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Fixed Point Theorems in M-Fuzzy metric space

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ABSTRACT
In this paper, we mainly prove a coincidence theorem and common fixed point theorem in M-fuzzy metric spaces which improve the results of Som [11] who proved his results on Metric and Banach spaces. Also we improve and extend some known results of Veerapandi et al. [12] by extending three mappings to four weak compatible mappings. Mathematics Subject Classification: 47H10, 54H25.

Keywords: Common fixed point theorem, metric space, M-fuzzy metric space, weakly compatible mappings

INTRODUCTION

After introduction of fuzzy sets by Zadeh [5], Kramosil and Michalek [4] introduced the concept of fuzzy metric space in 1975. Consequently in due course of time many researchers have defined a fuzzy metric space in different ways. Researchers like George and Veeramani [1], Grabiec [6], Subrahmanyam [7], Vasuki [9] used this concept to generalize some metric fixed point results. Recently, Sedghi and Shobe [10] introduced M-fuzzy metric space which is based on D*-metric concept.

In this paper, we prove a common fixed point theorem for four weakly compatible mappings in M-fuzzy metric space. First we give some known definitions and results in M-fuzzy metric space given by Sedghi and Shobe [10] and then prove our main result.

Definition 1.1 ([2]) A binary operation $*$: $[0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if it satisfies the following conditions.

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0, 1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a*b = ab$ and $a*b = \min\{a, b\}$.

Definition 1.2 ([10]) A 3-tuple $(X, M, *)$ is called a M-fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^3 \times (0, \infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$,
(1) $M(x, y, z, t) > 0$,
(2) $M(x, y, z, t) = 1$ if and only if $x = y = z$,
(3) $M(x, y, z, t) = M(p\{x, y, z\}, t)$, (symmetry) where p is a permutation function,
(4) $M(x, y, a, t) * M(a, z, z, s) \leq M(x, y, z, t + s)$,
(5) $M(x, y, z, .): (0, \infty) \to [0, 1]$ is continuous.

Remark 1.3 ([10]) Let $(X, M, *)$ be a M-fuzzy metric space. Then for every $t > 0$ and for every $x, y \in X$ we have $M(x, x, y, t) = M(x, y, y, t)$.

Definition 1.4 ([10]) Let (X, M, $*$) be a M-fuzzy metric space. For $t > 0$, the open ball $BM(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$BM(x, r, t) = \{y \in X: M(x, y, y, t) > 1 - r\}$.

A subset A of X is called open set if for each x. A there exist $t > 0$ and $0 < r < 1$ such that $BM(x, r, t) \subseteq A$.

Definition 1.5 ([10]) A sequence {xn} in X converges to x if and only if $M(x, x, xn, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in$ **N** such that $M (xn, xn, xm, t) > 1 - \varepsilon$ for each $n, m \geq n0$. The M-fuzzy metric space (X, M, $*$) is said to be complete if every Cauchy sequence is convergent.

Lemma 1.6 ([10]). Let (X, M, $*$) be a M-fuzzy metric space. Then M(x, y, z, t) is nondecreasing with respect to t, for all x, y, z in X.

Lemma 1.7 ([10]). Let (X, M, $*$) be a M-fuzzy metric space. Then M is continuous function on $X^3 \times (0, \infty)$.

In 1998, Jungck and Rhoades [3] introduced the concept of weakly compatibility of pair of self mappings in a metric space.

Definition 1.9 ([10]) Let f and g be two self maps of (X, M, $*$).Then f and g are said to be weakly compatible if there exists u in X with fu = g u implies f gu = gfu.

RESULTS

Theorem 2.1 Let S and T be two continuous self mappings of a complete M-fuzzy metric space (X, M, $*$). Let A and B be two self mappings of X satisfying

(i) $A(X) \cup B(X) \subseteq S(X) \cap T(X)$,

(ii) {A, T} and {B, S} are weakly compatible pairs, and

(iii) $aM(Tx,Sy,Bz, t) + bM(Tx, Ax,Sy, t) + cM(Sy,Bz,Ax, t)$
$+ \max\{M(Ax,Sy,Bz,t),M(Bz,Tx,Sy, t)\} \leq qM(Ax,Sy,Bz, t)$

for all x, y ,z $\in$ X and $t > 0$, where $a, b, c \geq 0$ with $0 < q < a+ b + c < 1$. Then A, B, S and T have a coincidence point.

Proof. Let $x_0 \in X$ be any arbitrary point. Since $A(X) \subseteq S(X)$, there must exists a point $x_1 \in X$ such that $Ax_0 = Sx_1$. Also since $B(X) \subseteq T(X)$, then there exists another point $x_2 \in X$ such that $Bx_1 = Tx_2$ and so on. In general, we get a sequence $\{y_n\}$ recursively as

$y_{2n} = Sx_{2n+1} = Ax_{2n}$ and $y_{2n+1} = Tx_{2n+2} = Ax_{2n+1}$, n = 0,1,2,3,….

Let $Mn = M (y_n, y_{n+1}, y_{n+2}, t) < 1$ for all n. Putting $x = x_{2n}, y = x_{2n+1}$ and $z = x_{2n+2}$ in (iii) we get

$aM (Tx_{2n}, Sx_{2n+1},Bx_{2n+2} , t) + bM(Tx_{2n} , Ax_{2n}, Sx_{2n+1}, t) + c M(Sx_{2n+1} , Bx_{2n+2}, Ax_{2n}, t) +$
$\max\{M(Ax_{2n} , Sx_{2n+1}, Bx_{2n+2}, t), M(Bx_{2n+2} , Tx_{2n}, Sx_{2n+1}, t) \}$
$\leq q M(Ax_{2n} , Sx_{2n+1}, Bx_{2n+2}, t)$,

i.e., $a M (y2n-1, y2n, y2n +2 , t) + b M(y2n-1, y2n, y2n , t) + c M(y2n, y2n +2, y2n , t) +\max\{ M(y2n, y2n, y2n +2 , t), M(y2n+2, y2n-1, y2n , t) \} \leq q M(y2n, y2n, y2n +2 , t)$,

i.e., $(a + b) M2n-1 + (c+ 1) M2n \leq qM2n$,

i.e., $(q - c-1) M2n \geq (a + b) M2n-1$, $(a + b)M2n-1 \leq (q - c-1) M2n < (q-c) M2n$,

$(q-c) M2n > (a + b) M2n-1$, $M2n > \{(a + b) / (q-c)\} M2n-1$

Let $(a + b) / (q-c) = r$ then $r >1$ which implies

$M2n > r M2n -1 > M2n -1$. ………..(1)

Thus $\{M2n, n \geq 0\}$ is an increasing sequence of positive real numbers in [0, 1] and therefore tends to a limit $m \leq 1$.
We claim m = 1, for m < 1, taking limit in (1) we get m < m, which is a contradiction. Therefore m = 1.
For any positive integer r,

$M(yn, yn ,yn+r, t) \geq M(yn, yn ,yn+1, t /r) \ast \cdots \ast M(yn+1, yn+1 ,yn+2, t /r) \ast \ldots$

$\ast M (yn+r-1, yn+r -1, yn+r, t /r) > (1-\varepsilon)$

$\ast (1-\varepsilon) \ast (1-\varepsilon) \ast \ldots$.r times = $(1-\varepsilon)$.

Thus $M(yn, yn ,yn+r, t) > (1-\varepsilon)$ which implies $M(yn, yn ,yn+s , t) > (1-\varepsilon)$ for all $n,s \geq n_0$ where $n_0 \in N$.

Thus {yn} is a Cauchy sequence in X. Since X is complete, there is a point $p \in X$ such that $yn \rightarrow p$, implying the sequences {Ax2n} and {Bx2n+1} converge to p, as such the subsequences {Sx2n+1} and {Tx2n+2} also converge to p .
Hence the sequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n+1}\}$ and $\{Tx_{2n+2}\}$ are Cauchy and converge to same limit, say p.
Now we prove that p is coincidence point of A, B, S and T under the given condition of weak compatibility.

Since $A(X) \subseteq S(X)$ and $A(X) \subseteq T(X)$ so there must exist points $u, v \in X$ such that p = Su and p = Tv .
Put $x = x_{2n,}$ y = u and z = u in (iii)
$aM(Tx_{2n} ,Su,Bu, t) + bM(Tx_{2n} , Ax_{2n} ,Su, t) + cM(Su,Bu,Ax_{2n} , t)$
$+ \max\{M(Ax_{2n} ,Su,Bu,t),M(Bx_{2n} ,Tx_{2n} ,Su, t)\} \leq qM(Ax_{2n} ,Su,Bu, t)$,
$aM(p ,p,Bu, t) + bM(p , p ,p, t) + cM(p,Bu,p, t) + \max\{M(p, p ,Bu ,t),M(p, p ,p, t)\} \leq qM (p ,p,Bu, t)$, implies $aM(p ,p,Bu, t) + b+ cM(p,Bu,p, t) + 1 \leq qM(p ,p,Bu, t)$,implies
$M (p, p, Bu, t) \geq (b+1)/(q-a-c) > 1$, therefore Bu = p = Su .
Put x = v, $y = x_{2n+1,}$ z = u in (iii)
$aM(Tv ,Sx_{2n+1},Bu, t) + bM(Tv , Av ,Sx_{2n+1}, t) + cM(Sx_{2n+1},Bu,Av , t)$
$+ \max\{M(Av ,Sx_{2n+1},Bu,t),M(Bu,Tv,Sx_{2n+1}, t)\} \leq qM(Av ,Sx_{2n+1},Bu, t)$, implies
$aM(p ,p,p, t) + bM(p , Av ,p, t) + cM(p,p,Av , t) + \max\{M(Av ,p,p,t),M(p,p,p, t)\} \leq qM(Av ,p,p, t)$, implies $a +1+(b+c) M(p , Av ,p, t) \leq q M(p , Av ,p, t)$,implies
$M (p, Av ,p, t) \geq \{(a + 1)/ (q-b-c)\} > 1$,implies Av = p .
Therefore Av = Tv = p.
Since {A, T} and {B, S} are weakly compatible therefore ATv =TAv which gives Ap = Tp and BSu = SBu which gives Bp = Sp, which proves that p is the coincidence point of A,B,S and T.

Theorem 2.2 Let S and T be two continuous self mappings of a complete M-fuzzy metric space $(X, M, \ast)$. Let A and B be two self mappings of X satisfying

(i) $A(X) \cup B(X) \subseteq S(X) \cap T(X)$,
(ii) {A, T} and {B, S} are weakly compatible pairs, and
(iii) $aM(Tx,Sy,Bz, t) + bM(Tx, Ax,Sy, t) + cM(Sy,Bz,Ax, t) + \max\{M(Ax,Sy,Bz,t),M(Bz,Tx,Sy, t)\} \leq qM(Ax,Sy,Bz, t)$

for all $x, y,z \in X$ and $t > 0$, where $a, b, c \geq 0$ with $0 < q < a+ b + c < 1$. Then A, B, S and T have a unique common fixed point.

Proof. As proved in Theorem 2.1, p is a coincidence point of A, B, S and T. Now, we will show that p is common unique fixed point of A, B, S and T. For this, first we prove that p is a fixed point of B.

Suppose $Bp \neq p$ then by putting $x = v$, $y = u$, $z = p$ in (iii), we get
$aM(Tv,Su,Bp,t)+bM(Tv,Av,Su,t)+cM(Su,Bp,Av,t)+\max\{M(Av,Su,Bp,t),M(Bp,Tv,Su, t\} \leq qM(Av,Su,Bp, t)$,implies
$aM(p,p,Bp,t)+bM(p,p,p,t)+cM(p,Bp,p,t)+\max\{M(p,p,Bp,t),M(Bp,p,p,t\} \leq qM(p,p,Bp, t)$,implies
$b +(a +c+1) M(p,p,Bp,t) \leq qM(p,p,Bp, t)$, implies $q M (p, p,Bp, t) \geq \{b / (q-a-c-1)\} > 1$.
Therefore $Bp = p$ and consequently $Sp = p$. Thus p is a common fixed point of B and S.
Similarly p is a common fixed point of A and T. Hence p is a common fixed point of A, B, S and T.
Now for the uniqueness of p, suppose p, let p= w , is another common fixed point
of A,B, S and T; i.e. $Ap = Bp = p = Sp = Tp$ and $Aw = Bw = w= Sw = Tw$. Then put $x = p$, $y = p$, $z = w$ in (iii) we have
$aM(Tp,Sp,Bw, t) + bM(Tp, Ap,Sp, t) + cM(Sp,Bw,Ap, t) + \max\{M(Ap,Sp,Bw,t),M(Bw,Tp,Sp, t)\} \leq qM(Ap,Sp,Bw, t)$,which implies
$aM(p,p,w, t) + bM(p, p,p,t) + cM(p,w,p, t)+\max\{M(p,p,w,t),M(w,p,p,t)\}\leq qM(p,p,w,t)$ which implies $(a + c + 1) M(p,w,p, t) + b \leq qM(p,p,w,t)$ implies $b \leq (q -a-c-1) M (p,p,w,t)$, implies
$M (p, p, w, t) \geq (b/ q-a-c-1) > 1$.
Therefore $p = w$.
Hence p is unique common fixed point of A, B, S and T.

Corollary 2.1 Let S and T be two continuous self mappings of a complete M-fuzzy metric space $(X, M, \ast)$. Let A be a self mapping of X satisfying

(a) $A(X) \subseteq S(X) \cap T(X)$,
(b) {A, T} and {A, S} are weakly compatible pairs, and
(c) $aM(Tx,Sy,Az, t) + bM(Tx,Ax,Sy, t) + cM(Sy,Az,Ax, t) + \max\{M(Ax,Sy,Az,t),M(Az,Tx,Sy, t)\} \leq qM(Ax,Sy,Az, t)$

for all $x, y ,z \in X$ and $t > 0$, where $a, b, c \geq 0$ with $0 < q < a+ b + c < 1$. Then A, S and T have a unique common fixed point.

Proof. Taking B =A in theorem 2.2, we get the required result.

Corollary 2.2 Let T be a continuous self mapping of a complete M-fuzzy metric space $(X, M, \ast)$. Let A and B be two self mappings of X satisfying

(d) $A(X) \cup B(X) \subseteq T(X)$,
(e) {A, T} and {B, T} are weakly compatible pairs and
(f) $aM(Tx,Ty,Bz, t) + bM(Tx, Ax,Ty, t) + cM(Ty,Bz,Ax, t) + \max\{M(Ax,Ty,Bz,t),M(Bz,Tx,Ty, t)\} \leq qM(Ax,Ty,Bz, t)$

for all $x, y, z \in X$ and $t > 0$, where $a, b, c \geq 0$ with $0 < q < a+ b + c < 1$. Then A, B and T have a unique common fixed point.
Proof .Taking S = T in theorem 2.2, we get the required result.

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