satisfying $T_i x_n(\omega) = S_j x_{n+1}(\omega)$ for $n = 0,1,2, \dots\dots\dots\dots\dots\dots\dots\dots$ and the following conditions:

For given $\varepsilon > 0, \exists\ \delta(\varepsilon) > 0$ such that

$\varepsilon \leq p\left(S_j x(\omega), S_j y(\omega)\right) < \varepsilon + \delta \Longrightarrow p(T_i x(\omega), T_i y(\omega)) < \varepsilon \dots\dots\dots$ (3.3.1)

And $p\left(S_j x(\omega), S_j y(\omega)\right) < \varepsilon$

$\Longrightarrow p(T_i x(\omega), T_i y(\omega)) < \frac{1}{2} p\left(S_j x(\omega), S_j y(\omega)\right) \dots\dots\dots\dots\dots\dots\dots\dots\dots$ (3.3.2)

Then

(3.3.1) for any arbitrary $\varepsilon > 0$, there exists positive integer $m$, such that $m \leq n < s$

Implies $p(T_i x_n(\omega), T_i x_s(\omega)) < \varepsilon$.

(3.3.2) The sequence $\{T_i x_n(\omega)\}$ is a Cauchy sequence.

**Proof:** By (3.3.1) & (3.3.2) we have

$p(T_i x(\omega), T_i y(\omega)) < p\left(S_j x(\omega), S_j y(\omega)\right)$ for all $x, y \in X$ and $\omega \in \Omega$.

Thus $\{p(T_i x_n(\omega), T_i x_{n+1}(\omega))\}$ is monotone decreasing sequence of non-negative real numbers.

Then there is non-negative real number $r$ such that

$\lim_{n\to\infty} p(T_i x_n(\omega), T_i x_{n+1}(\omega)) = r$

We show that $r = 0$.

Suppose that $r > 0$. then, for given $\delta > 0$, there exists a positive interger $m$ such that

For each $m' \geq m$, we have

$r = p(T_i m'(\omega), T_i x_{m'+1}(\omega)) = p\left(S_j x_{m'+1}(\omega), S_j x_{m'+2}(\omega)\right) < r + \delta$ … … … … … (3.3.3)

From (3.3.1) & (3.3.3), we obtain $p(T_i m'(\omega), T_i x_{m'+1}(\omega)) < r$,

Which contradicts (3.3.3). Therefore we have

$\lim_{n\to\infty} p(T_i x_n(\omega), T_i x_{n+1}(\omega)) = p\left(S_j x_{n+1}(\omega), S_j x_{n+2}(\omega)\right) = 0$

Similarly we obtain s

$\lim_{n\to\infty} p(T_i x_{n+1}(\omega), T_i x_{n+2}(\omega)) = p\left(S_j x_{n+2}(\omega), S_j x_{n+3}(\omega)\right) = 0$

To prove (3.3.1), suppose (3.3.1), does not hold. Then, there exists $\varepsilon > 0$ such that for all sufficiently large positive integer $k$, there exists positive integers $s_k, n_k$ with $k \leq n_k < s_k$ satisfying (3.2.3) From above results and (3.2.3), we have

$p\left(T_i x_{n_k}(\omega), T_i x_{s_k}(\omega)\right) \to \varepsilon$ and $p\left(T_i x_{n_k}(\omega), T_i x_{s_k-1}(\omega)\right) \to \varepsilon$ and

$p\left(T_i x_{n_k-1}(\omega), T_i x_{s_k}(\omega)\right) \to \varepsilon$ as $k \to \infty$ … … … … … … … … … … … … … … … … (3.3.4)

From (3.3.4), for $\varepsilon > 0$, there exists a positive integer $m$ such that for each $k \geq m$,

We have

$\frac{\varepsilon}{2} < p\left(T_i x_{n_k}(\omega), T_i x_{s_k-1}(\omega)\right) = p\left(S_j x_{n_k+1}(\omega), S_j x_{s_k}(\omega)\right) < \frac{\varepsilon}{2} + \varepsilon$ … … … … … … (3.3.5)

From (3.3.1) and (3.3.5) we obtain $p\left(T_i x_{n_k+1}(\omega), T_i x_{n_k}(\omega)\right) < \varepsilon$,

Which contradicts (3.3.4). Therefore (3.3.1) holds. The proof of (3.3.2) is same as (3.3.1) of Lemma 3.3.

## IV. MAIN RESULTS

**THEOREM 4.1:** Let $(X, d)$ be a Polish Space with $w - distance\ p$ and let $T_i$ and $S_j$ be $S - JSR\ (p)\ mappings$ from $\Omega \times X \to X$ satisfying

(4.1.1) $T(X) \subset S(X)$

(4.1.2) $p(T_i x(\omega), T_i y(\omega)) \leq \wp\{p\left(S_j x(\omega), S_j y(\omega)\right)\}$ for all $x, y \in X$

(4.1.3) for each $t > 0$, $\wp(t) < t$

(4.1.4) for each $z \in X$ with $z(\omega) \neq T_i z(\omega)$ or $z(\omega) \neq S_j z(\omega)$

$$inf \left\{ \begin{matrix} p(T_i x(\omega), z(\omega)) + p\left(S_j x(\omega), z(\omega)\right) + \\ p\left(S_j T_i x(\omega), T_i x(\omega)\right) + p\left(S_j S_j x(\omega), S_j x(\omega)\right) \end{matrix} \right\} > 0 \ \dots\dots\dots (4.1.1.1)$$

For any $x, y, z \in X$, $\omega \in \Omega$ be a selector.

Then $T_i$ and $S_j$ have a unique common fixed point.

**PROOF:** By the assumption, we have all the conditions of 3.2.2

Thus by (3.3.2) $\{T_i x(\omega)\}$ and $\{S_j x(\omega)\}$ have a limit point $z$ in $X$.

Suppose that $z(\omega) \neq T_i z(\omega)$ or $z(\omega) \neq S_j z(\omega)$.

Now, since $\lim_{n\to\infty} S_j x_n(\omega) = \lim_{n\to\infty} T_i x_n(\omega) = z(\omega)$

Therefore by (3.3.1) and the lower semi continuity, we have

$\lim_{n\to\infty} p(T_i x_n(\omega), z(\omega)) = p\left(S_j x_n(\omega), z(\omega)\right) = 0.$

Now,

$$0 < \inf \left\{ \begin{array}{c} p(T_i x_n(\omega), z(\omega)) + p\left(S_j x_n(\omega), z(\omega)\right) + \\ p\left(S_j T_i x(\omega), T x(\omega)\right) + p\left(S_j S_j x(\omega), S_j x_n(\omega)\right), x \in X \end{array} \right\}$$

$$\leq inf \left\{ \begin{array}{c} p(T_i x_n(\omega), z(\omega)) + p\left(S_j x_n(\omega), z(\omega)\right) + \\ p\left(S_j T_i x(\omega), T_i x_n(\omega)\right) + p\left(S_j S_j x(\omega), S_j x_n(\omega)\right) \end{array} \right\}$$

$$\leq inf \left\{ \begin{array}{c} p(T_i x_n(\omega), z(\omega)) + p\left(S_j x_n(\omega), z(\omega)\right) + \\ max \left\{ \begin{array}{c} \alpha p\left(S_j x_n(\omega), S_j S_j x_n(\omega)\right), \\ \alpha p(S_j S_j x_n(\omega), S_j x_n(\omega)) \end{array} \right\} + p(S_j S_j x_n(\omega), S_j x_n(\omega)) \end{array} \right\}$$

$= 0$

Which is contradiction. Thus $z(\omega)$ is a common fixed point of $T_i$ and $S_j S_j$. the uniqueness can be proved by the use of (3.3.1), (3.3.2) and (3.3.1) of the lemma 3.3.

**THEOREM 4.2:** Let $(X, d)$ be a Polish Space with $w - distance\ \ p$ and let $T_i$ and $S_j$ be $S_j - JSR\ (p)\ mappings$ from $\Omega \times X \to X$ satisfying

(4.2.1) $T_i(X) \subset S_j(X)$

(4.2.2) $p(T_i x(\omega), T_i y(\omega)) \leq \wp\{p\left(S_j x(\omega), S_j y(\omega)\right)\}$ for all $x, y \in X$

(4.2.3) for each $t > 0$, $\wp(t) < t$

(4.2.4) for each $z \in X$ with $z(\omega) \neq T_i z(\omega)$ or $z(\omega) \neq S_j z(\omega)$

$$inf \left\{ \begin{array}{c} p(T_i x(\omega), z(\omega)) + p\left(S_j x(\omega), z(\omega)\right) + \\ p\left(S_j T_i x(\omega), T_i z(\omega)\right) + p\left(S_j T_i x(\omega), S_j z(\omega)\right) + \\ p\left(S_j S_j x(\omega), S_j x(\omega)\right) + p\left(S_j S_j x(\omega), T_i z(\omega)\right) \\ + p\left(S_j S_j x(\omega), S_j z(\omega)\right), x \in X \end{array} \right\} > 0 \ldots\ldots\ldots (4.2.1.1)$$

For any $x, y, z \in X$, $\omega \in \Omega$ be a selector.

Then $T_i$ and $S_j$ have a unique common fixed point.

**PROOF:** Since $T_i(X) \subset S_j(X)$, we obtain a sequences $\{x_n(\omega)\}$ in $X$ such that

$T_i x_n(\omega) = S_j x_{n+1}(\omega)$

$\{T_i x_n(\omega)\}$ is a Cauchy sequence.

Since $X$ is complete and $T_i x_n(\omega) = S_j x_{n+1}(\omega)$, there exists $z(\omega) \in \Omega \times X$ such that $T_i x_n(\omega) \to z(\omega)$ and $S_j x_{n+1}(\omega) \to z(\omega)$ .

Suppose that $z(\omega) \neq T_i z(\omega)$ or $z(\omega) \neq S_j z(\omega)$

Now, since by (3.2.1) and the lower semi continuity, we have

$\lim_{n\to\infty} p(T_i x_n(\omega), z(\omega)) = \lim_{n\to\infty} p\left(S_j x_n(\omega), z(\omega)\right) = 0.$

Now,

$$0 < inf \left\{ \begin{array}{c} p(T_i x(\omega), z(\omega)) + p\left(S_j x(\omega), z(\omega)\right) + \\ p\left(S_j T_i x(\omega), T_i z(\omega)\right) + p\left(S_j T_i x(\omega), S_j z(\omega)\right) + \\ \left(S_j S_j x(\omega), S_j x(\omega)\right) + p\left(S_j S_j x(\omega), T_i z(\omega)\right) \\ + + p\left(S_j S_j x(\omega), S_j z(\omega)\right), x \in X \end{array} \right\}$$

$$\leq inf\begin{Bmatrix} p(T_i x_n(\omega), z(\omega)) + p\left(S_j x_n(\omega), z(\omega)\right) + \\ p\left(S_j T_i x_n(\omega), T_i z(\omega)\right) + p\left(S_j T_i x_n(\omega), S_j z(\omega)\right) + \\ p\left(S_j S_j x_n(\omega), S_j x_n(\omega)\right) + p\left(S_j S_j x_n(\omega), T_i z(\omega)\right) \\ +p\left(S_j S_j x_n(\omega), S_j z(\omega)\right), x \in X \end{Bmatrix}$$

$$\leq inf\begin{Bmatrix} p(z(\omega), z(\omega)) + p(z(\omega), z(\omega)) + \\ p\left(S_j z(\omega), T_i z(\omega)\right) + p\left(S_j z(\omega), S_j z(\omega)\right) + \\ p\left(S_j z(\omega), z(\omega)\right) + p\left(S_j z(\omega), T_i z(\omega)\right) \\ +p\left(S_j z(\omega), S_j z(\omega)\right), x \in X \end{Bmatrix}$$

$$\leq 0p\left(S_j z(\omega), T_i z(\omega)\right)$$

$$\Longrightarrow S_j z(\omega) = T_i z(\omega)$$

Thus $z(\omega)$ is a common fixed point of $T_i$ and $S_j$. The uniqueness can be proved by the use of (3.2.1), (3.2.2) of the lemma 3.2.

**Corollary 4.3:** Let $(X, d)$ be a Polish Space with $w - distance\ p$ and let $T_i$ and $S_j$ be $p - weakly\ commuting\ mappings$ from $\Omega \times X \to X$ satisfying

(4.3.1) $T_i(X) \subset S_j(X)$

(4.3.2) $p(T_i x(\omega), T_i y(\omega)) \leq \wp\{p\left(S_j x(\omega), S_j y(\omega)\right)\}$ for all $x, y \in X$

(4.3.3) for each $t > 0, \wp(t) < t$

(4.3.4) for each $z \in X$ with $z(\omega) \neq T_i z(\omega)$ or $z(\omega) \neq S_j z(\omega)$

$$inf\begin{Bmatrix} p(T_i x(\omega), z(\omega)) + p\left(S_j x(\omega), z(\omega)\right) + \\ p\left(S_j T_i x(\omega), T_i z(\omega)\right) + p\left(S_j T_i x(\omega), S_j z(\omega)\right) + \\ \left(S_j S_j x(\omega), S_j x(\omega)\right) + p\left(S_j S_j x(\omega), T_i z(\omega)\right) \\ + + p\left(S_j S_j x(\omega), S z(\omega)\right), x \in X \end{Bmatrix} > 0$$

For any $x, y, z \in X$, $\omega \in \Omega$ be a selector.

Then $T_i$ and $S_j$ have a unique common fixed point.

**Proof:** The proof is similar to theorem 4.2.

Since a metric $d$ is a $w - distance$, from the theorem 4.1 and simple calculation, we obtain the following corollary.

**Corollary 4.4:** A continuous self map of a Polish Space $(X, d)$ has a fixed point iff there exists $\gamma \in (0,1)$ and a mapping $S_j\colon \Omega \times X \to X$ which commutes with $T_i$ and satisfies

(4.4.1) $T_i(X) \subset S_j(X)$

(4.4.2) $d(S_j x(\omega), S_j x(\omega)) \leq \gamma d(T_i x(\omega), T_i x(\omega))$ for $x, y \in X$, $\omega \in \Omega$.

Then, $S$ and $T$ have unique fixed point.

**Corollary 4.5:** Let $(X, d)$ be a Polish Space with $w - distance\ p$ and let $T_i$ and $S_j$ be $p - weakly\ commuting\ mapping$ from $\Omega \times X \to X$ satisfying $T_i(X) \subset S_j(X)$, (3.3.1), (3.3.2). Then $T_i$ and $S_j$ have a unique common fixed point.

**Proof:** The proof is similar to theorem 4.1.

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