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Some inequalities for the ratio and difference of moments

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ABSTRACT

The generalized bounds for the ratio and difference of rth order moment μ_r' and $(r/s)^{th}$ power of sth order moment μ_s' are obtained here, when the random variable, discrete or continuous, takes values in the given finite positive real interval. Further the corresponding bounds for standard power means are derived from these generalized bounds.

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INTRODUCTION

The rth order moment μ_r' of a continuous random variable, which takes values on the interval [a, b] with pdf $\phi(x)$, is defined as

$$\mu_r' = \int_a^b x^r \phi(x)\,dx. \qquad (1.1)$$

For a random variable which takes discrete set of finite values x_i, with corresponding probabilities p_i, (i =1, 2, …, n), we define

$$\mu_r' = \sum_{i=1}^{n} p_i x_i^r. \qquad (1.2)$$

Further, power mean of order r is defined as

$$M_r = \left(\mu_r'\right)^{\frac{1}{r}} \quad r \neq 0, \qquad (1.3)$$

and

$$M_r = \lim_{r\to 0}\left(\mu_r'\right)^{\frac{1}{r}}, \qquad r = 0. \qquad (1.4)$$

It may be noted here that M_{-1}, M_o, M_1 and M_2 correspond to harmonic mean, geometric mean, arithmetic mean and root mean square mean. These means are generally known as standard power means.

The bounds for the rth order moment μ_r' when sth order moment μ_s' is prescribed and random variable, discrete or continuous, takes values in the interval [a, b] are given in [1]. The generalization of these results for the case when r and s can take any real values are reported in [2]. The bounds obtained in [1, 2] are applicable only when sth order moment μ_s' is prescribed. In the present paper we try to obtain inequalities, involving μ_r' and μ_s', which will be applicable even when sth order moment μ_s' is not prescribed. Further, the ratio and difference of two quantities also provide some information about the relationship between the concerned quantities. The ratio and difference are used in defining various parameters which characterize the distribution. For example variance of the distribution is the difference between square of root mean square and square of mean and Karl pearson measure of dispersion is defined as the ratio of the standard deviation to mean. The results obtained herein also give the minimum and maximum values of the difference and ratio of μ_r' and $\left(\mu_s'\right)^{\frac{r}{s}}$ when random variable, discrete or continuous, takes values in the given finite positive real interval. As special cases of these generalized inequalities we have obtained corresponding bounds for the ratio and difference of standard power means.

RESULTS

Theorem 2.1. For a random variable, discrete or continuous, which takes values on finite interval [a, b] where a > 0, we have

$$1 \le \frac{\mu_r^{'}}{\left(\mu_s^{'}\right)^{\frac{r}{s}}} \le \frac{s}{r}\frac{b^r - a^r}{b^s - a^s}\left\{\frac{r-s}{r}\frac{b^r - a^r}{a^s b^r - a^r b^s}\right\}^{\frac{r}{s}-1}, \tag{2.1}$$

where r is a positive real number and s is any non zero real number such that r > s.
If r is a negative real number with r > s then inequality (2.1) reverses its order.
Proof. For a random variable, discrete or continuous, which takes values on a finite interval [a, b] where a>0, we have [2],

$$\mu_r^{'} \le \frac{\left(b^r - a^r\right)\mu_s^{'} + a^r b^s - a^s b^r}{b^s - a^s}, \tag{2.2}$$

And $\mu_r^{'} \ge \left(\mu_s^{'}\right)^{\frac{r}{s}}$, (2.3)

where r is a positive real number and s is any non zero real number such that r>s. Since $\mu_s^{'}$ takes only positive real values we get from inequality (2.2) that,

$$\frac{\mu_r^{'}}{\left(\mu_s^{'}\right)} \le \frac{\left(b^r - a^r\right)\mu_s^{'} + a^r b^s - a^s b^r}{\left(b^s - a^s\right)\left(\mu_s^{'}\right)^{\frac{r}{s}}}. \tag{2.4}$$

Consider a function f $\left(\mu_s^{'}\right)$ defined by

$$\mathrm{f}\left(\mu_s^{'}\right) = \frac{\left(b^r - a^r\right)\mu_s^{'} + a^r b^s - a^s b^r}{\left(b^s - a^s\right)\left(\mu_s^{'}\right)^{\frac{r}{s}}}. \tag{2.5}$$

The derivative of the function f $\left(\mu_s^{'}\right)$ is given by

$$\mathrm{f'}\left(\mu_s^{'}\right) = \frac{r-s}{s}\frac{b^r - a^r}{b^s - a^s}\left(\mu_s^{'}\right)^{-\frac{r+s}{s}}\left[\frac{r}{r-s}\frac{a^s b^r - a^r b^s}{b^r - a^r} - \mu_s^{'}\right]. \tag{2.6}$$

From (2.6) we see that there is only one extremum point of f $\left(\mu_s^{'}\right)$, namely

$$\mu_s^{'} = \frac{r}{r-s}\frac{a^s b^r - a^r b^s}{b^r - a^r}. \tag{2.7}$$

It follows from Rolle's theorem that the value of $\mu_s^{'}$ given by equation (2.7) lies in the open interval (a^s, b^s) or (b^s, a^s) according as s is positive or negative. Further since $f'\left(\mu_s^{'}\right)$ changes its sign from positive to negative while $\mu_s^{'}$ passes through the value given by (2.7), we conclude that

$$f\left(\mu_s^{'}\right) \le \frac{s}{r}\frac{b^r - a^r}{b^s - a^s}\left[\frac{(r-s)\left(b^r - a^r\right)}{r\left(a^s b^r - a^r b^s\right)}\right]^{\frac{r}{s}-1}. \tag{2.8}$$

This gives the upper bound in inequality (2.1) whereas the lower bound is a consequence of inequality (2.3). If r is a negative real number with r > s, we have, [2],

$$\mu_r^{'} \geq \frac{\left(b^r - a^r\right)\mu_s^{'} + a^r b^s - a^s b^r}{b^s - a^s}, \quad (2.9)$$

and

$$\mu_r^{'} \leq \left(\mu_s^{'}\right)^{\frac{r}{s}}. \tag{2.10}$$

From inequality (2.9), we have

$$\frac{\mu_r^{'}}{\left(\mu_s^{'}\right)^{\frac{r}{s}}} \geq \frac{\left(b^r - a^r\right)\mu_s^{'} + a^r b^s - a^s b^r}{\left(b^s - a^s\right)\left(\mu_s^{'}\right)^{\frac{r}{s}}}. \tag{2.11}$$

From (2.6) we find that when r is negative with r > s, $f'\left(\mu_s^{'}\right)$ changes its sign from negative to positive while $\mu_s^{'}$ passes through the value given by (2.7) and we have

$$f\left(\mu_s^{'}\right) \geq \frac{s}{r}\frac{b^r - a^r}{b^s - a^s}\left[\frac{(r-s)\left(b^r - a^r\right)}{r\left(a^s b^r - a^r b^s\right)}\right]^{\frac{r}{s}-1}. \tag{2.12}$$

Also from (2.10),

$$\frac{\mu_r^{'}}{\left(\mu_s^{'}\right)^{\frac{r}{s}}} \leq 1. \tag{2.13}$$

From (2.12) and (2.13) we conclude that if r is a negative real number with r > s then inequality (2.1) reverses its order.

Theorem 2.2. For a random variable, discrete or continuous, which takes values on a finite interval [a, b] where a > 0, we have

$$1 \leq \frac{\mu_r^{'}}{M_0^r} \leq \frac{b^r - a^r}{r\left(\log b - \log a\right)} \frac{1}{\left(e^{\frac{b^r - a^r - r\left(a^r \log b - b^r \log a\right)}{r\left(b^r - a^r\right)}}\right)^r}, \tag{2.14}$$

where r is any non zero real number.

Proof. For a random variable, discrete or continuous, which takes values on a finite interval [a, b] where a > 0, we have [2],

$$M_0^r \leq \mu_r^{'} \leq \frac{\left(b^r - a^r\right)\log M_0 + a^r \log b - b^r \log a}{\log b - \log a}, \tag{2.15}$$

where r is any non zero real number.

Since M_0 takes only positive real value we see from inequality (2.15) that the ratio $\frac{\mu_r^{'}}{M_0^r}$ is bounded by the following inequality:

$$1 \leq \frac{\mu_r^{'}}{M_0^r} \leq \frac{\left(b^r - a^r\right)\log M_0 + a^r \log b - b^r \log a}{\left(\log b - \log a\right)M_0^r}. \quad (2.16)$$

Consider a function $f(M_0)$ defined by

$$f\left(M_0\right)=\frac{\left(b^r-a^r\right)\log M_0+a^r\log b-b^r\log a}{\left(\log b-\log a\right)M_0^r}$$

The function $f(M_0)$ has a maximum in the interval [a, b] at

$$M_0 = e^{\frac{b^r-a^r-r\left(a^r\log b-b^r\log a\right)}{r\left(b^r-a^r\right)}},$$

therefore

$$f\left(M_0\right)\le\frac{b^r-a^r}{r\left(\log b-\log a\right)}\frac{1}{\left[e^{\frac{b^r-a^r-r\left(a^r\log b-b^r\log a\right)}{r\left(b^r-a^r\right)}}\right]^r}. \tag{2.17}$$

Inequality (2.14) now follows from (2.16) and (2.17).

Theorem 2.3. For a random variable, discrete or continuous, which takes values on a finite interval [a, b] where a > 0, we have

$$0\le\mu_r^{'}-\mu_s^{'\frac{r}{s}}\le\frac{a^rb^s-a^sb^r}{b^s-a^s}+\frac{r-s}{r}\frac{b^r-a^r}{b^s-a^s}\left[\frac{s}{r}\frac{b^r-a^r}{b^s-a^s}\right]^{\frac{s}{r-s}}, \tag{2.18}$$

where r is a positive real number and s is any non zero real number such that r > s.
If r is a negative real number with r > s then inequality (2.18) reverses its order.

Proof. From inequalities (2.2) and (2.3) we see that the difference $\mu_r^{'}-\mu_s^{'\frac{r}{s}}$ is bounded by the following inequality:

$$0\le\mu_r^{'}-\mu_s^{'\frac{r}{s}}\le\frac{\left(b^r-a^r\right)\mu_s^{'}+a^rb^s-a^sb^r}{b^s-a^s}-\mu_s^{'\frac{r}{s}}, \tag{2.19}$$

where r is a positive real number and s is any non zero real number such that r > s.
Consider the function $f\left(\mu_s^{'}\right)$ defined by

$$f\left(\mu_s^{'}\right)=\frac{\left(b^r-a^r\right)\mu_s^{'}+a^rb^s-a^sb^r}{b^s-a^s}-\mu_s^{'\frac{r}{s}}. \tag{2.20}$$

The derivative of the function $f\left(\mu_s^{'}\right)$ is given by

$$f^{/}\left(\mu_s^{'}\right)=\frac{r}{s}\left[\frac{s}{r}\frac{b^r-a^r}{b^s-a^s}-\mu_s^{'\frac{r-s}{s}}\right]. \tag{2.21}$$

From (2.21) we see that there is only one extremum point of $f\left(\mu_s^{'}\right)$ namely

$$\mu_s^{'}=\left[\frac{s}{r}\frac{b^r-a^r}{b^s-a^s}\right]^{\frac{s}{r-s}}. \tag{2.22}$$

The function $f\left(\mu_s^{'}\right)$ is continuous in the interval [a, b] derivable in open interval (a, b) and $f(a^s)=f(b^s)$, therefore, it follows from Rolle's theorem that the value of $\mu_s^{'}$ given by (2.22) lies in the open interval (a^s, b^s) or (b^s, a^s) according as s is positive or negative. Further since $f^{'}\left(\mu_s^{'}\right)$

changes its sign from positive to negative while $\mu_s^{'}$ passes through the value given by (2.22), we conclude that

$$f\left(\mu_s^{'}\right) \leq \frac{a^r b^s - a^s b^r}{b^s - a^s} + \frac{r-s}{r}\frac{b^r - a^r}{b^s - a^s}\left[\frac{s}{r}\frac{b^r - a^r}{b^s - a^s}\right]^{\frac{s}{r-s}}. \tag{2.23}$$

Inequality (2.18) now follows from (2.19), (2.20) and (2.23). Proceeding essentially as above it can be easily seen that inequality (2.18) reverses its order when r is a negative real number with r > s.

Theorem 2.4. For a random variable, discrete or continuous, which takes values on a finite interval [a, b] where a > 0, we have

$$0 \leq \mu_r^{'} - M_0^r \leq \frac{1}{\log b - \log a}\left\{a^r \log b - b^r \log a + \frac{b^r - a^r}{r}\left(\log\frac{b^r - a^r}{r(\log b - \log a)} - 1\right)\right\}, \tag{2.24}$$

where r is any non zero real number.

Proof. From inequality (2.15) we see that the difference $\mu_r^{'} - M_0^r$ is bounded by the following inequality:

$$0 \leq \mu_r^{'} - M_0^r \leq \frac{\left(b^r - a^r\right)\log M_0 + a^r \log b - b^r \log a}{\log b - \log a} - M_0^r. \tag{2.25}$$

Consider a function f(M_0) defined by

$$f(M_0) = \frac{\left(b^r - a^r\right)\log M_0 + a^r \log b - b^r \log a}{\log b - \log a} - M_0^r. \tag{2.26}$$

The function f(M_0) has the maximum in the interval [a, b] at

$$M_0 = \left[\frac{b^r - a^r}{r\left(\log b - \log a\right)}\right]^{\frac{1}{r}},$$

therefore

$$f(M_0) \leq \frac{1}{\log b - \log a}\left[a^r \log b - b^r \log a + \frac{b^r - a^r}{r}\left(\log\frac{b^r - a^r}{r(\log b - \log a)} - 1\right)\right]. \tag{2.27}$$

Inequality (2.24) now follows from (2.25), (2.26) and (2.27).

APPLICATIONS

On using the results derived in section 2 and giving particular values to r and s it is possible to derive a host of results which give bounds for the ratio and difference of Harmonic mean (H), Geometric mean (G), Arithmetic mean (A) and root mean square (R) when random variable, discrete or continuous, takes values in the interval [a, b]. If we put r = 1 and s = –1 in inequality (2.1) we get bounds for the ratio A/H, if we put these values of r and s in inequality (2.18) we get bounds for the difference A – H and so on. In particular the following bounds are deduced from the general results.

$$1 \leq \frac{R}{A} \leq \frac{a+b}{2\sqrt{ab}}, \tag{3.1}$$

$$1 \leq \frac{R}{H} \leq \frac{2}{ab\left(a+b\right)}\left[\frac{a^2 + ab + b^2}{3}\right]^{\frac{3}{2}}, \tag{3.2}$$

$$1 \le \frac{R}{G} \le \left(\frac{b^2 - a^2}{2e \log \frac{b}{a}} \right)^{\frac{1}{2}} \left(\frac{b^{a^2}}{a^{b^2}} \right)^{\frac{1}{b^2 - a^2}}, \tag{3.3}$$

$$1 \le \frac{A}{H} \le \frac{(a+b)^2}{4ab}, \tag{3.4}$$

$$1 \le \frac{A}{G} \le \frac{b-a}{e \log \frac{b}{a}} \left(\frac{b^a}{a^b} \right)^{\frac{1}{b-a}}, \tag{3.5}$$

$$1 \le \frac{G}{H} \le \frac{b-a}{e(\log b - \log a)ab} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}}, \tag{3.6}$$

$$0 \le R - A \le \frac{1}{a+b} \left(\frac{b-a}{2} \right)^2, \tag{3.7}$$

$$0 \le A - H \le \left(\sqrt{b} - \sqrt{a} \right)^2, \tag{3.8}$$

and

$$0 \le A - G \le \frac{1}{\log \frac{b}{a}} \log \frac{b^a}{a^b} \left[\frac{b-a}{e \log \frac{b}{a}} \right]^{b-a}. \tag{3.9}$$

Further the bounds for the ratios A/R, H/R, G/R, H/A, G/A and H/G can be easily obtained from inequalities (3.1)-(3.6). The bounds for the difference A–R, H–A and G–A can also be obtained from inequalities (3.7), (3.8) and (3.9) respectively.

Karl pearson's coefficient of dispersion "V" is an important and widely used measure of dispersion. It is defined as

$$V = \frac{\sigma}{\mu_1'}, \tag{3.10}$$

where σ is the standard deviation. Kendall and Stuart [3] have remarked that there is in general no bound for this measure. We here find a bound on this measure "V" when random variable takes values in the interval [a, b] with a > 0. From inequality (3.1), we have

$$1 \le \frac{R^2}{A^2} \le \frac{(a+b)^2}{4ab},$$

Or

$$0 \le \frac{\mu_2'}{\mu_1'^2} - 1 \le \frac{(b-a)^2}{4ab},$$

Or

$$0 \le \frac{\sigma^2}{\mu_1'^2} \le \frac{(b-a)^2}{4ab}. \tag{3.11}$$

From equation (3.10) and (3.11) we conclude that

$$0 \le V \le \frac{b-a}{2\sqrt{ab}}. \tag{3.12}$$

Inequality (3.12) gives the bound on Karl pearson's coefficient of dispersion V.

We now consider the case of a discrete frequency distribution in which variable takes values x_i with corresponding frequencies f_i , (i = 1,2, …, n). For discrete frequency distribution the mean $\mu_1^{'}$ and variance σ^2 are respectively defined as,

$$\mu_1^{'} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i, \qquad (3.13)$$

and

$$\sigma^2 + \mu_1^{'2} = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2, \qquad (3.14)$$

where

(3.15) $$\text{N} = \sum_{i=1}^{n} f_i. \qquad (3.15)$$

For a = $x_1 \le x_2 \le \ldots \le x_n = b$ we can write equation (3.14) as

$$\sigma^2 + \mu_1^{'2} = \frac{a^2+b^2}{N} + \frac{(f_1-1)a^2 + f_2x_2^2 + \ldots + (f_n-1)b^2}{N}$$

$$\ge \frac{a^2+b^2}{N} + \frac{N-2}{N}\left[\frac{(f_1-1)a + f_2x_2 + \ldots + (f_n-1)b}{N-2}\right]^2. \qquad (3.16)$$

From equation (3.13) we have

$$(f_1 - 1)\, a + f_2x_2 + \ldots + (f_n - 1)\, b = N\mu_1^{'} - a - b\,. \qquad (3.17)$$

Combining (3.16) and (3.17) we get that

$$\sigma^2 \ge \frac{N-1}{N(N-2)}(b-a)^2 - \frac{2}{(N-2)}(b-\mu_1^{'})(\mu_1^{'}-a). \qquad (3.18)$$

Kendall and Stuart [3] have given following upper bound on V for discrete frequency distribution,

$$V \le \sqrt{N-1}\,. \qquad (3.19)$$

It is clear that for discrete frequency distribution the coefficient of dispersion V is bounded above by inequality (3.12) when the random variable takes values in the interval [a, b] with a > 0. The lower bound on V can be obtained from inequality (3.18). Consider a function $f(\mu_1^{'})$ defined by

$$f(\mu_1^{'}) = \frac{(N-1)(b-a)^2 - 2N(b-\mu_1^{'})(\mu_1^{'}-a)}{N(N-2)\mu_1^{'2}}\,. \qquad (3.20)$$

The function $f(\mu_1^{'})$ has minimum in the interval [a, b] when

$$\mu_1^{'} = \frac{(N-1)(a^2+b^2)+2ab}{N(a+b)}, \qquad (3.21)$$

therefore from (3.20) and (3.21) we have

$$f(\mu_1^{'}) \ge \frac{(b-a)^2}{(N-1)(a^2+b^2)+2ab}. \qquad (3.22)$$

From (3.10), (3.20), (3.21) and (3.22) we conclude that

$$V \geq \frac{b-a}{\sqrt{(N-1)\left(a^2+b^2\right)+2ab}}. \tag{3.23}$$

Inequality (3.23) gives the lower bound on Karl pearson coefficient of dispersion; cf. also [4]. For more detail and related results one may also see [5-7]

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REFERENCES

[1] J N Kapur and A Rani, "Testing the consistency of given values of a set of moments of a probability distribution, *J .Bihar Math*. SOC16,51-63, 1995.

[2] R Sharma, R G Shandil, S Devi and M Dutta, "Some inequalitiesbetween moments of probability distribution," *J Inequal. Pure & Appl. Math*;5(2),Art.86,2004 .

[3] Kendall and stuart, Vol. 1, the Advanced Theory of statistics,fourth edition, Charles Griffin & Co. London,1977.

[4] R.Sharma, A. Kaura, M.Gupta and S.Ram, Some Bounds on Sample Parameters with Refinements of Samuelson and Brunk inequalities", *Journal of Mathematical Inequalities*;Volume3,no.1(2009),99-106.

[5] B.C.Rennie, On a class of Inequalities. *J. Austral. Math. Soc.* 3, 442-448, (1963).

[6] J.N. Kapur, and H.K. Kesavan, Entropy Optimization Principles and their Applications, Academic Press USA (1994).

[7] Sita Ram, Ph.D. Thesis, On a class of inequalities involving moments and weighted power means, H.P. University, July, (2005).